MAX PLANCK INSTITUTE
FOR DYNAMICS OF COMPLEX
TECHNICAL SYSTEMS
MAGDEBURG

CSC
COMPUTATIONAL METHODS IN
SYSTEMS AND CONTROL THEORY

Model Reduction for Dynamical Systems

–Lecture 5–

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Outline

1. Mathematical Basics IV
   Systems and control theory

System Norms

Definition

The $L_2^n(-\infty, +\infty)$ space is the vector-valued function space $f : \mathbb{R} \mapsto \mathbb{R}^n$, with the norm

$$\|f\|_{L_2^n} = \left(\int_{-\infty}^{\infty} ||f(t)||^2 dt \right)^{1/2}.$$

Here and below, $|| \cdot ||$ denotes the Euclidean vector or spectral matrix norm.

Definition

The frequency domain $\mathcal{L}_2(\jmath\mathbb{R})$ space is the matrix-valued function space $F : \mathbb{C} \mapsto \mathbb{C}^{p\times m}$, with the norm

$$||F||_{\mathcal{L}_2} = \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} ||F(\jmath\omega)||^2 d\omega \right)^{1/2},$$

System Norms

Definition

The $L_\infty^n(-\infty, +\infty)$ space is the vector-valued function space $f : \mathbb{R} \mapsto \mathbb{R}^n$, with the norm

$$\|f\|_{L_\infty^n} = \sup_t \|f(t)\|_\infty .$$

System Norms

The maximum modulus theorem will be used repeatedly.

Theorem

Let $f(z) : \mathbb{C}^n \mapsto \mathbb{C}$ be a regular analytic, or holomorphic, function of n complex variables $z = (z_1, \ldots, z_n), n \geq 1$, defined on an (open) domain $\mathbb{D}$ of the complex space $\mathbb{C}^n$, which is not a constant, $f(z) \neq$ const. Let

$$max_f = \max\{|f(z)| : z \in \mathbb{D}\}.$$

If $f(z)$ is continuous in a bounded domain $\mathbb{D}$, then max_f can only be attained on the boundary of $\mathbb{D}$, i.e. $max_f = \max\{|f(z)| : z \in \partial\mathbb{D}\}$.

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Consider the transfer function $G(s) = C(sI - A)^{-1}B + D$, and input functions $u \in \mathcal{L}_2(\jmath\mathbb{R})$, with the $\mathcal{L}_2$-norm

$$||u||^2_{\mathcal{L}_2} := \frac{1}{2\pi}\int_{-\infty}^{\infty} u(\jmath\omega)^H u(\jmath\omega)\, d\omega.$$

System Norms

Assume A is (asymptotically) stable: $\Lambda(A) \subset \mathbb{C}^- := \{z \in \mathbb{C} : \mathrm{Re}\,(z) < 0\}$. Then G is analytic in $\mathbb{C}^+ \cup \jmath\mathbb{R}$, and following the maximal modulus theorem, $G(s)$ is bounded: $||G(s)||_F \leq M < \infty$, $\forall s \in \mathbb{C}^+ \cup \jmath\mathbb{R}$. Thus we have

$$\begin{aligned}\int_{-\infty}^{\infty} y(\jmath\omega)^H y(\jmath\omega)\, d\omega &= \int_{-\infty}^{\infty} u(\jmath\omega)^H G(\jmath\omega)^H G(\jmath\omega) u(\jmath\omega)\, d\omega \\ &= \int_{-\infty}^{\infty} ||G(\jmath\omega)u(\jmath\omega)||^2\, d\omega \leq \int_{-\infty}^{\infty} M^2 ||u(\jmath\omega)||^2 d\omega \\ &= M^2 \int_{-\infty}^{\infty} u(\jmath\omega)^H u(\jmath\omega)\, d\omega \ < \ \infty,\end{aligned}$$

