MAX PLANCK INSTITUTE FOR DYNAMICS OF COMPLEX TECHNICAL SYSTEMS MAGDEBURG

CSC COMPUTATIONAL METHODS IN SYSTEMS AND CONTROL THEORY

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Model Reduction for Dynamical Systems
Lecture 6

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Outline

- MOR: Balanced truncation

Overlook

Balanced truncation: first balancing, then truncate.

Given a LTI system:

$$dx(t)/dt = Ax(t) + Bu(t)$$

$$y(t) = L^T x(t)$$

For convenience of discussion, we denote the system as a block form:

$$\left(\begin{array}{c|c} A & B \\ \hline L^T & \end{array}\right) \xrightarrow{\text{Balancing } T} \left(\begin{array}{c|c} \tilde{A} & \tilde{B} \\ \hline \tilde{L}^T & \end{array}\right) = \left(\begin{array}{cc|c} \tilde{A}_{11} & \tilde{A}_{12} & \tilde{B}_1 \\ \tilde{A}_{21} & \tilde{A}_{22} & \tilde{B}_2 \\ \hline \tilde{L}_1^T & \tilde{L}_2^T & \end{array}\right)$$

truncate

$$\left(\begin{array}{c|c} \tilde{A}_{11} & \tilde{B}_1 \\ \hline \tilde{L}_1^T & \end{array}\right)$$

reduced model

The unimportant part is truncated

Eigenspaces of P and Q

——— make the two measures practically computable

Till now it seems we could do the truncation by finding subspace spanned by the eigenvectors corresponding to the small eigenvalues of P or Q.

However, it could happen that states which are difficult to reach produce the maximal energy of observation; states which produce the smallest energy of observation are nevertheless the easiest to reach!

For such system, we do not know which states to truncate!

Eigenspaces of P and Q

—— make the two measures practically computable

Example: Consider the following LTI system

$$dx(t)/dt = Ax(t) + Bu(t)$$
$$y(t) = L^T x(t)$$

$$A = \begin{pmatrix} 1 & 3 \\ -1 & -2 \end{pmatrix}, B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, L = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

The two Gramians are:

$$P = \begin{pmatrix} 2.5 & -1 \\ -1 & 0.5 \end{pmatrix}, Q = \begin{pmatrix} 0.5 & 0.5 \\ 0.5 & 1 \end{pmatrix}$$

Their eigenvalues and eigenvectors are:

$$\xi_1^P = \begin{pmatrix} 0.92388 \\ -0.38268 \end{pmatrix}, \lambda_1 = 2.91421 \qquad \xi_1^Q = \begin{pmatrix} 0.52573 \\ 0.85865 \end{pmatrix}, \lambda_1^Q = 1.30901$$

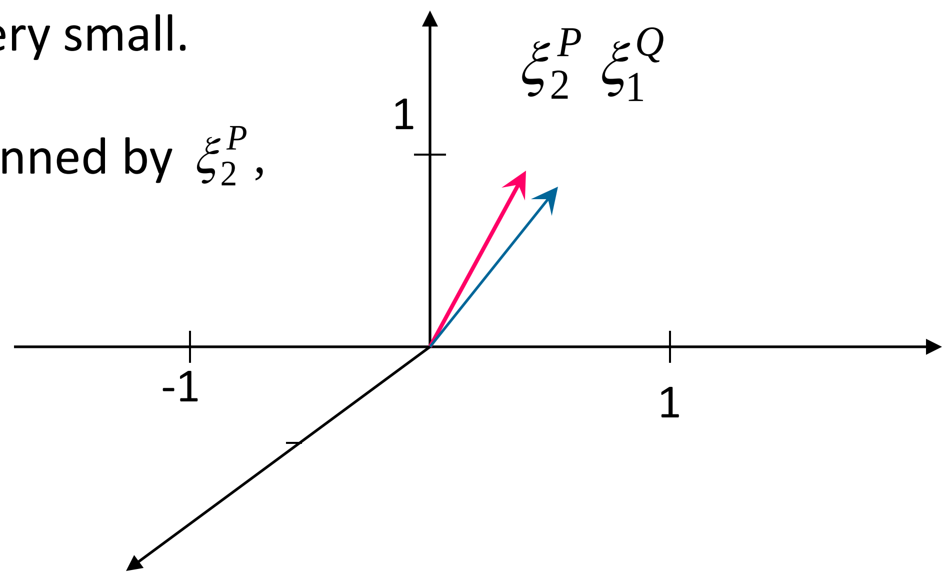
$$\xi_2^P = \begin{pmatrix} 0.38268 \\ 0.92388 \end{pmatrix}, \lambda_2^P = 0.08578 \qquad \xi_2^Q = \begin{pmatrix} -0.85865 \\ 0.52573 \end{pmatrix}, \lambda_2^Q = 0.19098$$

