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Advanced Topics in Numerical Linear Algebra – 3. Exercise

These problems will be discussed in the exercise course of May 17, 2018. Please prepare their solutions before the class, you will be asked to perform one of the questions on the board. You may send your MATLAB® implementations via email to *penke@mpi-magdeburg.mpg.de*.

Problem 1

Prove Theorem II.13:
Let $A \in \mathbb{R}^{n\times n}$ be *strictly dissipative*, i. e., $A + A^T < 0$ (strictly negative definite) and $Q_k \in \mathbb{R}^{n\times k}$ with $Q_k^T Q_k = I_k$. Then for $H_k := Q_k^T A Q_k$ it holds $\Lambda(H_k) \subset \mathbb{C}^-$ and the reduced Lyapunov equation $H_k Y + Y H_k^T + Q_k^T B B^T Q_k = 0$ has a unique solution.

Problem 2

We consider the Lyapunov equation

$$AX + XA^T + bb^T = 0, \quad A \in \mathbb{R}^{n\times n} \text{ (large, sparse)}, \; b \in \mathbb{R}^n. \tag{1}$$

Let $\mathcal{P} = \{p_1, \dots, p_k\} \subset \mathbb{C}_-$ be a (given) set of shift parameters.

a) Implement the low-rank ADI iteration from the lecture and test it using the provided data (`large_lyap.mat`, includes shifts) on the homepage. The occurring linear systems should be solved by the backslash "\" command. Exploit the low-rank structure of R_k (see lecture) and use the normalized residual norm corresponding to (1) as stopping criterion via $\|R_k\|_2/\|b\|_2^2 < 10^{-8}$.
Hint: Do not form $X \approx X_k = Z_k Z_k^T$, i.e., let the implementation deliver Z_k!

b) Show that $\operatorname{range}(Z_k) = \operatorname{span}\big\{(A + p_1 I)^{-1} b, \dots, (A + p_k I)^{-1} b\big\}$ for $p_i \neq p_j, \forall i, j$.

Problem 3 (Hamiltonian Matrices)

Let $J = \left[\begin{smallmatrix} 0 & I_c \\ -I_n & 0 \end{smallmatrix}\right]$ and $H \in \mathbb{R}^{2n\times 2n}$.

a) Prove that the following statements are equivalent:

  i) H is Hamiltonian ($(HJ)^T = HJ$).

  ii) $H = JS$ for some matrix $S = S^T \in \mathbb{R}^{2n\times 2n}$.

  iii) It holds $(JH)^T = JH$.

  iv) H has the structure

$$H = \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & -H_{11}^T \end{bmatrix}$$

  with $H_{12} = H_{12}^T$, $H_{21} = H_{21}^T$.

b) Let p_H be the characteristic polynomial of H. Show that:

  i) If H is Hamiltonian, then $p_H(\lambda) = p_H(-\lambda)$ for all $\lambda \in \mathbb{C}$.

  ii) If H is Hamiltonian and for $\lambda \in \mathbb{C}$ it holds $p_H(\lambda) = 0$, then $p_H(-\lambda) = p_H(-\overline{\lambda}) = p_H(\overline{\lambda}) = 0$.

Problem 4 (Spectral Mapping Theorem)

Prove the *spectral mapping theorem*.
Let $A \in \mathbb{C}^{n\times n}$ and $f : D \to \mathbb{C}$ be an analytic function and let $\Lambda(A) = \{\lambda_1, \dots, \lambda_n\} \subset D$ where D is a domain in $\mathbb{C}$. Show that $\Lambda(f(A)) = \{f(\lambda_1), \dots, f(\lambda_n)\}$.