So that $y = Gu \in \mathcal{L}_2(\jmath\mathbb{R})$. $(||Ax||_\infty \leq ||Ax||_2 \leq ||A||_F ||x||_2)$
Consequently, the $\mathcal{L}_2$-induced operator norm is well defined:

$$||G||_{\mathcal{L}_\infty} := \sup_{||u||_2 \neq 0} \frac{||Gu||_{\mathcal{L}_2}}{||u||_{\mathcal{L}_2}}. \tag{1}$$

System Norms

It can be further proved that

$$||G||_{\mathcal{L}_\infty} = \sup_{\omega\in\mathbb{R}} ||G(\jmath\omega)|| = \sup_{\omega\in\mathbb{R}} \sigma_{max}\left(G(\jmath\omega)\right).$$

With the above defined $\mathcal{L}_\infty$-norm, the frequency domain $\mathcal{L}_\infty$ space is defined as

Definition

The frequency domain $\mathcal{L}_\infty(\jmath\mathbb{R})$ space is the matrix-valued function space $F : \mathbb{C} \mapsto \mathbb{C}^{p\times m}$, with the norm

$$||F||_{\mathcal{L}_\infty} = \sup_{\omega\in\mathbb{R}} ||F(\jmath\omega)|| = \sup_{\omega\in\mathbb{R}} \sigma_{max}\left(F(\jmath\omega)\right).$$

System Norms

Error bound 1

$$||Gu||_{\mathcal{L}_2} \leq ||G||_{\mathcal{L}_\infty} ||u||_{\mathcal{L}_2}$$

Consequently,

$$||y - \hat{y}||_{\mathcal{L}_2} = ||Gu - \hat{G}u||_{\mathcal{L}_2} \leq ||G - \hat{G}||_{\mathcal{L}_\infty} ||u||_{\mathcal{L}_2}$$

System Norms

When the funtion has better property, for example analytic, then we can define $\mathcal{H}$ norms for these functions.

Definition

The Hardy space $\mathcal{H}_\infty$ is the function space of matrix-, scalar-valued functions that are analytic and bounded in $\mathbb{C}^+ := \{z \in \mathbb{C} : \mathrm{Re}(z) > 0\}$.

The $\mathcal{H}_\infty$-norm is defined as

$$||F||_{\mathcal{H}_\infty} := \sup_{z\in\mathbb{C}^+} ||F(z)|| = \sup_{\omega\in\mathbb{R}} ||F(\jmath\omega)|| = \sup_{\omega\in\mathbb{R}} \sigma_{max}\left(F(\jmath\omega)\right).$$

The second equality follows the maximum modulus theorem.

System Norms

Definition

The Hardy space $\mathcal{H}_2(\mathbb{C}^+)$ is the function space of matrix-, scalar-valued functions that are analytic in $\mathbb{C}^+$ and bounded w.r.t. the $\mathcal{H}_2$-norm defined as

$$\begin{aligned} ||F||_{\mathcal{H}_2} &:= \frac{1}{2\pi}\left(\sup_{\sigma>0}\int_{-\infty}^{\infty}||F(\sigma+\jmath\omega)||_F^2 d\omega\right)^{\frac{1}{2}} \\ &= \frac{1}{2\pi}\left(\int_{-\infty}^{\infty}||F(\jmath\omega)||_F^2\, d\omega\right)^{\frac{1}{2}}. \end{aligned}$$

The last equality follows maximum modulus theorem.

System Norms

Theorem [[Antoulas '05]](Section 5.5.1)

Practical Computation of the $\mathcal{H}_2$-norm of the transfer function

$$||G||^2_{\mathcal{H}_2} = \mathrm{tr}(B^T QB) = \mathrm{tr}(CPC^T),$$

where P, Q are the controllability and observability Gramians (the infinite Gramians) of the corresponding LTI system.

