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Advanced Topics in Numerical Linear Algebra – Bartels Stewart Algorithm.

We want to compute the solution X of the Lyapunov equation

$$AX + XA^* = W, \quad A \in \mathbb{C}^{n\times n}, \quad W = W^* \in \mathbb{C}^{n\times n}. \tag{1}$$

Algorithm 1 Bartels-Stewart (complex)

Input: $A, W \in \mathbb{C}^{n\times n}, W = W^*$
Output: $X = X^*$ solving (1)

Compute (complex) Schur decomposition $T = Q^*AQ$ with QR-algorithm
if $\operatorname{diag}(T) \cap \operatorname{diag}(-T^*) \neq \emptyset$ **then**
  STOP (no unique solution)
end if
$\begin{bmatrix} W_1 & W_2 \\ W_2^* & W_3 \end{bmatrix} \leftarrow Q^*WQ$
$k = n - 1$
while $k > 1$ **do**
  Solve $X_3 = \frac{W_3}{T_3 + \overline{T}_3}$ with $T_3 = T_{kk}$
  Solve

$$T_1X_2 + X_2\overline{T}_3^* = W_1$$

  with $T_1 = T(1:k, 1:k), W_1 = W_1 - T_2X_2^* - X_2T_2^*$
  $k = k - 1$
end while
Solve $T_1X_1 + X_1T_1^* = W_2 - T_2X_2^* - X_2T_2^*$ in $\mathbb{C}^{1\times 1}$
Back transformation $X \leftarrow QXQ^*$

- total operations $\approx 32n^3 \approx \underbrace{25n^3}_{\text{Schur}} + \underbrace{3n^3}_{\text{Premult}} + \underbrace{3n^3}_{\text{Postmult}} + \underbrace{n^3}_{\text{While}}$
- algorithm uses only numerically backward stable parts and unitary transformations $\Rightarrow$ can be considered backward stable,
- real version for $A \in \mathbb{R}^{n\times n}$, $W \in \mathbb{R}^{n\times n}$.
- version for Sylvester equations works analogously (see exercise),
- implemented, e.g., in the routine `lyap` of the MATLAB Control Toolbox

Literatur

[1] R. Bartels and G. Stewart, *Solution of the Matrix Equation $AX + XB = C$: Algorithm 432*, Comm. ACM, 15 (1972), pp. 820–826.

[2] D. Sorensen and Y. Zhou, *Direct methods for matrix Sylvester and Lyapunov equations*, J. Appl. Math, 2003 (2003), pp. 277–303.