Otto-von-Guericke-Universität Magdeburg
Max Planck Institute for Dynamics of Complex Technical Systems
Prof. Dr. Peter Benner, Dr. Patrick Kürschner, Carolin Penke

summer term 2018

Advanced Topics in Numerical Linear Algebra – Projection Methods for Lyapunov Equations.

Consider the large and sparse Lyapunov equation

$$AX + XA^T = -BB^T, \tag{1}$$

i.e., $A \in \mathbb{R}^{n\times n}$ sparse, $\Lambda(A) \in \mathbb{C}_-$, $A + A^T \prec 0$, n "large", $B \in \mathbb{R}^{n\times m}$ for $m \ll n$.
The following pseudocode represents a (very basic) joint framework for the standard (block) Krylov via Lanczos/Arnoldi [1, 2], Extended Krylov subspace method (EKSM) [3], and rational Krylov subspace method (RKSM) [4, 5].

Algorithm 1 Projection Method for Lyapunov Equations

Input: A, B forming (1), only for RKSM shifts $\{s_1, \ldots, s_{\text{maxiter}}\} \subset \mathbb{C}_+$
Output: Q_j, Y_j such that $X = Q_j Y_j Q_j^T$, $Q_j \in \mathbb{R}^{n\times n_Q}$, $Y_j = Y_j^T \in \mathbb{R}^{n_Q\times n_Q}$ (approx.) solves (1).

1: $j = 1, \gamma_0 = 1$, $Q_0 = q_0 = \texttt{orth}(B)$. Only EKSM: $Q_0 := \texttt{orth}(q_0,\ \tilde{q}_0 := A^{-1}B)$.
2: **while** $\gamma_{j-1} \geq$tol. **do**
3: Generate new basis vectors:

$$\begin{cases} w &= Aq_j \ : \ \text{Standard Krylov via Arnoldi} \\ w &= [Aq_j(:,1:m), A^{-1}q_j(:,m+1:2m)] \ : \ \text{Extended Krylov} \\ w &= (A - s_jI_n)^{-1}q_j \ : \ \text{Rational Krylov} \end{cases}$$

4: Orthogonal expansion: $Q_j = [Q_{j-1}, q_j] = \texttt{orth}([Q_{j-1}, w])$.
5: $H_j = Q_j^* A Q_j$, $B_j = Q_j^* B$;
6: Solve

$$H_j Y_j + Y_j H_j^* + B_j B_j^* = 0$$

for Y_j by, e.g., Bartels-Stewart algorithm.
7: $\gamma_j = \|AQ_jY_jQ_j^* + Q_jY_jQ_j^*A^T + BB^T\|/\|B\|^2$.
8: $j = j + 1$
9: **end while**

Remarks

- In line 1 and 4, `orth` should be understood as any stable orthogonalization routine, e.g., modified (block) Gram-Schmidt, to orthogonally expand Q_{j-1} by the new basis vectors.
- The dimension of the small scale Lyapunov equation in line 6 grows linearly in m leading to growing costs $\mathcal{O}(m^3 j^3)$. Some savings are possible by solving it only every couple of steps, e.g., every 5th step. The condition $A + A^T \prec 0$ ensures that the projected Lyapunov equation is uniquely solvable (Theorem II.13).
- There are efficient formula to get the Lyapunov residual norm $\|AQ_jY_jQ_j^* + Q_jY_jQ_j^*A^T + BB^T\|$ (cf. Theorem II.12).

- For pairs of complex conjugated shifts ($\xi_i,\ \xi_{i+1} = \overline{\xi_i} \in \mathbb{C}$) in RKSM, the amount of complex operations can be greatly reduced.
- Generalized equations $AXE^T + EXA^T = -BB^T$, $E \in \mathbb{R}^{n\times n}$ nonsingular, are dealt with by implicitly applying Algorithm 1 to the equivalent Lyapunov equation defined by $M^{-1}A, M^{-1}B$ or, if $M = LL^T \succ 0$, by $L^{-1}AL^{-T},\ L^{-1}B$. Requires additional solves with M or L.
- Sylvester and discrete-time Lyapunov equations can be dealt with similarly.

References

[1] Y. Saad, *Numerical Solution of Large Lyapunov Equations*, Signal Processing, Scattering, Operator Theory and Numerical Methods (1990), pp. 503–511.

[2] I. Jaimoukha and E. Kasenally, *Krylov subspace methods for solving large Lyapunov equations*, SIAM J. Numer. Anal., 31 (1994), pp. 227–251.

[3] V. Simoncini, *A new iterative method for solving large-scale Lyapunov matrix equations*, SIAM J. Sci. Comput., 29 (2007), pp. 1268–1288.

[4] V. Druskin, L. Knizhnerman, and V. Simoncini, *Analysis of the rational Krylov subspace and ADI methods for solving the Lyapunov equation*, SIAM J. Numer. Anal., 49 (2011), pp. 1875–1898.

[5] V. Druskin and V. Simoncini, *Adaptive rational Krylov subspaces for large-scale dynamical systems*, Systems & Control Letters, 60 (2011), pp. 546–560.