System Norms

Following [Antoulas, Beattie, Gugercin '10][1] (pp. 15-16), the $\mathcal{H}_2$ approximation error gives the following bound

$$
\begin{aligned}
&\max_{t>0} \|y(t) - \hat{y}(t)\|_\infty \\
&= \max_{t>0} \left|\left|\frac{1}{2\pi}\int_{-\infty}^{\infty}(y(\jmath\omega) - \hat{y}(\jmath\omega))e^{\jmath\omega t}d\omega\right|\right|_\infty \quad (\text{inverse Fourier transform}) \\
&\leq \max_{t>0} \frac{1}{2\pi}\int_{-\infty}^{\infty} ||(y(\jmath\omega) - \hat{y}(\jmath\omega))||_\infty |\cos\omega t + \jmath\sin\omega t| d\omega \\
&\leq \frac{1}{2\pi}\int_{-\infty}^{\infty} ||(G(\jmath\omega) - \hat{G}(\jmath\omega))||_F ||u(\jmath\omega)||_2 d\omega \quad (*) \\
&\leq \left(\frac{1}{2\pi}\int_{-\infty}^{\infty} ||(G(\jmath\omega) - \hat{G}(\jmath\omega))||_F^2\right)^{1/2} \left(\frac{1}{2\pi}\int_{-\infty}^{\infty} ||u(\jmath\omega)||_2^2\right)^{1/2} d\omega \\
&\qquad\qquad (\text{ Cauchy-Schwarz inequality}) \\
&= \|G - \hat{G}\|_{\mathcal{H}_2} \|u\|_{\mathcal{L}_2},
\end{aligned}
$$

where (*) uses the facts $y(\jmath\omega) = G(\jmath\omega)u(\jmath\omega)$ and $||Ax||_\infty \leq ||Ax||_2 \leq ||A||_F ||x||_2$ (http://de.wikipedia.org/wiki/Frobeniusnorm). G and $\hat{G}$ are original and reduced transfer functions. $||\cdot||_\infty$ is the vector norm in Euclidean space for any fixed t.

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[1] A. C. Antoulas, C. A. Beattie, S. Gugercin. Interpolatory Model Reduction of Large-scale Dynamical Systems.

System Norms

Then

Error bound 2

$$||y - \hat{y}||_{\infty} \leq ||G - \hat{G}||_{\mathcal{H}_2} ||u||_{\mathcal{L}_2}.$$

System Norms

(Plancherel Theorem)

The Fourier transform of $f \in L_2^n(-\infty, \infty)$:

$$F(\xi) = \int_{-\infty}^{\infty} f(t) e^{-\xi t} dt$$

is a Hilbert space isomorphism between $L_2^n(-\infty, \infty)$ and $\mathcal{L}_2(\jmath\mathbb{R})$. Furthermore, the Fourier transform maps $L_2^n(0, \infty)$ onto $\mathcal{H}_2(\mathbb{C}^+)$. In addition it is an isometry, that is, it preserves distances:

$$L_2^n(-\infty, \infty) \cong \mathcal{L}_2(\jmath\mathbb{R}), \quad L_2^n(0, \infty) \cong \mathcal{H}_2(\mathbb{C}^+).$$

Consequently, L_2^n-norm in time domain and $\mathcal{L}_2$-norm, $\mathcal{H}_2$-norm in frequency domain coincide.

Approximation Problems

Therefore the Error bound 1,

$$||y - \hat{y}||_2 = ||Gu - \hat{G}u||_2 \leq ||G - \hat{G}||_{\mathcal{L}_\infty} ||u||_2, \qquad (2)$$

holds in time and frequency domain due to Plancherel theorem, i.e. the $||\cdot||_2$ in (2) can be the L_2^n-norm in time domain, or the $\mathcal{L}_2$-norm in frequency domain.

