MAX PLANCK INSTITUTE
FOR DYNAMICS OF COMPLEX
TECHNICAL SYSTEMS
MAGDEBURG

CSC COMPUTATIONAL METHODS IN
SYSTEMS AND CONTROL THEORY

Model Reduction for Dynamical Systems

–Lecture 9–

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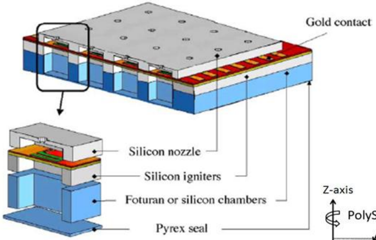
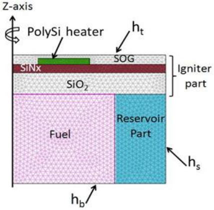
Example 1

A microthruster

Upper-left[1]: the structure of an array of pyrotechnical thrusters. Lower-right: the structure of a 2D-axisymmetric model.





Example 1

- When the PolySilicon (green) in the middle is excited by a current, the fuel below is ignited and the explosion will occur through the nozzle.
- The thermal process can be modeled by a heat transfer partial differential equation, while the heat exchange through device interfaces is modeled by convection boundary conditions with different film coefficients h_t, h_s, h_b.
- The film coefficients h_t, h_s, h_b respectively describe the heat exchange on the top, side, and bottom of the microthruster with the outside surroundings. The values of the film coefficients can change from 1 to 10^9

Example 1

After finite element discretization of the 2D-axisymmetric model, a parameterized system is derived,

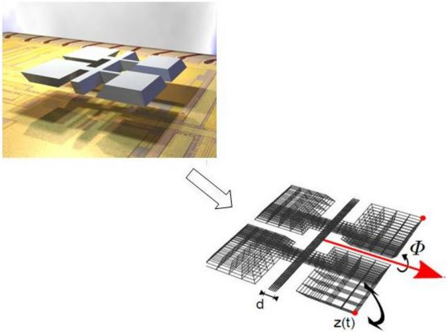
$$\begin{aligned} E\dot{x} &= (A - h_t A_t - h_s A_s - h_b A_b)x + B \\ y &= Cx. \end{aligned} \tag{1}$$

Here, h_t, h_s, h_b are the parameters and the dimension of the system is $n = 4,257$. We observe the temperature at the center of the PolySilicon heater changing with time and the film coefficient, which defines the output of the system[2].

[2]Detailed description of the parameterized system can be find at http://simulation.uni-freiburg.de/downloads/benchmark

Example 2

The second example is a butterfly gyroscope. The parameterized system is obtained by finite element discretization of the model for the gyroscope (The details of the model can be found in [Moosmann07]).



Example 2

- The paddles of the device are excited to a vibration $z(t)$, where all paddles vibrate in phase. With the external rotation ϕ, the Coriolis force acts upon the paddles, which causes an out-of-phase movement measured as the z-displacement difference δz between the two red dotted nodes.
- The interesting output of the system is δz, the difference of the displacement $z(t)$ between the two end nodes depicted as red dots on the same side of the bearing.

Example 2

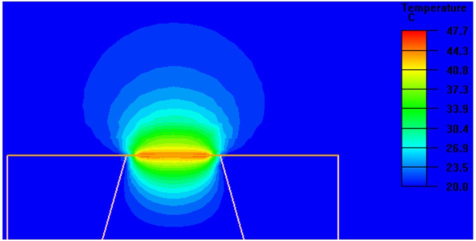
The system is of the following form:

$$\begin{aligned} M(d)\ddot{x} + D(\theta,\alpha,\beta,d)\dot{x} + T(d)x &= Bu(t) \\ y &= Cx. \end{aligned} \qquad (2)$$

- $M(d) = (M_1 + dM_2)$, $D(\theta,\alpha,\beta,d) = \theta(D_1 + dD_2) + \alpha M(d) + \beta T(d)$, and $T(d) = (T_1 + \frac{1}{d}T_2 + dT_3)$.
- Parameters d,θ,α,β. d is the width of the bearing, and θ is the rotation velocity along the x axis. α,β are used to form the Rayleigh damping matrices $\alpha M(d), \beta T(d)$ in $D(\theta,\alpha,\beta,d)$.
- The dimension of the system is $n = 17913$.

