MAX PLANCK INSTITUTE
FOR DYNAMICS OF COMPLEX
TECHNICAL SYSTEMS
MAGDEBURG

CSC

COMPUTATIONAL METHODS IN
SYSTEMS AND CONTROL THEORY

Model Reduction for Dynamical Systems

–Lecture 4–

Peter Benner   Lihong Feng

Otto-von-Guericke Universitaet Magdeburg
Faculty of Mathematics
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Max Planck Institute for Dynamics of Complex Technical Systems
Computational Methods in Systems and Control Theory
Magdeburg, Germany

benner@mpi-magdeburg.mpg.de   feng@mpi-magdeburg.mpg.de

http://www.mpi-magdeburg.mpg.de/csc/teaching/17ss/mor

Outline

1. Mathematical Basics III
   Systems and control theory

Realizations of Linear Systems (with $E = I_n$ for simplicity)

Definition

The **McMillan degree** of Σ is the unique minimal number $\hat{n} \geq 0$ of states necessary to describe the input-output behavior completely.
A **minimal realization** is a realization $(\hat{A}, \hat{B}, \hat{C}, \hat{D})$ of Σ with order $\hat{n}$.

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For stable systems, the infinite Gramians are defined as

Infinite Gramians

$$P = \int_0^\infty e^{At} B B^T e^{A^T t} dt.$$

$$Q = \int_0^\infty e^{A^T t} C^T C e^{At} dt.$$

Balanced Realizations

Definition

A realization (A, B, C, D) of a linear system Σ is **balanced** if its infinite controllability/observability Gramians P/Q satisfy

$$P = Q = \operatorname{diag}\{\sigma_1, \ldots, \sigma_n\} \quad (\text{w.l.o.g. } \sigma_j \geq \sigma_{j+1},\ j = 1, \ldots, n-1).$$

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When does a balanced realization exist?
Assume A to be Hurwitz, i.e. $\Lambda(A) \subset \mathbb{C}^-$. Then:

Theorem

Given a **stable** minimal linear system $\Sigma : (A, B, C, D)$, a balanced realization is obtained by the state-space transformation with

$$T_b := \Sigma^{-\frac{1}{2}} V^T R,$$

where $P = S^T S$, $Q = R^T R$ (e.g., Cholesky decompositions) and $SR^T = U \Sigma V^T$ is the SVD of SR^T.

Proof. Exercise!

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$\sigma_1, \ldots, \sigma_n$ are the **Hankel singular values** of Σ.

Note: $\sigma_1, \ldots, \sigma_n \geq 0$ as $P, Q \geq 0$ by definition, and $\sigma_1, \ldots, \sigma_n > 0$ in case of minimality! For nonbalanced systems, the Hankel singular values can be computed by $\{\sigma_1, \ldots, \sigma_n\} = (\Lambda(PQ))^{\frac{1}{2}}$. I.e. they are the square roots of the eigenvalues of PQ.

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The infinite controllability/observability Gramians P/Q satisfy the **Lyapunov equations**

$$AP + PA^T + BB^T = 0, \quad A^T Q + QA + C^T C = 0.$$

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Proof. (For controllability Gramian only, observability case is analogous!)

$$\begin{aligned}
AP + PA^T + BB^T &= A\int_0^\infty e^{At}BB^Te^{A^Tt}dt + \int_0^\infty e^{At}BB^Te^{A^Tt}dt\,A^T + BB^T \\
&= \int_0^\infty \underbrace{Ae^{At}BB^Te^{A^Tt} + e^{At}BB^Te^{A^Tt}A^T}_{=\frac{d}{dt}e^{At}BB^Te^{A^Tt}}dt + BB^T \\
&= \underbrace{\lim_{t\to\infty} e^{At}BB^Te^{A^Tt}}_{=0} - \underbrace{e^{A\cdot 0}}_{=I_n}BB^T\underbrace{e^{A^T\cdot 0}}_{=I_n} + BB^T \\
&= 0.
\end{aligned}$$

Balanced Realizations

Theorem

The Hankel singular values (HSVs) of a stable minimal linear system are system invariants, i.e. they are unaltered by state-space transformations!

Proof. The HSVs of a stable minimal linear system are $(\Lambda(PQ))^{\frac{1}{2}}$. Now let

$$(\hat{A}, \hat{B}, \hat{C}, D) = (TAT^{-1}, TB, CT^{-1}, D)$$

be any transformed realization with associated controllability Lyapunov equation

$$0 = \hat{A}\hat{P} + \hat{P}\hat{A}^T + \hat{B}\hat{B}^T = TAT^{-1}\hat{P} + \hat{P}T^{-T}A^TT^T + TBB^TT^T.$$

This is equivalent to

$$0 = A(T^{-1}\hat{P}T^{-T}) + (T^{-1}\hat{P}T^{-T})A^T + BB^T.$$

The uniqueness of the solution of the Lyapunov equation (for stable systems) implies that $\hat{P} = TPT^T$ and, analogously, $\hat{Q} = T^{-T}QT^{-1}$.

Balanced Realizations

Therefore,

$$\hat{P}\hat{Q} = TPQT^{-1},$$

showing that $\Lambda(\hat{P}\hat{Q}) = \Lambda(PQ) = \{\sigma_1^2, \ldots, \sigma_n^2\}$.

Remark

For non-minimal systems, the Gramians can also be transformed into diagonal matrices with the leading $\hat{n} \times \hat{n}$ submatrices equal to $\mathrm{diag}(\sigma_1, \ldots, \sigma_{\hat{n}})$, and

$$\hat{P}\hat{Q} = \mathrm{diag}(\sigma_1^2, \ldots, \sigma_{\hat{n}}^2, 0, \ldots, 0).$$

see [Laub/Heath/Paige/Ward 1987, Tombs/Postlethwaite 1987].