The transfer function is analytic, therefore $||G||_{\mathcal{H}_\infty}$ is defined. Futhermore, from their definitions, we have

$$||G||_{\mathcal{L}_\infty} = ||G||_{\mathcal{H}_\infty},$$

so that,

$$||y - \hat{y}||_2 \leq ||G - \hat{G}||_{\mathcal{H}_\infty} ||u||_2.$$

Approximation Problems

Similarly, the Error bound 2 holds as

$$||y - \hat{y}||_\infty \leq ||G - \hat{G}||_{\mathcal{H}_2}||u||_2,$$

where $||\cdot||_2$ can be the L_2^n-norm in time domain, or the $\mathcal{L}_2$-norm in frequency domain.
Finally, we get two error bounds,

Output errors bounds

$$\begin{aligned} ||y - \hat{y}||_2 &\leq ||G - \hat{G}||_{\mathcal{H}_\infty}||u||_2 & \Longrightarrow \left|\left|G - \hat{G}\right|\right|_\infty < \mathrm{tol} \\ ||y - \hat{y}||_\infty &\leq ||G - \hat{G}||_{\mathcal{H}_2}||u||_2 & \Longrightarrow ||G - \hat{G}||_{\mathcal{H}_2} < \mathrm{tol} \end{aligned}$$

Goal of MOR: $||G - \hat{G}||_\infty < tol$ or $||G - \hat{G}||_{\mathcal{H}_2} < tol$.

Approximation Problems

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$$||y - \hat{y}||_\infty \leq ||G - \hat{G}||_{\mathcal{H}_2}||u||_2,$$

where $||\cdot||_2$ can be the L_2^n-norm in time domain, or the $\mathcal{L}_2$-norm in frequency domain.

Finally, we get two error bounds,

Output errors bounds

$$\begin{aligned} ||y - \hat{y}||_2 &\leq ||G - \hat{G}||_{\mathcal{H}_\infty}||u||_2 \qquad \Longrightarrow \left|\left|G - \hat{G}\right|\right|_\infty < \mathrm{tol} \\ ||y - \hat{y}||_\infty &\leq ||G - \hat{G}||_{\mathcal{H}_2}||u||_2 \qquad \Longrightarrow ||G - \hat{G}||_{\mathcal{H}_2} < \mathrm{tol} \end{aligned}$$

Goal of MOR: $||G - \hat{G}||_\infty < tol$ or $||G - \hat{G}||_{\mathcal{H}_2} < tol$.

Computable error measures

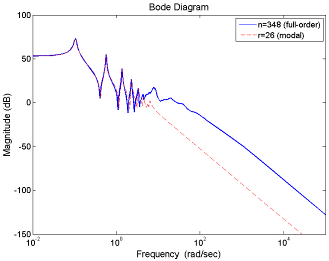
Evaluating system norms is computationally very (sometimes too) expensive.

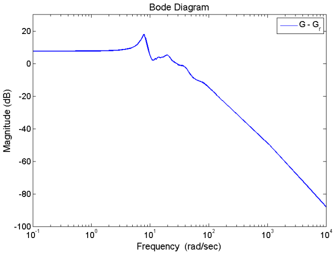
Other measures

- absolute errors $\left|\left|G(\jmath\omega_j) - \hat{G}(\jmath\omega_j)\right|\right|_2$, $\left|\left|G(\jmath\omega_j) - \hat{G}(\jmath\omega_j)\right|\right|_\infty$ $(j = 1, \ldots, N_\omega)$;
- relative errors $\frac{\left|\left|G(\jmath\omega_j) - \hat{G}(\jmath\omega_j)\right|\right|_2}{||G(\jmath\omega_j)||_2}$, $\frac{\left|\left|G(\jmath\omega_j) - \hat{G}(\jmath\omega_j)\right|\right|_\infty}{||G(\jmath\omega_j)||_\infty}$;
- "eyeball norm", i.e. look at **frequency response/Bode (magnitude) plot**: for SISO system, log-log plot frequency vs. $|G(\jmath\omega)|$ (or $|G(\jmath\omega) - \hat{G}(\jmath\omega)|$) in decibels, 1 dB $\simeq 20\log_{10}$(value).

Computable error measures

For MIMO systems, $q \times m$ array of plots G_{ij}.





References

1. A.C. Antoulas.
   Approximation of Large-Scale Dynamical Systems.
   SIAM Publications, Philadelphia, PA, 2005.