Example 3

The third example is a silicon-nitride membrane[3]. This structure resembles a micro-hotplate similar to other micro-fabricated devices such as gas sensors [GrafBT04] and infrared sources [SpannSH05].



Temperature distribution over the silicon-nitride membrane.

[3] Picture courtesy of T. Bechtold, IMTEK, University of Freiburg, Germany.

Example 3

The model of the silicon-nitride membrane is a system with four parameters [BechtoldHRG10].

$$\begin{aligned} (E_0 + \rho c_p E_1)\dot{x} + (K_0 + \kappa K_1 + hK_2)x &= Bu(t) \\ y &= Cx. \end{aligned} \tag{3}$$

- The mass density ρ in $\mathrm{kg/m^3}$, the specific heat capacity c_p in $\mathrm{J/kg}/K$, the thermal conductivity in $\mathrm{W/m/K}$, and the heat transfer coefficient h in $\mathrm{W/m^2/K}$.
- The dimension of the system is $n = 60020$.

In frequency domain

Using Laplace transform, the system in time domain is transformed into

$$\begin{aligned} E(s_1, \ldots, s_p)x &= Bu(s_p), \\ y &= L^{\mathrm{T}}x, \end{aligned} \tag{4}$$

where the matrix $E \in \mathbb{R}^{n \times n}$ is parametrized. The new parameter s_p is in fact the frequency parameter s, which corresponds to time t.

In case of a nonlinear and/or non-affine dependence of the matrix E on the parameters, the system in (4) is first transformed to an affine form

$$\begin{aligned} (E_0 + \tilde{s}_1 E_1 + \tilde{s}_2 E_2 + \ldots + \tilde{s}_p E_p)x &= Bu(s_p), \\ y &= L^{\mathrm{T}}x. \end{aligned} \tag{5}$$

Here the newly defined parameters $\tilde{s}_i, i = 1, \ldots, p$, might be some functions (rational, polynomial) of the original parameters s_i in (4).

To obtain the projection matrix V for the reduced model, the state x in (5) is expanded into a Taylor series at an expansion point $\tilde{s}_0 = (\tilde{s}_1^0, \ldots, \tilde{s}_p^0)^T$ as below,

$$
\begin{aligned}
x &= [I - (\sigma_1 M_1 + \ldots + \sigma_p M_p)]^{-1} \tilde{E}^{-1} B u(s_p) \\
&= \sum_{m=0}^{\infty} [\sigma_1 M_1 + \ldots + \sigma_p M_p]^m \tilde{E}^{-1} B u(s_p) \\
&= \sum_{m=0}^{\infty} \sum_{k_2=0}^{m-(k_3+\ldots+k_p)} \cdots \sum_{k_{p-1}=0}^{m-k_p} F^m_{k_2,\ldots,k_p}(M_1, \ldots, M_p)
\end{aligned}
\qquad (6)
$$

where $\sigma_i = \tilde{s}_i - \tilde{s}_i^0$, $\tilde{E} = E_0 + \tilde{s}_1^0 E_1 + \ldots + \tilde{s}_p^0 E_p$, $M_i = -\tilde{E}^{-1} E_i$, $i = 1, 2, \ldots p$, and $B_M = \tilde{E}^{-1} B$.

