Problem 1. Model reduction by interpolation
(a) Consider the LTI, MIMO system given by,

\begin{align*}
  E\dot{x} &= Ax + Bu, \\
  y &= Cx,
\end{align*}

where \( E, A \in \mathbb{R}^{n \times n} \), \( B \in \mathbb{R}^{n \times m} \) and \( C \in \mathbb{R}^{p \times n} \) are the system matrices. Assume the projection matrices \( V, W \in \mathbb{R}^{n \times r} \) satisfy \( W^T V = V^T W = I_{r \times r} \). Let the corresponding reduced model be,

\begin{align*}
  \tilde{E}\dot{x} &= \tilde{A}\dot{x} + \tilde{B}u, \\
  \tilde{y} &= \tilde{C}x,
\end{align*}

Let,

\begin{align*}
  \text{range}(V) &= \text{span}\{ \tilde{B}(s_0), \tilde{A}_B(s_0)\tilde{B}(s_0), \ldots, \tilde{A}_B^{q-1}(s_0)\tilde{B}(s_0) \}, \\
  \text{range}(W) &= \text{span}\{ \tilde{C}^T(s_0), \tilde{A}_C(s_0)\tilde{C}^T(s_0), \ldots, \tilde{A}_C^{q-1}(s_0)\tilde{C}^T(s_0) \},
\end{align*}

where \( \tilde{B}(s_0) = (s_0E - A)^{-1}B \), \( \tilde{A}_B(s_0) = (s_0E - A)^{-1}E \), \( \tilde{A}_C(s_0) = (s_0E - A)^{-T}E^T \) and \( \tilde{C}^T(s_0) = (s_0E - A)^{-T}C^T \).

Assume that the matrix pair \((E, A)\) is regular. Prove that, there are at least \( 2q \) moments being equal between original systems and the reduced systems, \( M_i = M_i, i = 0, 1, \ldots, 2q - 1 \). That is,

\[ C[(s_0E - A)^{-1}E]^i(s_0E - A)^{-1}B = \tilde{C}[(s_0\tilde{E} - \tilde{A})^{-1}\tilde{E}]^i(s_0\tilde{E} - \tilde{A})^{-1}\tilde{B}, i = 0, 1, \ldots, 2q - 1 \]

\textbf{Hint:} The following holds true,

\begin{align*}
  (\tilde{A}_B(s_0))^i\tilde{B}(s_0) &= [(s_0E - A)^{-1}E]^i(s_0E - A)^{-1}B = VW^T[(s_0E - A)^{-1}E]^i(s_0E - A)^{-1}B, \\
  (\tilde{A}_C(s_0))^i\tilde{C} &= [(s_0E - A)^{-T}E^T]^i(s_0E - A)^{-T}C^T = WV^T[(s_0E - A)^{-T}E^T]^i(s_0E - A)^{-T}C^T.
\end{align*}

(b) Implement a function for interpolating a SISO LTI system

\begin{align*}
  \dot{x}(t) &= Ax(t) + Bu(t), \\
  y(t) &= Cx(t),
\end{align*}

at specified interpolation points, assuming the interpolation points are pairwise distinct and closed under conjugation. Use projection-based approach, i.e. project using matrices \( V \) and \( W \) such that

\[ \text{range}(V) = \text{span}\{ (\sigma_1I - A)^{-1}B, \ldots, (\sigma_rI - A)^{-1}B \} \]
and
\[ \text{range}(W) = \text{span}\{(\sigma_1 I - A)^{-T}C^T, \ldots, (\sigma_r I - A)^{-T}C^T\}, \]

where \(\sigma_1, \ldots, \sigma_r\) are the interpolation points. Ensure that \(V\) and \(W\) are real matrices. Test your function on the clamped beam model \texttt{beam.mat} [1].

**Problem 2. Iterative rational Krylov algorithm**
Implement the iterative rational Krylov algorithm (IRKA) discussed in the course for SISO LTI systems. Use the clamped beam model \texttt{beam.mat} [1] for testing. Compare the transfer functions of the original and the reduced-order Hermite interpolant and compute the relative \(H_2\)-error for different reduced orders.

**References**