Eigenspaces of P and Q

—— make the two measures practically computable

$$\xi_2^P = \begin{pmatrix} 0.38268 \\ 0.92388 \end{pmatrix}, \lambda_2^P = 0.08578 \qquad \xi_1^Q = \begin{pmatrix} 0.52573 \\ 0.85865 \end{pmatrix}, \lambda_1^Q = 1.30901$$

The angle between ξ_2^P, and ξ_1^Q is very small.



This means if S is the subspace spanned by ξ_2^P, then the easily observable states

$$x = \alpha_1 \xi_1^Q + \alpha_2 \xi_2^Q, \alpha_1 >> \alpha_2$$

might also be in S.

It tells us if we truncate the states which are difficult to reach (the states locate in S), we risk truncating the states which are easy to observe (produce the maximal energy of observation) , because they might also be in S).

Eigenspaces of P and Q

—— make the two measures practically computable

However, if *P* and *Q* have the same eigenvalues and eigenvectors, then the problems is solved.

The states in the subspace spanned by the eigenvectors of P corresponding to the small eigenvalues always in the subspace spanned by the eigenvectors of Q corresponding to the small eigenvalues, because the eigenvalues are the same and eigenvectors are the same, therefore the subspaces are the same.

Can we achieve this?

Yes.

We can achieve it by balancing.

Basic idea of balancing transformation:

Use state space transformation $\tilde{x} = Tx$ to get another realization of the same system, so that the transformed Gramians are diagonal matrices.

Definition of Balancing transformation:

Finding a nonsingular matrix *T*, such that $\tilde{P} = TPT^T, \tilde{Q} = T^{-T}QT^{-1}$ and $\tilde{P} = \tilde{Q}$.

Definition of Balanced system:

The reachable, observable and stable LTI system is balanced, if its two Gramians are equal $P = Q$, it is principal-axis balanced if .

$$P = Q = \Sigma = diag(\sigma_1, \cdots, \sigma_n).$$

Basic idea of balancing transformation:

Use state space transformation $\tilde{x} = Tx$ to get another realization of the same system, so that the transformed Gramians are equal and are diagonal matrices. I.e.

$$\tilde{P} = TPT^{T} = \Sigma, \quad \tilde{Q} = T^{-T}QT^{-1} = \Sigma$$

How to construct T?

Recall that $\tilde{P}\tilde{Q} = TPQT^{-1}$.

Since $\tilde{P}\tilde{Q} = \Sigma^2$, we have $TPQT^{-1} = \Sigma^2$, which means $PQ = T^{-1}\Sigma^2 T$.

T should be the inverse of the matrix Y of eigenvectors of PQ.

Check : $\tilde{P} = TPT^T$ = ? How to make $TPT^T = \Sigma$?

If $P = UU^T$, then $TPT^T = TUU^TT^T = \tilde{T}U^{-1}UU^TU^{-T}\tilde{T}^T = I$ if $\tilde{T}\tilde{T}^T = I$.

Here we must have the relation $T = \tilde{T}U^{-1}$.

If further $T = \Sigma^{1/2}\tilde{T}U^{-1}$, then $TPT^T = \Sigma^{1/2}\tilde{T}U^{-1}UU^TU^{-T}\tilde{T}^T\Sigma^{1/2} = \Sigma$.

How to compute $\tilde{T}$?

Substitute $T = \Sigma^{1/2}\tilde{T}U^{-1}$ into $TPQT^{-1} = \Sigma^2$, we get

$$\Sigma^{1/2}\tilde{T}U^{-1}PQU\tilde{T}^{-1}\Sigma^{-1/2} = \Sigma^2$$

The left hand side $= \Sigma^{1/2}\tilde{T}U^{-1}UU^TQU\tilde{T}^{-1}\Sigma^{-1/2} = \Sigma^{1/2}\tilde{T}U^TQU\tilde{T}^{-1}\Sigma^{-1/2}$.

