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Advanced Topics in Numerical Linear Algebra – 2. Exercise

These problems will be discussed in the exercise course of April 26, 2018. Please prepare their solutions before the class, you will be asked to perform one of the questions on the board. You may send your MATLAB® implementations via email to *penke@mpi-magdeburg.mpg.de*.

Problem 1 (Bartels-Stewart)

Now we consider the Sylvester equation

$$AX + XB = Q, \tag{1}$$

with $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{m\times m}$, $\Lambda(A) \cap \Lambda(-B) = \emptyset$, $Q \in \mathbb{R}^{n\times m}$ and the sought solution $X \in \mathbb{R}^{m\times n}$.

(a) Based on the derivation of the complex Bartels-Stewart method for Lyapunov equations from the lecture, derive and implement the Bartels-Stewart algorithm in MATLAB for solving (1) using complex Schur forms of A and B (e.g., `schur(A,'complex')`). Test your implementation with the Sylvester equation given in `sylv_eqn.mat` (look, e.g., at the residual $\|AX + XB - Q\|_2$).

(b) Now assume that one of the matrices is much smaller than the other one, for instance, $m \ll n$. Give an algorithm that solves (1) without transforming the larger matrix into a Schurform. Would this algorithm be also viable if one matrix is small, dense and the other one very large, sparse?

Problem 2 (Review Arnoldi)

Now let $\mathcal{Q}_k = \mathcal{K}_k(A, b) = \operatorname{span}\{b, Ab, \ldots, A^{k-1}b\}$, i.e., the kth Krylov subspace of A, b. The Arnoldi process can be used to construct a basis for $\mathcal{K}_k(A, b)$:

Algorithm 1 Arnoldi process

```
q_1 = b/||b||_2.
for k = 1, 2, ... do
  w = A q_k
  for j = 1, 2, ..., k do
    h_{j,k} = q_j^T w
    w = w - h_{j,k} q_j
  end for
  h_{k+1,k} = ||w||, q_{k+1} = w/h_{k+1,k}.
end for
```

(a) Validate that the Arnoldi-recursion $AQ_k = Q_k H_k + h_{k+1,k} e_k^T$ holds.

(b) Show that indeed $\operatorname{range}(Q_k) = \mathcal{K}_k(A, b)$ provided $h_{j+1,j} \neq 0$, $j \leq k$. What happens if $h_{k+1,k} = 0$?

Problem 3 (Galerkin projection methods for Lyapunov)

Consider the Lyapunov equation

$$AX + XA^T + bb^T = 0, \quad A \in \mathbb{R}^{n\times n} \text{ (large, sparse, Hurwitz)}, \ b \in \mathbb{R}^n \tag{2}$$

and a $k \ll n$-dimensional subspace $\mathcal{Q}_k \subset \mathbb{R}^n$. Let the columns of $Q_k = [q_1, \ldots, q_k] \in \mathbb{R}^{n\times k}$, $Q_k^T Q_k = I$ form a basis of $\mathcal{Q}_k$. Let $X_k := Q_k Y_k Q_k^T \approx X$ be an approximate solution of (2) for some $Y_k = Y_k^T \in \mathbb{R}^{k\times k}$.

(a) Use the inner product of two matrices $\langle V, W\rangle := \mathrm{tr}\big(VW^T\big)$ to show that

$$R_k := AX_k + X_kA^T + bb^T \ \perp_{\langle\cdot,\cdot\rangle}\ \mathcal{Z}_k \quad \Leftrightarrow \quad H_kY_k + Y_kH_k^T + Q_k^Tbb^TQ_k = 0, \tag{3}$$

where $\mathcal{Z}_k := \{X_k := Q_kY_kQ_k^T \in \mathbb{R}^{n\times n} \ : \ Q_k^TQ_k = I_k,\ \mathrm{range}(Q_k) = \mathcal{Q}_k,\ Y_k \in \mathbb{R}^{k\times k}\}$ and $H_k := Q_k^TAQ_k$.

(b) Assume $\mathcal{Q}_k = \mathcal{K}_k(A, b)$ with Q_k generated by the Arnoldi process (**Problem 2**) and we carry out the Galerkin projection for (2) described in problem **(a)**. Show that the Lyapunov residual $R_k = AX_k + X_kA^T + bb^T$ has rank 2 and that

$$\|R_k\|_2 = \|e_k^Th_{k+1,k}Y_k\|_2 \quad (e_k = k\text{th unit vector}).$$

(c) Implement an Arnoldi method for (2) MATLAB which expands $\mathcal{Q}_k$ and solves the small Lyapunov equation (3) in each iteration step. Use the `lyap` routine to solve the projected Lyapunov equations. Run this algorithm using the provided data (`problem3.mat`) on the homepage until $\|R_k\|_2/\|b\|_2^2 < 10^{-8}$ and plot $\|R_k\|_2/\|b\|_2^2$ against k. Discuss the numerical efficiency of this approach w.r.t. your observations.

In part **(c)**, do not explicitly form $X_k = Q_kY_kQ_k^T$! Instead, let your routine deliver the *low-rank solution factors* Q_k, Y_k as output data.

Problem 4 (Lyapunov stabilization)

Let $(A, B) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m}$ be controllable with $\Lambda(A) \subset \mathbb{C}_+ \cup \mathbb{C}_-$ and $\beta > \rho(A) = \max(|z| \ : \ z \in \Lambda(A))$. Consider the solution X of the Lyapunov equation

$$(A + \beta I)X + X(A + \beta I)^T = 2BB^T.$$

Show that $A - BB^TX^{-1}$ is Hurwitz ($\Lambda(A - BB^TX^{-1}) \subset \mathbb{C}_-$). In that case $F := -B^TX^{-1}$ is called *stabilizing feedback matrix* for (A, B). **Hint:** Theorem II.10 from lecture.