PMOR based on multi-moment-matching

- σ^0: $L^{\mathrm{T}} B_M$: the 0th order multi-moment; the columns in B_M: the 0th order moment vectors.
- σ^1: $L^{\mathrm{T}} M_i B_M, \ i = 1, 2, \ldots, p$: the first order multi-moments; the columns in $M_i B_M, \ i = 1, 2, \ldots, p$: the first order moment vectors.
- σ^2: $\ldots$; the columns in $M_i^2 B_M, \ i = 1, 2, \ldots, p$, $(M_1 M_i + M_i M_1) B_M, \ i = 2, \ldots, p$, $(M_2 M_i + M_i M_2) B_M, \ i = 3, \ldots, p$, $\ldots$, $(M_{p-1} M_p + M_p M_{p-1}) B_M$: the second order moment vectors.
- $\ldots$.

Since the coefficients corresponding not only to $s = s_p$, but also to those associated with the other parameters $s_i, i = 1, \ldots, p-1$ are, we call them as **multi-moments** of the transfer function.

For the general case, the projection matrix V is constructed as

$$
\begin{aligned}
&\operatorname{range}\{V\} \\
&= \operatorname{colspan}\{\bigcup_{m=0}^{m_q} \bigcup_{k_2=0}^{m-(k_p+\ldots+k_3)} \cdots \bigcup_{k_{p-1}=0}^{m-k_p} \bigcup_{k_p=0}^{m} F_{k_2,\ldots,k_p}^{m}(M_1,\ldots,M_p)B_M\} \\
&= \operatorname{colspan}\{B_M, M_1B_M, M_2B_M, \ldots, M_pB_M, (M_1)^2B_M, (M_1M_2+M_2M_1)B_M, \ldots, \\
&\qquad (M_1M_p+M_pM_1)B_M, (M_2)^2B_M, (M_2M_3+M_3M_2)B_M, \ldots\}.
\end{aligned}
\tag{7}
$$

By observing the power series expansion of x in (6), we get the following equivalent, but different formulation,

$$\begin{aligned} x &= [I - (\sigma_1 M_1 + \ldots + \sigma_p M_p)]^{-1} \tilde{E}^{-1} B u \\ &= \sum_{m=0}^{\infty} [\sigma_1 M_1 + \ldots + \sigma_p M_p]^m B_M u \\ &= B_M u + [\sigma_1 M_1 + \ldots + \sigma_p M_p] B_M u \\ &\quad + [\sigma_1 M_1 + \ldots + \sigma_p M_p]^2 B_M u + \ldots \\ &\quad + [\sigma_1 M_1 + \ldots + \sigma_p M_p]^j B_M u + \ldots \end{aligned} \tag{8}$$

By defining

$$\begin{aligned} x_0 &= B_M, \\ x_1 &= [\sigma_1 M_1 + \ldots + \sigma_p M_p] B_M, \\ x_2 &= [\sigma_1 M_1 + \ldots + \sigma_p M_p]^2 B_M, \ldots, \\ x_j &= [\sigma_1 M_1 + \ldots + \sigma_p M_p]^j B_M, \ldots, \end{aligned}$$

we have $x = (x_0 + x_1 + x_2 + \cdots + x_j + \cdots)u$ and obtain the recursive relations

$$
\begin{aligned}
x_0 &= B_M, \\
x_1 &= [\sigma_1 M_1 + \ldots + \sigma_p M_p] x_0, \\
x_2 &= [\sigma_1 M_1 + \ldots + \sigma_p M_p] x_1, \ldots \\
x_j &= [\sigma_1 M_1 + \ldots + \sigma_p M_p] x_{j-1}, \ldots.
\end{aligned}
$$

If we define a vector sequence based on the coefficient matrices of x_j, $j = 0, 1, \ldots$ as below,

$$
\begin{aligned}
R_0 &= B_M, \\
R_1 &= [M_1 R_0, M_2 R_0, \ldots, M_p R_0], \\
R_2 &= [M_1 R_1, M_2 R_1, \ldots, M_p R_1], \\
&\vdots \\
R_j &= [M_1 R_{j-1}, M_2 R_{j-1}, \ldots, M_p R_{j-1}], \\
&\vdots
\end{aligned}
\tag{9}
$$

and let R be the subspace spanned by the vectors in $R_j,\ j = 0, 1, \cdots, m$:

$$R = \mathrm{colspan}\{R_0, \dots, R_j, \dots, R_m\},$$

then there exists $z \in \mathbb{R}^q$, such that $x \approx Vz$. Here the columns in $V \in \mathbb{R}^{n\times q}$ is a basis of R. We see that the terms in $R_j,\ j = 0, 1, \dots, m$ are the coefficients of the parameters in the series expansion (8). They are also the j-th order moment vectors.