Look at the right hand side, we get

$$\Sigma^{1/2}\tilde{T}U^TQU\tilde{T}^{-1}\Sigma^{-1/2} = \Sigma^2,$$

i.e. $\tilde{T}U^TQU\tilde{T}^{-1} = \Sigma^2$. Therefore $\tilde{T}$ is the inverse of the matrix of eigenvectors of U^TQU.

Furtunately, U^TQU is a s.p.d. matrix. therefore the inverse of the matrix of eigenvectors is exactly the transpose of the matrix itself. So that we do not have to compute the inverse.

The above analysis clearly shows that:

Existence of balancing transformation: $dx(t)/dt = Ax(t) + Bu(t)$

$$y(t) = L^T x(t)$$

Given a reachable, observable and stable LTI system
and the corresponding Gramians *P* and *Q*, a (principal axis) balancing
transformation is given as follows:

$$T = \Sigma^{1/2} K^T U^{-1} \qquad and \qquad T^{-1} = U K^{-T} \Sigma^{-1/2}$$

Here, $P = UU^T$ is the Cholesky factorization of *P*. $U^T Q U = K \Sigma^2 K^T$
is the eigen-decomposition of $U^T Q U$. (Symmetric positive semi-definite matrix has real non-negtive eigenvalues and orthogonal eigenvectors. Here, the Eigenvectors in *K* are taken as orthonormal)

What is the corresponding balanced system?

Apply the state space tansformation: $\tilde{x} = Tx$ to the original realization:

$$\begin{aligned} dx(t)/dt &= Ax(t) + Bu(t) \\ y(t) &= L^T x(t) \end{aligned} \quad \xrightarrow{\tilde{x} = Tx} \quad \begin{aligned} d\tilde{x}(t)/dt &= TAT^{-1}\tilde{x}(t) + TBu(t) \\ y(t) &= L^T T^{-1}\tilde{x}(t) \end{aligned}$$

Balancing :

- Given
$$dx(t)/dt = Ax(t) + Bu(t)$$
$$y(t) = L^T x(t)$$

- Compute *P*, *Q*.

- Compute $P = UU^T \quad U^T Q U = K\Sigma^2 K^T$ The eigenvalues are ordered from the largest to the smallest

- $$dx(t)/dt = Ax(t) + Bu(t)$$
$$y(t) = L^T x(t)$$
$$\xrightarrow[T^{-1} = UK\Sigma^{-1/2}]{T = \Sigma^{1/2} K^T U^{-1}}$$
$$d\tilde{x}(t)/dt = TAT^{-1}\tilde{x}(t) + TBu(t)$$
$$y(t) = L^T T^{-1}\tilde{x}(t)$$

MOR: Balanced truncation

Truncate

balanced system: $$d\tilde{x}(t)/dt = TAT^{-1}\tilde{x}(t) + TBu(t)$$
$$y(t) = L^T T^{-1}\tilde{x}(t)$$

$\tilde{P} = \tilde{Q} = \Sigma \Rightarrow$ the unit vectors e_i are the eigenvectors of $\Sigma : \Sigma e_i = \sigma_i e_i, i = 1, \dots, n.$

Assume that the elements on the diagonal of Σ is already ordered as : $\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_n$.
Therefore $e_1, \dots e_r$ span the subspace containing (the first r) easily controllable and easily observable states.

Truncate the difficult - to - observe and difficult - to - control states means :

$\tilde{x} = e_1\tilde{x}_1 + \dots e_n\tilde{x}_n \approx e_1\tilde{x}_1 + \dots e_r\tilde{x}_r = (\tilde{x}_1, \dots \tilde{x}_r, 0, \dots, 0)^T$.

