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Computational Methods in Systems and Control Theory
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Introduction to Model Order Reduction

Exercise 1 (Controllability of dynamical systems)
Let $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$. Show that the following statements are equivalent:

a) the pair (A, B) is controllable
(i.e., for all times $t_0, t_1 \in \mathbb{R}$, $t_0 < t_1$, and states $x_0, x_1 \in \mathbb{R}^n$, there exists $u(t)$ such that the solution of the initial value problem $\dot{x}(t) = Ax(t) + Bu(t)$, $x(t_0) = x_0$, satisfies $x(t_1) = x_1$),

b) the controllability matrix $\mathcal{C} = \begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix} \in \mathbb{R}^{n\times nm}$ has full rank n,

c) the controllability Gramian

$$P(t) = \int_0^t e^{A\tau} B B^T e^{A^T\tau} \,\mathrm{d}\tau$$

is positive definite for all $t > 0$.

Exercise 2 (The (infinite) controllability Gramian and a Lyapunov equation)
Let $A \in \mathbb{R}^{n\times n}$ be stable and $W \in \mathbb{R}^{n\times n}$. Prove that

$$X = \int_0^\infty e^{At} W e^{A^T t} \,\mathrm{d}t$$

is the unique solution of the Lyapunov equation

$$AX + XA^T + W = 0.$$

Exercise 3 (Properties of the matrix sign function)
Let $Z \in \mathbb{C}^{n\times n}$ be a matrix with no eigenvalues on the imaginary axis and a Jordan canonical form

$$Z = S \begin{bmatrix} J^+ & 0 \\ 0 & J^- \end{bmatrix} S^{-1},$$

where $S \in \mathbb{C}^{n\times n}$ is an invertible transformation matrix, $J^+ \in \mathbb{C}^{k\times k}$ and $J^- \in \mathbb{C}^{(n-k)\times(n-k)}$ are also in Jordan canonical form and contain the k eigenvalues in $\mathbb{C}_+$ and $n-k$ eigenvalues in $\mathbb{C}_-$, respectively. The matrix sign function is defined as

$$\operatorname{sign}(Z) := S \begin{bmatrix} I_k & 0 \\ 0 & -I_{n-k} \end{bmatrix} S^{-1}.$$

Show that

a) $\operatorname{sign}(T^{-1}ZT) = T^{-1}\operatorname{sign}(Z)T$ for all nonsingular $T \in \mathbb{C}^{n\times n}$,

b) if Z is stable, then $\operatorname{sign}(Z) = -I_n$ and $\operatorname{sign}(-Z) = I_n$,

c) $\operatorname{sign}(Z)^2 = I_n$, i.e., $\operatorname{sign}(Z)$ is a square root of the identity matrix,

d) the sign function iteration

$$Z_{k+1} = \frac{1}{2}(Z_k + Z_k^{-1}), \quad Z_0 = Z,$$

for $k = 0, 1, 2, \ldots$, is a Newton iteration applied to the function $F(X) = X^2 - I_n$.

Exercise 4 (Solving Lyapunov equations via the matrix sign function)
Consider the Lyapunov equation

$$AX + XA^T + W = 0, \tag{1}$$

with $A \in \mathbb{R}^{n\times n}$ and $W \in \mathbb{R}^{n\times n}$. Assume that A is a stable matrix and X is the solution of the equation (1).

a) Show that

$$\operatorname{sign}\left(\begin{bmatrix} A & W \\ 0 & -A^T \end{bmatrix}\right) = \begin{bmatrix} -I_n & 2X \\ 0 & I_n \end{bmatrix}.$$

Hint: Compute $T^{-1}\begin{bmatrix} A & W \\ 0 & -A^T \end{bmatrix} T$, with $T = \begin{bmatrix} I_n & X \\ 0 & I_n \end{bmatrix}$.

b) Show that instead of iterating on $\begin{bmatrix} A & W \\ 0 & -A^T \end{bmatrix}$, one can compute X via an iteration on A and W.

c) Implement the iteration from part b) as Lyapunov equation solver in MATLAB. The function should have the following head

```
[X, err] = lyap_sgn(A, W, maxit, tol)
```

with `A` and `W` the matrices from (1), `maxit` the maximum number of iteration steps and the tolerance `tol` for the sopping criterion $\|A_k + I_n\|_F <$ `tol`. The outputs are `X`, the solution of the Lyapunov equation and `err` a vector containing the deviations $\|A_k + I_n\|_F$ for all made iteration steps.

d) Test your implementation on random stable examples by computing the normalized error

$$\frac{\|AX + XA^T + W\|_F}{\|W\|_F}$$

and plotting how $\|A_k + I_n\|_F$ varies across the iterations. For W symmetric, check if the approximate solution you find is also symmetric (e.g., by computing $\|X - X^T\|_F$).