How to compute an orthonormal basis V?

A Robust Algorithm

Algorithm 1: Compute $V = [v_1, v_2, \ldots, v_q]$ [Benner, Feng'14]

Initialize $a_1 = 0$, $a_2 = 0$, $sum = 0$.
Compute $R_0 = \tilde{E}^{-1}B$.
if multiple input **then**
  Orthogonalize the columns in R_0 using MGS: $[v_1, v_2, \ldots, v_{q_1}] = \mathrm{orth}\{R_0\}$ with respect to a user given tolerance $\varepsilon > 0$ specifying the deflation criterion for numerically linearly dependent vectors.
  $sum = q_1$ (% q_1 is the number of columns remained after deflation w.r.t. ε.)
else
  $v_1 = R_0/||R_0||_2$
  $sum = 1$
end if
Compute the orthonormal columns in $R_1, R_2, \ldots, R_m$ iteratively as below:

A Robust Algorithm

continued

for $i = 1, 2, \ldots, m$ **do**
  $a_2 = sum$;
  for $t = 1, 2, \ldots, p$ **do**
    if $a_1 = a_2$ **then**
      stop
    else
      for $j = a_1 + 1, \ldots a_2$ **do**
        $w = \tilde{E}^{-1} E_t v_j$; $col = sum + 1$;
        for $k = 1, 2, \ldots, col - 1$ **do**
          $h = v_k^T w$; $w = w - h v_k$
        end for
        if $\|w\|_2 > \varepsilon$ **then**
          $v_{col} = \frac{w}{\|w\|_2}$; $sum = col$;
        end if
      end for
    end if
  end for
  $a_1 = a_2$;
end for

Orthogonalize the columns in V by MGS w.r.t. ε.

Adaptively select expansion points

Let $\mu = (\tilde{s}_1, \ldots, \tilde{s}_p)$, $\Delta(\mu)$ is an error estimation, or error bound for $\hat{x}/\hat{y}$, the state/output of the system computed from ROM.

Greedy algorithm: Adaptive selection of the expansion points μ^i

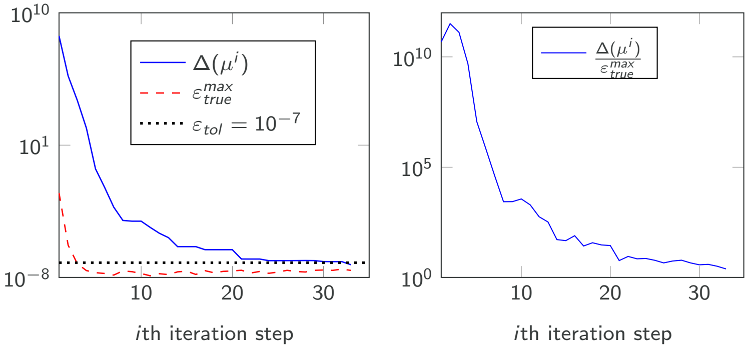
$V = []$; $\epsilon = 1$;
Initial expansion point: μ^0; $i = -1$;
Ξ_{train}: a large set of the samples of μ
WHILE $\epsilon > \epsilon_{tol}$
  i=i+1;
  $\mu^i = \hat{\mu}$;
  Use Algorithm 1 to compute $V_i = span\{R_0, \ldots, R_q\}_{\mu^i}$;
  $V = [V, V_i]$;
  $\hat{\mu} = arg \max_{\mu \in \Xi_{train}} \Delta(\mu)$;
  $\epsilon = \Delta(\hat{\mu})$;
END WHILE.