I.e. $\tilde{x} \approx (\tilde{x}_1, \dots \tilde{x}_r, 0, \dots, 0)^T =: x_T$. Replace $\tilde{x}$ with x_T in the balanced system :

$$dx_T(t)/dt = TAT^{-1}x_T(t) + TBu(t)$$
$$y(t) = L^T T^{-1}x_T(t)$$

$$\xrightarrow{z := (\tilde{x}_1, \dots \tilde{x}_r)}$$

$$d\begin{pmatrix} z(t) \\ 0 \end{pmatrix} = \begin{pmatrix} \tilde{A}_{11}z(t) \\ 0 \end{pmatrix} + \begin{pmatrix} \tilde{B}_1 u(t) \\ \tilde{B}_2 u(t) \end{pmatrix}$$
$$y(t) = \begin{pmatrix} \tilde{L}_1^T z(t) & 0 \end{pmatrix}$$

MOR: Balanced truncation

——Truncate

$$\tilde{A} = TAT^{-1} = \begin{pmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ \tilde{A}_{21} & \tilde{A}_{22} \end{pmatrix} \qquad \tilde{B} = TB = \begin{pmatrix} \tilde{B}_1 \\ \tilde{B}_2 \end{pmatrix} \qquad \tilde{L}^T = LT^{-1} = \begin{pmatrix} \tilde{L}_1^T & \tilde{L}_2^T \end{pmatrix}$$

$$d\begin{pmatrix} z(t) \\ 0 \end{pmatrix} = \begin{pmatrix} \tilde{A}_{11} z(t) \\ 0 \end{pmatrix} + \begin{pmatrix} \tilde{B}_1 u(t) \\ \tilde{B}_2 u(t) \end{pmatrix}$$

$$y(t) = \begin{pmatrix} \tilde{L}_1 z(t) & 0 \end{pmatrix}$$

is a non-minimal realization of a system.

Another realization of the same system is:

$$dz(t) = \tilde{A}_{11} z(t) + \tilde{B}_1 u(t)$$

$$\hat{y}(t) = \tilde{L}_1 z(t)$$

The reduced-order model (ROM)

Therefore we have the following simple steps for truncation:

MOR: Balanced truncation

——Truncate

Balancing:

$$dx(t)/dt = Ax(t) + Bu(t)$$
$$y(t) = L^T x(t)$$

$$\xrightarrow[T^{-1} = UK\Sigma^{-1/2}]{T = \Sigma^{1/2}K^T U^{-1}}$$

$$d\tilde{x}(t)/dt = TAT^{-1}\tilde{x}(t) + TBu(t)$$
$$y(t) = L^T T^{-1}\tilde{x}(t)$$

$$\Downarrow$$

$$dz(t)/dt = \tilde{A}_{11} z(t) + \tilde{B}_1 u(t)$$
$$\hat{y}(t) = \tilde{L}_1^T z(t)$$

$$\Downarrow$$

ROM!

Truncate:

$$TPT^T = \Sigma \quad and \quad T^{-T} Q T^{-1} = \Sigma$$

$$\Sigma = \begin{pmatrix} \Sigma_1 & \\ & \Sigma_2 \end{pmatrix}$$

Σ_2: Small part

$$\tilde{A} = TAT^{-1} = \begin{pmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ \tilde{A}_{21} & \tilde{A}_{22} \end{pmatrix}$$

$$\tilde{B} = TB = \begin{pmatrix} \tilde{B}_1 \\ \tilde{B}_2 \end{pmatrix}$$

$$\tilde{L}^T = LT^{-1} = \begin{pmatrix} \tilde{L}_1^T & \tilde{L}_2^T \end{pmatrix}$$

Separated according to the separation of Σ.

MOR: Balanced truncation

Balancing:

$$dx(t)/dt = Ax(t) + Bu(t) \qquad y(t) = L^T x(t)$$

$$T = \Sigma^{1/2} K^T U^{-1} \qquad T^{-1} = UK\Sigma^{-1/2}$$

$$d\tilde{x}(t)/dt = TAT^{-1}\tilde{x}(t) + TBu(t) \qquad y(t) = L^T T^{-1}\tilde{x}(t)$$

- Does it make sense if we do model reduction on the balanced system rather than the original system?

Yes. As a state transformation, balancing does not change the transfer function, and the HSVs. The balanced system is only a different realization of the system.

