MAX PLANCK INSTITUTE FOR DYNAMICS OF COMPLEX TECHNICAL SYSTEMS MAGDEBURG

CSC COMPUTATIONAL METHODS IN SYSTEMS AND CONTROL THEORY

Otto-von-Guericke Universität Magdeburg
Faculty of Mathematics
Summer term 2019

Model Reduction for Dynamical Systems
-Lecture 1-

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CSC COMPUTATIONAL METHODS IN SYSTEMS AND CONTROL THEORY

- **Lecture: Lihong Feng**
- **Exercise: Sridhar Chellappa**
- **Lecture and exercises/homework information:**

  https://www.mpi-magdeburg.mpg.de/3668354/mor_ss19

- **Ask questions.**
- **Take notes when necessary, and only when necessary.**

What problem does this Lecture try to solve?

- **Not** $ax = b.$
- **Not** $e\dfrac{dx}{dt} = ax + b.$
- **Not** $a_1 x_1 = c_1,$
  $a_2 x_2 = c_2.$
- **Not** $a_1 \dfrac{dx_1(t)}{dt} = c_1,$
  $a_2 \dfrac{dx_2(t)}{dt} = c_2.$
- **But:**

$$
\begin{aligned}
e_{11}\frac{dx_1}{dt} + \cdots e_{1n}\frac{dx_n}{dt} &= a_{11}x_1 + \cdots a_{an}x_n + b_1,\\
&\vdots\\
e_{n1}\frac{dx_1}{dt} + \cdots e_{nn}\frac{dx_n}{dt} &= a_{n1}x_1 + \cdots a_{nn}x_n + b_n.
\end{aligned}
$$

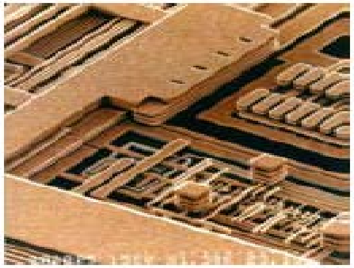
$$n \geq 10^4$$

Motivation

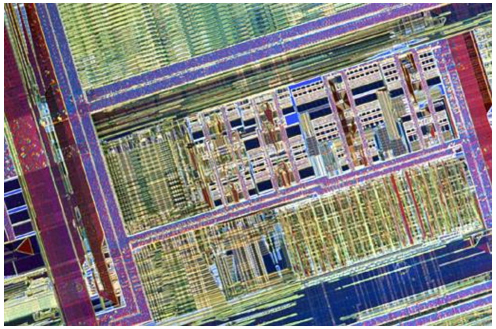
Large-scale dynamical systems are omnipresent in science and technology.

Example 1.

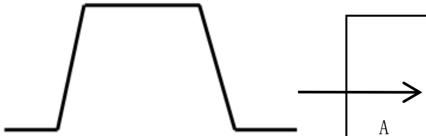
Copper interconnect pattern(IBM)

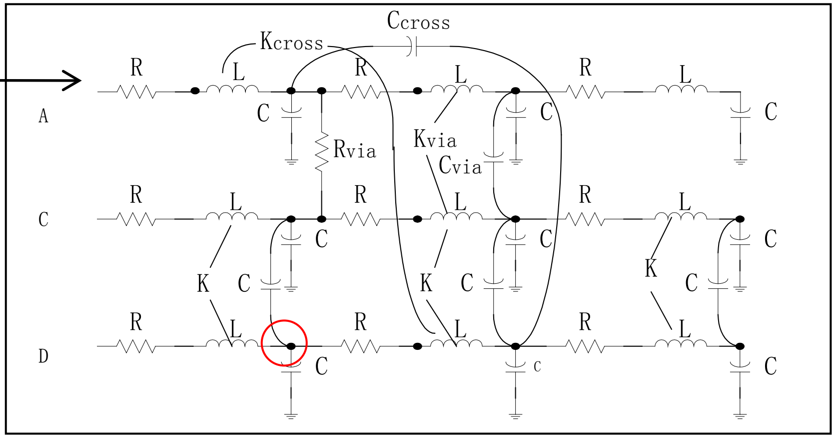


Picture from [N.P can der Meijs’01]

