
Scientific Computing 1 9th Homework

Handout: 12/06/2018

Return: 12/14/2018

Exercise 1:

(2 Points)

Let V be an n -dimensional pre-Hilbert space over \mathbb{R} . The inner product is defined as

$$\langle x, y \rangle := x^T y.$$

Show that this is a proper definition of an inner product and prove that

$$\|x\| = \sqrt{\langle x, x \rangle}$$

defines a norm on this space.

Exercise 2:

(2 Points)

Prove that the linear system $Ax = b$ is solvable if and only if

$$\text{rank}(A) = \text{rank}([A \ b])$$

Does this condition guarantee uniqueness?

Exercise 3:

(6 Points)

Show that the set of unit lower triangular matrices, i.e. all lower triangular matrices with only ones on their diagonal, is a group with respect to the matrix multiplication.

Exercise 4:

(3 Points)

a.) Let $Q, \tilde{Q} \in \mathbb{C}^{n \times n}$ be two unitary matrices. Show that the product $Q\tilde{Q}$ is also an unitary matrix.

b.) The Givens rotation is defined as

$$G(i, j, \theta) = \begin{bmatrix} 1 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & & \vdots & & \vdots \\ 0 & \cdots & c & \cdots & -s & \cdots & 0 \\ \vdots & & \vdots & \ddots & \vdots & & \vdots \\ 0 & \cdots & s & \cdots & c & \cdots & 0 \\ \vdots & & \vdots & & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & 0 & \cdots & 1 \end{bmatrix}$$

where $c = \cos(\theta)$ and $s = \sin(\theta)$ appear at the intersections of the i -th and j -th rows and columns. Prove that $G(i, j, \theta)$ is a unitary matrix for arbitrary parameters.

Exercise 5:**(3 Points)**

Let $I \in \mathbb{R}^{n \times n}$ be the identity matrix.

- a.) Show that $\|I\| = 1$ holds for all induced matrix norms.
- b.) Why is the Frobenius norm $\|\cdot\|_F$ not an induced matrix norm?

Exercise 6:**(2 Points)**

The 2-norm of a matrix $A \in \mathbb{C}^{n \times n}$ can be computed as

$$\|A\|_2 = \sqrt{\varrho(A^H A)}.$$

Why is it reasonable to avoid the computation of the matrix 2-norm?

Exercise 7:**(6 Points)**

Let $x \in \mathbb{R}^n$. Prove the following inequalities:

- a.) $\|x\|_2 \leq \|x\|_1 \leq \sqrt{n}\|x\|_2$
- b.) $\|x\|_\infty \leq \|x\|_2 \leq \sqrt{n}\|x\|_\infty$
- c.) $\frac{1}{n}\|x\|_1 \leq \|x\|_\infty \leq \|x\|_1$

Use the results to show the corresponding inequalities for a matrix $A \in \mathbb{C}^{n \times n}$. Finally, show that all norms are finite.

Overall Points: 24