Otto von Guericke University Magdeburg
Institute for Analysis and Numerics
Max Planck Institute for Dynamics of Complex Technical Systems
Computational Methods in Systems and Control Theory

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Dr. Lihong Feng
Sridhar Chellappa

Model Reduction of Dynamical Systems
Homework 1

Deadline: 25 April 2019

Problem 1. *Properties of the Singular Value Decomposition (SVD)*
Let $A = U\Sigma V^T \in \mathbb{R}^{m\times n}$ be the singular value decomposition with orthogonal matrices $U = \begin{bmatrix} u_1 & \dots & u_m \end{bmatrix} \in \mathbb{R}^{m\times m}$ and $V = \begin{bmatrix} v_1 & \dots & v_n \end{bmatrix} \in \mathbb{R}^{n\times n}$, and a diagonal matrix $\Sigma \in \mathbb{R}^{m\times n}$ with $\sigma_1 \geq \dots \geq \sigma_{\min\{m,n\}} \geq 0$ on the diagonal. Additionally, let $r = \operatorname{rank}(A) < \min\{m,n\}$, i.e. $\sigma_r > 0$ and $\sigma_{r+1} = 0$.

(a) Prove that

$$\ker(A) = \operatorname{span}\{v_{r+1},\dots,v_n\}\,,$$
$$\operatorname{range}(A) = \operatorname{span}\{u_1,\dots,u_r\}\,.$$

(b) Prove that

$$\|A\|_F = \sqrt{\sigma_1^2 + \ldots + \sigma_r^2}.$$

(c) (Schmidt-Eckart-Young-Mirsky theorem). Let $k < r$ and $A_k = \sum_{i=1}^k \sigma_i u_i v_i^T$. Prove that

$$\min_{\operatorname{rank}(B)=k} \|A - B\|_2 = \|A - A_k\|_2 = \sigma_{k+1}.$$

Hint: Show that, for a matrix $B \in \mathbb{R}^{m\times n}$ with $\operatorname{rank}(B) = k$, there exists $\|z\|_2 = 1$ such that $Bz = 0$ and $z \in \operatorname{span}\{v_1,\dots,v_{k+1}\}$. Then show that $\|A - B\|_2 \geq \|Az - Bz\|_2$ and $\|Az - Bz\|_2 = \|Az\|_2 \geq \sigma_{k+1}$.

(d) What is the distance (in terms of the spectral norm) of a square invertible matrix $A \in \mathbb{R}^{n\times n}$ (with singular values $\sigma_1,\dots,\sigma_n$) to the set of singular matrices?

(e) (Least squares approximation) Consider the system of linear equation $Ax = b$. When the system is overdetermined ($m > n$) or underdetermined ($m < n$), one typically looks for a solution that minimizes the residual $r(x) := \|b - Ax\|$. If the Euclidean norm $\|\cdot\|_2$ is used, this is called a least-squares problem. Prove that,

$$x_{\text{LS}} = \sum_{i=1}^{r} \frac{u_i^T b}{\sigma_i} v_i$$

minimizes the residual $r(x) = \|b - Ax\|_2$ and that it has the smallest $2-$norm among all minimizers.

Problem 2. *Uniqueness of transfer function*
Consider the following state-space representation,

$$\Sigma : \begin{cases} \dot{x}(t) &= Ax(t) + Bu(t), \\ y(t) &= Cx(t) + Du(t) \end{cases}$$

Let T be an invertible state transformation matrix which transforms the state as, $x = T\tilde{x}$. Let the transformed state-space representation be,

$$\tilde{\Sigma} : \begin{cases} \dot{\tilde{x}}(t) &= \tilde{A}\tilde{x}(t) + \tilde{B}u(t), \\ \tilde{y}(t) &= \tilde{C}\tilde{x}(t) + Du(t) \end{cases}$$

Show that Σ and $\tilde{\Sigma}$ have the same transfer functions.

Problem 3. *Controllability of LTI systems*
Let $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$, and $\mathcal{C} = \begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix}$ the associated Kalman controllability matrix.

(a) Show that the controllability subspace range($\mathcal{C}$) is an A-invariant subspace, i.e. $A\,\mathrm{range}(\mathcal{C}) \subseteq \mathrm{range}(\mathcal{C})$.

(b) Let $T \in \mathbb{R}^{n\times n}$ be an invertible matrix. Show that the controllability matrix of $(T^{-1}AT, T^{-1}B)$ is $T^{-1}\mathcal{C}$.

(c) (Kalman controllability form). Let $r = \mathrm{rank}(\mathcal{C})$, $0 < r < n$. Show that there exists an invertible matrix $T \in \mathbb{R}^{n\times n}$ such that

$$\tilde{A} := T^{-1}AT = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} \text{ and } \tilde{B} := T^{-1}B = \begin{bmatrix} B_1 \\ 0 \end{bmatrix},$$

with $A_{11} \in \mathbb{R}^{r\times r}$ and $B_1 \in \mathbb{R}^{r\times m}$, where (A_{11}, B_1) is a controllable matrix pair.

Problem 4. *Observability of LTI systems*
Consider the LTI system Σ as defined in Problem 2. Show that the system is observable if and only if the finite-time observability gramian

$$W_o(t) = \int_0^t e^{A^T\tau} C^T C e^{A\tau} d\tau$$

is nonsingular for any $t > 0$.

Problem 5. *Properties of LTI systems*
Consider the dynamical system given by

$$A = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -4 & -2 \\ 0 & 2 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}.$$

Analyze the above system with respect to stability, controllability, and observability. Further, compute the transfer function $H(s) = C(sI - A)^{-1}B$ of the system. What can you say with regard to minimality of the system?

You can send your solutions as follows:

1. **by sending them to** `chellappa@mpi-magdeburg.mpg.de`,

Solutions should be written in a PDF file (created using LaTeX). When sending emails, please add `[mor19]` **to the subject line. The filename should include your name and the corresponding exercise sheet number as well as the problem number. In case of several files please hand in a compressed file. Moreover, please print the source code (in case of coding problems) of your routine and hand it in together with the other exercises.**