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Advanced Topics in Numerical Linear Algebra – 1. Exercise

These problems will be discussed in the first exercise class. Please prepare their solutions before the class, you will be asked to perform one of the questions on the board.

Problem 1

Let $Y \in \mathbb{C}^{j\times k}$ and $Z \in \mathbb{C}^{m\times n}$. Then the $mj \times kn$ matrix

$$Y \otimes Z = \begin{bmatrix} y_{11}Z & y_{12}Z & \dots & y_{1k}Z \\ y_{21}Z & y_{22}Z & \dots & y_{2k}Z \\ \vdots & \vdots & \ddots & \vdots \\ y_{j1}Z & y_{j2}Z & \dots & y_{jk}Z \end{bmatrix}$$

is called the *Kronecker product* or *tensor product* of Y and Z.

a) Let W, X, Y, Z be matrices such that the products WX and YZ are defined. Show that $(W \otimes Y)(X \otimes Z) = (WX) \otimes (YZ)$.

b) Let S, G be nonsingular matrices. Show that $S \otimes G$ is nonsingular, too, and that $(S \otimes G)^{-1} = S^{-1} \otimes G^{-1}$.

c) Show that if A and B, as well as, C and D are similar matrices then $A \otimes C$ and $B \otimes D$ are similar (A similar to B if $\exists Q$ nonsingular s.t. $A = Q^{-1}BQ$).

d) Let $A \in \mathbb{C}^{k\times k}$ and $B \in \mathbb{C}^{m\times m}$. Assume A and B have eigenvalues $\lambda_1, \dots, \lambda_k$ and $\mu_1, \dots, \mu_m$. Show that

$$\Lambda(A \otimes B) = \{\lambda_i \mu_j \,|\, i = 1, \dots, k;\ j = 1, \dots, m\}.$$

Extra: What are the eigenvectors of $A \otimes B$?

Problem 2

The vectorization operator vec( ) : $\mathbb{R}^{n\times m} \mapsto \mathbb{R}^{nm}$ vertically stacks the columns of a matrix $X = [x_1, \dots, x_m]$ via

$$\mathrm{vec}(X) = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix}.$$

Show that for $\mathcal{T} \in \mathbb{R}^{n\times m}$, $\mathcal{O} \in \mathbb{R}^{m\times p}$, $\mathcal{R} \in \mathbb{R}^{p\times r}$ it holds

$$\mathrm{vec}(\mathcal{TOR}) = (\mathcal{R}^T \otimes \mathcal{T})\,\mathrm{vec}(\mathcal{O})\,. \tag{1}$$

As an important consequence, it follows that the matrix equation $\mathcal{TOR} = \mathcal{B}$ and the linear system (1) are equivalent.

Problem 3

We consider the *(discrete-time) Sylvester equation*

$$AXB - X = -Q \text{ with } A \in \mathbb{R}^{n\times n},\ B \in \mathbb{R}^{m\times m},\ Q = \mathbb{R}^{n\times m},\ X \in \mathbb{R}^{n\times m}.$$

a) Use Kronecker product and vectorization to find an equivalent linear system to this matrix equation. Use this system to derive conditions on the spectra of A, B that ensure a unique solution X.

b) If $B = A^T$ and $Q = Q^T$, the above equation is also called *discrete-time Lyapunov or Stein* equation. Give a sufficient condition on (the spectrum of) A for the existence of a unique solution in this case.

c) Now show that if $A = (S - \rho I)(S + \rho I)^{-1}$, $Q = -2\rho(S + \rho I)^{-1}\hat{Q}(S + \rho I)^{-T}$ with S Hurwitz ($\mathrm{Re}(\lambda) < 0\ \forall\lambda \in \Lambda(S)$), $\hat{Q} = \hat{Q}^T$, and $\rho < 0$, the discrete-time Lyapunov is equivalent to a continuous-time Lyapunov equation.

Remark: The terms discrete/continuous-time indicate the connection of the matrix equations to discrete- and continuous-time linear dynamical systems $x_{k+1} = Ax_k$ and $\dot{x}(t) = Sx(t)$, respectively.

Problem 4

Let $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$. We say (A, B) is *controllable* if

$$\mathrm{rank}\big([B, AB, \ldots A^{n-1}B]\big) = n \quad\Leftrightarrow\quad y^*B \neq 0 \quad \forall y \in \mathbb{C}^n\backslash\{0\} \text{ satisfying } y^*A = y^*\lambda$$

(y are the "left" eigenvectors of A).
Show that for A Hurwitz, the continuous-time Lyapunov equation $AX + XA^T + BB^T = 0$ is solved by

$$X = \int_0^\infty \mathrm{e}^{At}\, BB^T\, \mathrm{e}^{A^T t}\, dt.$$

Show that this solution is positive definite if (A, B) is controllable.