MOR: Balanced truncation

Observe:

$$dx(t)/dt = Ax(t) + Bu(t) \qquad d\tilde{x}(t)/dt = TAT^{-1}\tilde{x}(t) + TBu(t) \qquad dz(t)/dt = \tilde{A}_{11}z(t) + \tilde{B}_1 u(t)$$
$$y(t) = L^T x(t) \quad \longrightarrow \quad y(t) = L^T T^{-1}\tilde{x}(t) \quad \longrightarrow \quad \hat{y}(t) = \tilde{L}_1^T z(t)$$

$x = T^{-1}\tilde{x} = Y\tilde{x}$. Here the columns in $Y := T^{-1}$ are the eigenvectors of the matrix product PQ. $Y = T^{-1} \Rightarrow T = Y^{-1} =: (W_1 \quad W_2)^T$

If separate Y as $Y = (Y_1, Y_2)$, and $\tilde{x}$ as $\tilde{x} = \begin{pmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{pmatrix}$, then $x = Y\tilde{x} = Y_1\tilde{x}_1 + Y_2\tilde{x}_2$.

$$\tilde{A} = TAT^{-1} = \begin{pmatrix} W_1^T \\ W_2^T \end{pmatrix} A \begin{pmatrix} Y_1 & Y_2 \end{pmatrix} = \begin{pmatrix} W_1^T AY_1 & W_1^T AY_2 \\ W_2^T AY_1 & W_2^T AY_2 \end{pmatrix} \Rightarrow \tilde{A}_{11} = W_1^T AY_1$$

$$\tilde{B} = TB = \begin{pmatrix} W_1^T \\ W_2^T \end{pmatrix} B = \begin{pmatrix} W_1^T B \\ W_2^T B \end{pmatrix} \Rightarrow \tilde{B}_1 = W_1^T B \qquad \tilde{L}^T = L^T T^{-1} = L^T \begin{pmatrix} Y_1 & Y_2 \end{pmatrix} = \begin{pmatrix} L^T Y_1 & L^T Y_2 \end{pmatrix} \Rightarrow \tilde{L}_1 = L^T Y_1$$

MOR: Balanced truncation

Therefore the two ROMs are the same:

$$dz(t)/dt = \tilde{A}_{11} z(t) + \tilde{B}_1 u(t)$$
$$\hat{y}(t) = \tilde{L}_1^T z(t)$$

$$d\tilde{x}_1(t)/dt = W_1^T A Y_1 \tilde{x}_1(t) + W_1^T B u(t)$$
$$y(t) = L^T Y_1 \tilde{x}_1(t)$$

Conclusion: balanced truncation is Petrov-Galerkin projection as below:

Let $x \approx Y_1 \tilde{x}_1$

$$\left.\begin{aligned} dx(t)/dt &= Ax(t) + Bu(t) \\ y(t) &= L^T x(t) \end{aligned}\right. \xrightarrow[\text{Petrov-Galerkin using } W_1^T]{x \approx Y_1 \tilde{x}_1} \left.\begin{aligned} d\tilde{x}_1(t)/dt &= W_1^T A Y_1 \tilde{x}_1(t) + W_1^T B u(t) \\ y(t) &= L^T Y_1 \tilde{x}_1(t) \end{aligned}\right.$$

Therefore, balanced truncation is equivalent to: finding the invariant subspace range(Y) of PQ, and remaining only the part (Y_1) which corresponds to the largest HSVs (square root of the eigenvalues of PQ).

MOR: Balanced truncation

Algorithm 1

Given $dx(t)/dt = Ax(t) + Bu(t)$

$y(t) = L^T x(t)$

- Balancing:

  1. Compute *P, Q.*

  2. Compute $P = UU^T \qquad U^T QU = K\Sigma^2 K^T$

  3. $T = \Sigma^{1/2} K^T U^{-1}, \qquad T^{-1} = UK\Sigma^{-1/2} \qquad \Sigma = \begin{pmatrix} \Sigma_1 & \\ & \Sigma_2 \end{pmatrix}$

  4. Balancing and separating A, B, L according to the separation of Σ :

$$\tilde{A} = TAT^{-1} = \begin{pmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ \tilde{A}_{21} & \tilde{A}_{22} \end{pmatrix} \quad \tilde{B} = TB = \begin{pmatrix} \tilde{B}_1 \\ \tilde{B}_2 \end{pmatrix} \quad \tilde{L}^T = L^T T^{-1} = \begin{pmatrix} \tilde{L}_1^T & \tilde{L}_2^T \end{pmatrix}$$

- Truncate:

  5. Form the reduced model:

$$d\hat{x}(t)/dt = \tilde{A}_{11}\hat{x}(t) + \tilde{B}_1 u(t)$$

$$\hat{y}(t) = \tilde{L}_1^T \hat{x}(t)$$

MOR: Balanced truncation

computational details

Recall:

$$AP + PA^T = -BB^T$$

$$A^T Q + QA = -LL^T$$

In MATLAB, use command:

$$P = lyap(A, B * B^{'})$$

$$Q = lyap(A^T, L * L^{'})$$

MOR: Balanced truncation

—— Numerical issues

The balancing matrix is: $T = \Sigma^{1/2} K^T U^{-1}, \; P = UU^T.$

Computation of U^{-1} may cause numerical instability, because *U* is usually near singular.