Example 1: A MEMS model with 4 parameters (benchmark available at `http://modlereduction.org`),

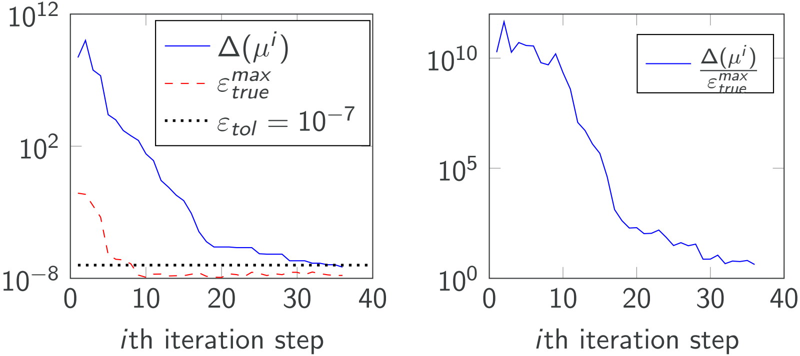
$$\begin{aligned} M(d)\ddot{x} + D(\theta,\alpha,\beta,d)\dot{x} + T(d)x &= Bu(t), \\ y &= Cx. \end{aligned}$$

Here, $M(d) = (M_1 + dM_2)$, $T(d) = (T_1 + \frac{1}{d}T_2 + dT_3)$,
$D(\theta,\alpha,\beta,d) = \theta(D_1 + dD_2) + \alpha M(d) + \beta T(d) \in R^{n\times n}$, n=17,913. Parameters, d, θ, α, β.

- $\theta \in [10^{-7}, 10^{-5}]$, $s \in 2\pi\sqrt{-1} \times [0.05, 0.25]$, $d \in [1, 2]$.
- Ξ_{train}: 3 random θ, 10 random s, 5 random d, $\alpha = 0$, $\beta = 0$ [Salimbahrami et al.' 06]. Totally 150 samples of μ.



$$V_{\mu^i} = span\{B_M, R_1, R_2\}_{\mu^i},\ i = 1, \ldots, 33.\ \epsilon_{tol} = 10^{-7},\ \varepsilon_{true}^{max} = \max_{\mu \in \Xi_{train}} |H(\mu) - \hat{H}(\mu)|, \text{ ROM size=804.}$$



$V_{\mu^i} = span\{B_M, R_1\}_{\mu^i}$, $i = 1, \dots, 36$. $\epsilon_{tol} = 10^{-7}$, ROM size=210.

Example 2: a silicon nitride membrane

$$\begin{aligned}(E_0 + \rho c_p E_1) dx/dt + (K_0 + \kappa K_1 + h K_2) x &= bu(t) \\ y &= Cx.\end{aligned}$$

Here, the parameters $\rho \in [3000, 3200]$, $c_p \in [400, 750]$, $\kappa \in [2.5, 4]$, $h \in [10, 12]$, $f \in [0, 25] Hz$

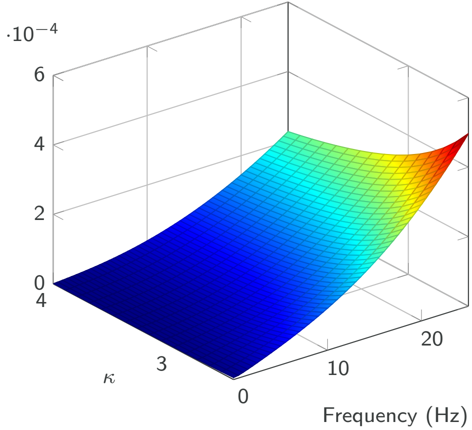
Ξ_{train}: 2250 random samples have been taken for the four parameters and the frequency.

