Otto-von-Guericke-Universität Magdeburg summer term 2018
Max Planck Institute for Dynamics of Complex Technical Systems
Prof. Dr. Peter Benner, Dr. Patrick Kürschner, Carolin Penke

Advanced Topics in Numerical Linear Algebra – Low Rank ADI Iteration.

Consider the Lyapunov equation

$$AX + XA^T = -BB^T, \quad \text{with} \quad A \in \mathbb{R}^{n\times n} \quad \text{large,sparse}, \quad \Lambda(A) \subset \mathbb{C}_-, \quad B \in \mathbb{R}^{n\times m},\ m \ll n. \tag{1}$$

The following algorithm [1, 2, 4] successively forms a low rank solution factor of the solution $X \approx ZZ^*$.

Algorithm 1 Low-rank ADI (LR-ADI) iteration for Lyapunov equations

Input: A, B forming (1), shifts $\mathcal{P} = \{p_1, \ldots, p_{\text{maxiter}}\} \subset \mathbb{C}_-$
Output: Z_j such that $X = Z_j Z_j^*$, $Z_j \in \mathbb{C}^{n\times mj}$ (approx.) solves (1).

1: $W_0 := B$, $j = 1$, $Z_0 = [\ ]$.
2: **while** $\|W_{j-1}\|_2^2 \geq$tol. **do**
3: Solve $(A + p_j I)V_j = W_{j-1}$ for $V_j \in \mathbb{C}^{n\times m}$.
4: $W_j = W_{j-1} - 2\,\mathrm{Re}(p_j)\,V_j$
5: $Z_j = \left[Z_{j-1},\ \sqrt{-2\,\mathrm{Re}(p_j)}V_j\right]$
6: $j = j + 1$
7: **end while**

Remarks:

- Major work are linear system solves with $(A+p_jI)$ and m right hand sides in each step in line 3. If linear solves (sparse-direct or iterative) are of $\approx$ linear complexity in n, complexity of LR-ADI is $\approx$ linear in n, linear in m.
- Requires complex arithmetic if the shifts are complex (can be reduced if $p_i \in \mathbb{C}_-$, $p_{i+1} = \overline{p_i}$ [3, 4]).
- For generalized Lyapunov equations $AXE^T + EXA^T = -BB^T$,$E \in \mathbb{R}^{n\times n}$ nonsingular, the lines 3 and 4 change to $V_j = (A + p_jE)^{-1}W_{j-1}$ and $W_j = W_{j-1} - 2\,\mathrm{Re}(p_j)\,EV_j$.
- Generalizations to Sylvester equations can be found in, e.g., [5, 6].

References

[1] T. Penzl, *A cyclic low rank Smith method for large sparse Lyapunov equations*, SIAM J. Sci. Comput., 21 (2000), pp. 1401–1418.

[2] J.-R. Li and J. White. Low Rank Solution of Lyapunov Equations. *SIAM J. Matrix Anal. Appl.*, 24(1):260–280, 2002.

[3] P. Benner, P. Kürschner, and J. Saak, *Efficient Handling of Complex Shift Parameters in the Low-Rank Cholesky Factor ADI Method*, Numer. Algorithms, 62 (2013), pp. 225–251.

[4] ——, *An Improved Numerical Method for Balanced Truncation for Symmetric Second Order Systems*, Math. Comput. Model. Dyn. Sys., 19 (2013), pp. 593–615.

[5] P. Benner, R.-C. Li, and N. Truhar, *On the ADI Method for Sylvester Equations*, J. Comput. Appl. Math., 233 (2009), pp. 1035–1045.

[6] P. Benner and P. Kürschner, *Computing Real Low-rank Solutions of Sylvester equations by the Factored ADI Method*, Comput. Math. Appl., 67 (2014), pp. 1656–1672.