Exercise 5 (Model reduction by balanced truncation)
Here we apply the balanced truncation method to the Clamped Beam model from the NICONET benchmark collection. You need to download `beam.mat` from

`http://slicot.org/20-site/126-benchmark-examples-for-model-reduction`

a) Compute the controllability and observability Gramians by solving the Lyapunov equations

$$AP + PA^T + BB^T = 0,$$
$$A^TQ + QA + C^TC = 0,$$

using the function `lyap_sgn` you implemented in Exercise 4.

b) Compute the factorizations $P = R^TR$ and $Q = L^TL$.

c) Compute the singular value decomposition $LR^T = U\Sigma V^T$.

d) Plot the Hankel singular values.

e) Compute the reduced-order model $(A_r, B_r, C_r) = (W_r^T A V_r, W_r^T B, C V_r)$, where

$$V_r = R^T V(:, 1:r)\Sigma(1:r, 1:r)^{-\frac{1}{2}},$$
$$W_r = L^T U(:, 1:r)\Sigma(1:r, 1:r)^{-\frac{1}{2}},$$

for different orders r.

f) Draw the log-log plots of $\omega \mapsto \|H(i\omega)\|_2$ and $\omega \mapsto \|H_r(i\omega)\|_2$, where

$$H(s) = C(sI_n - A)^{-1}B,$$
$$H_r(s) = C_r(sI_r - A_r)^{-1}B_r,$$

are the transfer functions of the original and reduced model. Use 1000 logarithmically distributed sample points over the frequency interval $\omega \in [10^{-2}, 10^4]$.

g) Draw the log-log plot of $\omega \mapsto \|H(i\omega) - H_r(i\omega)\|_2$, same as in f), with a horizontal line for the upper bound of the $\mathcal{H}_\infty$-error using the Hankel singular values σ_k and the error formula

$$\|H - H_r\|_{\mathcal{H}_\infty} \leq 2 \sum_{k=r+1}^{n} \sigma_k.$$

Exercise 6 (Model reduction by interpolation)
Implement a function

```
[Ar, Br, Cr] = rat_krylov(A, B, C, p)
```

for interpolating a SISO LTI system

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \\ y(t) &= Cx(t), \end{aligned}$$

with $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times 1}$ and $C \in \mathbb{R}^{1\times n}$ at specified interpolation points $\mathtt{p} = \{\sigma_1, \ldots, \sigma_r\}$, assuming the interpolation points are pairwise distinct and closed under conjugation.

a) Use the projection-based approach with matrices V and W such that

$$\begin{aligned} \operatorname{range}(V) &= \operatorname{span}\{(\sigma_1 I_n - A)^{-1}B, \ldots, (\sigma_r I_n - A)^{-1}B\}, \\ \operatorname{range}(W) &= \operatorname{span}\{(\sigma_1 I_n - A)^{-T}C^T, \ldots, (\sigma_r I_n - A)^{-T}C^T\}. \end{aligned}$$

Ensure that V and W are real matrices.

b) Test your function on the clamped beam example data from Exercise 5 for different sets of interpolation points by drawing the log-log plots of $\omega \mapsto \|H(i\omega)\|_2$ and $\omega \mapsto \|H_r(i\omega)\|_2$ as in Exercise 5 f). Use 1000 logarithmically distributed sample points over the frequency interval $\omega \in [10^{-2}, 10^4]$.

Exercise 7 (Optional: Balancing-free square root (BFSR) method)
For numerical reasons, the balancing-free square root (BFSR) algorithm is preferred to the method used in Exercise 5. The difference is in the part e).

a) Compute the reduced-order models as in Exercise 5 e) using the projection matrices

$$V_r = P_1 \quad \text{and} \quad W_r = Q_1 \left(P_1^T Q_1\right)^{-1},$$

with the QR factorizations

$$R^T V(:,1:r) = \begin{bmatrix} P_1 & P_2 \end{bmatrix} \begin{bmatrix} \widehat{R} \\ 0 \end{bmatrix} \quad \text{and} \quad L^T U(:,1:r) = \begin{bmatrix} Q_1 & Q_2 \end{bmatrix} \begin{bmatrix} \widetilde{R} \\ 0 \end{bmatrix},$$

where $P_1, Q_1 \in \mathbb{R}^{n\times r}$ have orthonormal columns and $\widehat{R}, \widetilde{R} \in \mathbb{R}^{r\times r}$ are upper-triangular.

b) Repeat the parts f) and g) from Exercise 5 using the new balancing-free square root implementation of the balanced truncation and compare the results to the standard square root method.

c) Show that the reduced-order system is equivalent to a balanced system and that it satisfies the same error bound as the one obtained by the standard square root balanced truncation method.

Exercise 8 (Optional: Low-rank Lyapunov equation solver)
It is possible to combine parts a) and b) in Exercise 5.

a) Considering the Lyapunov equation

$$AX + XA^T + BB^T = 0,$$

use Exercise 4 b) to derive an iteration method that is only working on A and B without forming the product BB^T explicitly.

b) Since B_{k+1} has twice the number of columns as B_k, it is necessary to include column compression in the iterations. Implement a function

```
W = col_comp(B, tol)
```

that performs the column compression using the pivoted QR decomposition or the SVD with a specified error tolerance `tol`, such that for $K \in \mathbb{R}^{n \times m_1}$ a matrix $K \in \mathbb{R}^{n \times m_2}$ is constructed, with $m_2 < m_1$ and $KK^T \approx BB^T$.

c) Implement a Lyapunov equation solver

```
[Z, err] = lyap_sgn_fac(A, B, maxit, tol)
```

using the above iteration from a) with the column compression from b) and test your implementation on random stable examples.