$\varepsilon_{true}^{re} = \max\limits_{\mu \in \Xi_{train}} |H(\mu) - \hat{H}(\mu)|/|H(\mu)|$, $\hat{\Delta}^{re}(\mu) = \hat{\Delta}(\mu)/|\hat{H}(\mu)|$

$$V_{\mu^i = \mathrm{span}\{B_M, R_1\}},\ \epsilon_{tol}^{re} = 10^{-2},\ n = 60,020,\ r = 8,$$

iteration	ε_{true}^{re}	$\hat{\Delta}^{re}(\mu^i)$
1	1×10^{-3}	3.44
2	1×10^{-4}	4.59×10^{-2}
3	2.80×10^{-5}	4.07×10^{-2}
4	2.58×10^{-6}	2.62×10^{-5}

- Ξ_{train}: 3 samples for κ, 10 samples for the frequency.
- Ξ_{var}: 16 samples for κ, 51 samples for the frequency.



Relative error of the final ROM over Ξ_{var}.

IRKA based PMOR

Consider a linear parametric system

$$\begin{aligned} C(p_1, p_2, \cdots, p_l)\frac{dx}{dt} &= G(p_1, p_2, \cdots, p_l)x + B(p_1, p_2, \cdots, p_l)u(t), \\ y(t) &= L(p_1, p_2, \cdots, p_l)^{\mathrm{T}} x, \end{aligned} \tag{10}$$

where the system matrices $C(p_1, p_2, \cdots, p_l)$, $G(p_1, p_2, \cdots, p_l)$, $B(p_1, p_2, \cdots, p_l)$, $LT(p_1, p_2, \cdots, p_l)$, are (maybe, nonlinear, non-affine) functions of the parameters p_1, p_2, p_l.

A straight forward way is [Baur, et.al'11]:
Set a group of samples of $\mu = (p_1, \ldots, p_l)$: $\mu^0, \ldots, \mu^l$.
For each sample $\mu^i = (p_1^i, \ldots, p_l^i)$, $i = 1 \ldots, l$, implement IRKA to get the projection matrices W_i, V_i.
The final projection matrices:

- $\mathrm{range}(V) = \mathrm{orth}(V_1, \ldots, V_l)$,
- $\mathrm{range}(W) = \mathrm{orth}(W_1, \ldots, W_l)$,
- $W = W(V^T W)^{-1}$.

The reduced parametric model is:

Parametric ROM

$$
\begin{aligned}
W^T C(p_1, p_2, \cdots, p_l) V \frac{dx}{dt} &= W^T G(p_1, p_2, \cdots, p_l) V x \\
&\quad + W^T B(p_1, p_2, \cdots, p_l) u(t), \\
y(t) &= L(p_1, p_2, \cdots, p_l)^{\mathrm{T}} V x,
\end{aligned}
$$

Question: How to select the samples of μ ?

How to deal with nonaffine matrices?

Nonafine matrices are those matrices that cannot be written as:

$$E(p_1, \ldots, p_l) = E_0 + p_1 E_1 + \ldots, p_l E_l.$$

- PMOR based on multi-moment-matching cannot directly deal with nonaffine case. We must first approximate with affine matrices.
- IRKA can deal with nonaffine matrices directly.

Steady parametric systems

$$E(p_1, \ldots, p_l)x = B(p_1, \ldots, p_l)$$

- Solving steady systems for multi-query tasks is also time-consuming.
- Application of PMOR based on multi-moment-matching to steady systems is straight forward.
- IRKA ?.

Applicable to nonlinear parametric systems?

Nonlinear parametric systems:

$$f(\mu, x) = b(\mu),$$

or

$$\begin{aligned} E(\mu)\tfrac{dx}{dt} &= A(\mu)x + f(\mu, x) = B(\mu)u(t), \\ y(t) &= L(\mu)^{\mathrm{T}}x, \end{aligned}$$

$\mu = (p_1, \ldots, p_m)$, $x = x(\mu, t)$.

- PMOR based on multi-moment matching or IRKA could deal with weakly nonlinear parametric systems.
- Good candidates for MOR of general nonlinear parametric systems are POD and reduced basis methods.
- **To be introduced**: POD and reduced basis method for linear and nonlinear parametric systems.

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   And many more...