U is usually near singular, because the matrix *P* has *numerically* low-rank, i.e. near singular.

P is near singular because in may cases, its eigenvalues decay rapidly to zero, some eigenvalues are very close to zero, e.g. $\lambda_i = 10^{-20}$.

Q and Σ behaves similarly as *P*.

However in algorithm 1, we need to compute:

$$T = \Sigma^{1/2} K^T U^{-1}, \quad T^{-1} = UK\Sigma^{-1/2}$$

Can we avoid computing $U^{-1}, \; \Sigma^{-1}$?

MOR: Balanced truncation

computational details

If using Choelsky factorization of both

$$P = Z_P Z_P^T, Q = Z_Q Z_Q^T$$

Observe

$$Z_p^T Q Z_p = Z_p^T Z_Q Z_Q^T Z_p$$

Use SVD instead of eigen- decomposition

$$Z_P^T Z_Q = \tilde{U} \Sigma \tilde{V}^T$$

Comparing with P defined in Algorithm 1, we immediately get

$$T = \Sigma^{1/2} \tilde{U}^T Z_p^{-1}$$

To avoid computing the inverse of Z_p , we have:

$$Z_P^T Z_Q = \tilde{U} \Sigma \tilde{V}^T \Rightarrow Z_p^{-1} = \tilde{U} \Sigma^{-1} \tilde{V}^T Z_Q^T \Longrightarrow T = \Sigma^{-1/2} \tilde{V}^T Z_Q^T$$

The balanced system which is balanced by $T=\Sigma^{-1/2}\tilde{V}^T Z_Q^T$ and $T^{-1}=Z_P\tilde{U}\Sigma^{-1/2}$ is:

$$\tilde{A}=TAT^{-1}=\begin{pmatrix}\tilde{A}_{11} & \tilde{A}_{12}\\ \tilde{A}_{21} & \tilde{A}_{22}\end{pmatrix} \quad \tilde{B}=TB=\begin{pmatrix}\tilde{B}_1\\ \tilde{B}_2\end{pmatrix} \quad \tilde{L}^T=LT^{-1}=\begin{pmatrix}\tilde{L}_1^T & \tilde{L}_2^T\end{pmatrix}, \quad \Sigma=\begin{pmatrix}\Sigma_1 & \\ & \Sigma_2\end{pmatrix}$$

Furthermore, if $\tilde{U}=\begin{pmatrix}\tilde{U}_1 & \tilde{U}_2\end{pmatrix}, \tilde{V}=(\tilde{V}_1,\tilde{V}_2)$, then

$$T=\Sigma^{-1/2}\tilde{V}^T Z_Q^T=\begin{pmatrix}\Sigma_1^{-1/2}\tilde{V}_1^T Z_Q^T\\ \Sigma_2^{-1/2}\tilde{V}_2^T Z_Q^T\end{pmatrix} \qquad \begin{aligned} T^{-1}&=Z_P\tilde{U}\Sigma^{-1/2}\\ &=\begin{pmatrix}Z_P\tilde{U}_1\Sigma_1^{-1/2} & Z_P\tilde{U}_2\Sigma_2^{-1/2}\end{pmatrix}\end{aligned}$$

Define: $W=Z_Q\tilde{V}_1\Sigma_1^{-1/2},\ V=Z_P\tilde{U}_1\Sigma_1^{-1/2},$

we have: $\tilde{A}_{11}=W^TAV, \tilde{B}_1=W^TB, \tilde{L}_1=L^TV.$

Algorithm 2 SR method (Getting the reduced model without computing Σ^{-1}, U^{-1}):

1. Do Cholesky factorization of the two Gramians: $P = Z_P Z_P^T, Q = Z_Q Z_Q^T$

   Z_P, Z_Q are lower triangular matrices.

