Otto von Guericke University Magdeburg
Institute for Analysis and Numerics
Max Planck Institute for Dynamics of Complex Technical Systems
Computational Methods in Systems and Control Theory

Summer Term 2019
Dr. Lihong Feng
Sridhar Chellappa

Model Reduction of Dynamical Systems
Homework 2

Deadline: 29 May 2019

Problem 1. *LTI system gramians*

(a) Consider the LTI finite time controllability and observability gramians, $P(t), Q(t)$ respectively,

$$P(t) = \int_0^t e^{A\tau} BB^T e^{A^T\tau} d\tau,$$
$$Q(t) = \int_0^t e^{A^T\tau} C^T C e^{A\tau} d\tau$$

Recall the notion that, if the LTI system is controllable then $P(t)$ is nonsingular and if the system is observable then $Q(t)$ is nonsingular. Now show that, if the system is controllable and observable, then the infinite gramians P, Q are nonsingular.

(b) Recall from the lecture that $P \geq P(t)\ \forall t > 0$. We also know that the minimal energy required to reach a desired state at time t is given by,

$$\|u\|^2 = x^T P(t)^{-1} x$$

Show that, $\|u\|^2 = x^T P(t)^{-1} x \geq x^T P^{-1} x$.

Problem 2. *Minimality of LTI systems*

Show that if (A, B) is controllable and (A, C) is observable, then the realization (A, B, C, D) is minimal.

Hint: Consider the Hankel matrix (the product of the observability and controllability matrix).

Problem 3. *Lyapunov equation*

Let $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$. Show that, if any two of the following three statements are true, then all of them are true:

(a) A is Hurwitz,

(b) (A, B) is controllable,

(c) there exists $P > 0$ such that $AP + PA^T + BB^T = 0$.

Problem 4. *Computation of system norms*

(a) Consider the LTI system

$$A = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -4 & -2 \\ 0 & 2 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}, \quad D = 0.$$

Analytically compute the $\mathcal{H}_\infty$-norm of the system.

(b) Consider the following LTI system:

$$A = \begin{bmatrix} -8 & 8 \\ -8 & -42 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 2 \end{bmatrix}, \quad D = 0.$$

Compute the $\mathcal{H}_2$-norm of the system by first solving a Lyapunov equation.

Problem 5. *Balanced realizations*

Let (A, B, C, D) be a minimal realization of an LTI system. Show that a balanced realization is given by the state-space transformation

$$T := \Sigma^{-\frac{1}{2}} V^T R,$$

where $P = S^T S$ and $Q = R^T R$ (e.g., Cholesky decompositions) satisfy the pair of Lyapunov equations

$$AP + PA^T + BB^T = 0,$$
$$A^T Q + QA + C^T C = 0$$

and

$$SR^T = U \Sigma V^T$$

is the SVD of SR^T.

Hint: First note that $T^{-1} = S^T U \Sigma^{-\frac{1}{2}}$, then the result follows by simple algebraic manipulations.

You can send your solutions as follows:

1. **by sending them to** `chellappa@mpi-magdeburg.mpg.de`,

Solutions should be written in a PDF file (created using LaTeX). When sending emails, please add `[mor19]` **to the subject line. The filename should include your name and the corresponding exercise sheet number as well as the problem number. In case of several files please hand in a compressed file. Moreover, please print the source code (in case of coding problems) of your routine and hand it in together with the other exercises.**