2. Do Singular value decomposition (SVD) of matrix $Z_P^T Z_Q$, i.e., there are two orthonormal matrices $\tilde{U}, \tilde{V}$, $\tilde{U}^T \tilde{U} = I$, $\tilde{V}^T \tilde{V} = I$, such that

$$Z_P^T Z_Q = \tilde{U} \Sigma \tilde{V}^T = \begin{pmatrix} \tilde{U}_1 & \tilde{U}_2 \end{pmatrix} \begin{pmatrix} \Sigma_1 & \\ & \Sigma_2 \end{pmatrix} \begin{pmatrix} \tilde{V}_1^T \\ \tilde{V}_2^T \end{pmatrix}.$$

3. Let $W = Z_Q \tilde{V}_1 \Sigma_1^{-1/2}, \quad V = Z_P \tilde{U}_1 \Sigma_1^{-1/2}.$

4. Let $\hat{A} = W^T A V, \hat{B} = W^T B, \hat{L}^T = L^T V.$

5. The reduced model is
$$d\hat{x}(t)/dt = \hat{A}\hat{x}(t) + \hat{B}u(t)$$
$$\hat{y}(t) = \hat{L}^T \hat{x}(t)$$

Algorithm 2 sometimes cannot continue either, because the Cholesky factorization of P, Q cannot be done. This is because that in some cases P and Q include too small eigenvalues like: $\lambda = 10^{-20}$, which is considered by the algorithm as a singular matrix, therefore Cholesky factorization cannot be continued.

Paper [BennerQ '05] provides an algorithm computing the numerically full rank factors of P and Q , which are in the forms $\tilde{Z}_P \in R^{n\times\hat{n}}, \tilde{Z}_Q \in R^{n\times\hat{n}} \quad \hat{n} << n$

The full rank factors numerically satisfy: $P = \tilde{Z}_P \tilde{Z}_P^T, Q = \tilde{Z}_Q \tilde{Z}_Q^T$.

[BennerQ '05] P. Benner, E.S. Quitana-Orti, Model reduction based on spectral projection methods. In: P. Benner, V.L. Mehrmann, D.C. Sorensen (eds.), "Dimension Redution of Large-Scale Systems", vol. 45 of Lecture Notes in Computational Science and Engineering, pp. 5-48, Springer-Verlag, Berlin/Heidelberg, 2005. (**Algorithm 4 in the paper**)

MOR: Balanced truncation

Numerical issues

Algorithm 3 Getting the reduced model using full-rank factors [BennerQ'05]:

1. Compute full-rank factors of the Gramians: $P = \tilde{Z}_P \tilde{Z}_P^T, Q = \tilde{Z}_Q \tilde{Z}_Q^T$
   $\tilde{Z}_P \in R^{n\times\hat{n}}, \tilde{Z}_Q \in R^{n\times\hat{n}}, \hat{n} << n.$
2. Compute SVD

$$\tilde{Z}_P^T \tilde{Z}_Q = U\Sigma V^T = \begin{pmatrix} U_1 & U_2 \end{pmatrix} \begin{pmatrix} \Sigma_1 & \\ & \Sigma_2 \end{pmatrix} \begin{pmatrix} V_1^T \\ V_2^T \end{pmatrix}.$$

3. Let $W = \tilde{Z}_Q V_1 \Sigma_1^{-1/2}, \;\; V = \tilde{Z}_P U_1 \Sigma_1^{-1/2}.$
4. Let $\hat{A} = W^T A V, \hat{B} = W^T B, \hat{L}^T = L^T V.$
5. The reduced model is

$$d\hat{x}(t)/dt = \hat{A}\hat{x}(t) + \hat{B}u(t)$$

$$\hat{y}(t) = \hat{L}^T \hat{x}(t)$$

Theorem [BennerQ'05]

If the original LTI system is stable, then the reduced model obtained by Algorithm 1, Algorithm 2, Algorithm 3 satisfies:

1) The reduced model is balanced, minimal and stable. It's Gramians are equal to the same diagonal matrix.

2) The absolute error bound (proof in [Antoulas '05] Chapter 7)

$$\| H(s) - \hat{H}(s) \|_{H_\infty} \leq 2 \sum_{k=r+1}^{n} \sigma_k$$

holds.