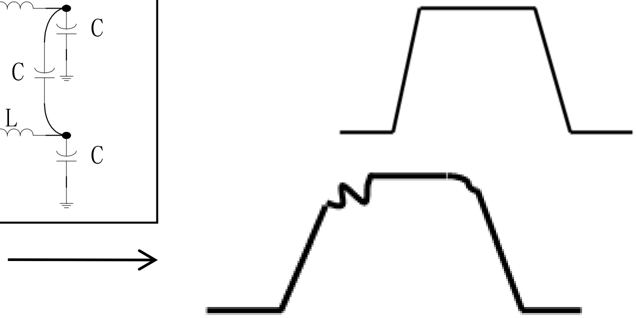


Picture from Encarta
http://encarta.msn.com/media_461519585/pentium_microprocessor.html





Signal integrity: noise, delay.



Motivation

The interconnect can be modelled by a mathematical model:

$$E\frac{dx(t)}{dt} = Ax(t) + Bu(t),$$
$$y(t) = Cx(t),$$

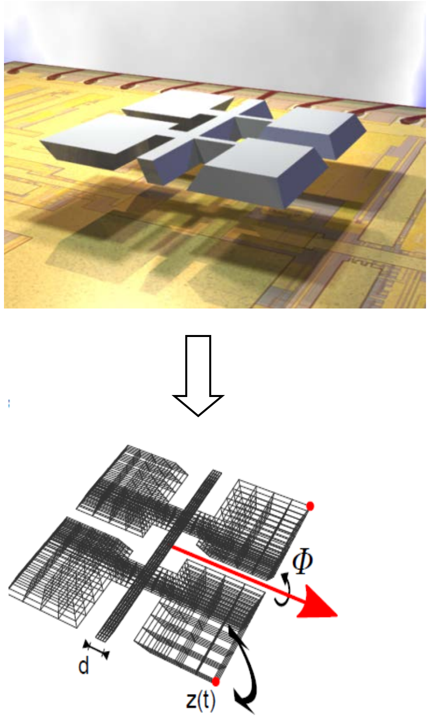
With O(10^6) ordinary differential equations.

Example 2

- A Gyroscope is a device for measuring or maintaining orientation and has been used in various automobiles (aviation, shipping and defense).

- The design of the device is verified by modelling and simulation.

https://morwiki.mpi-magdeburg.mpg.de/morwiki/index.php/Gyroscope



Pic. by Jan Lienemann and C. Moosmann, IMTEK

Motivation

- The butterfly Gyroscope can be modeled by partial differential equations (PDEs).

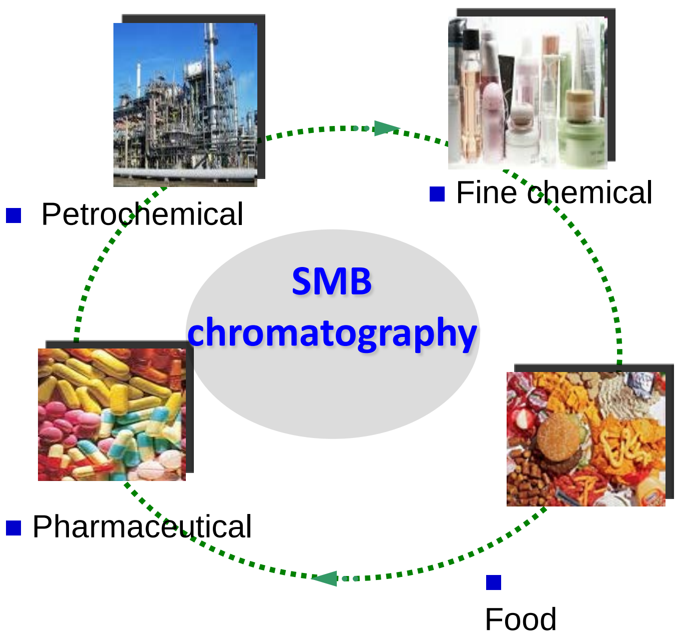
- Using finite element method, the PDE is discretized into (in space) ODEs:

$$M(\mu)\frac{d^2x}{dt^2}+K(\mu)\frac{dx(t)}{dt}+D(\mu)x(t)+Bu(t),$$
$$y(t)=Cx(t),$$

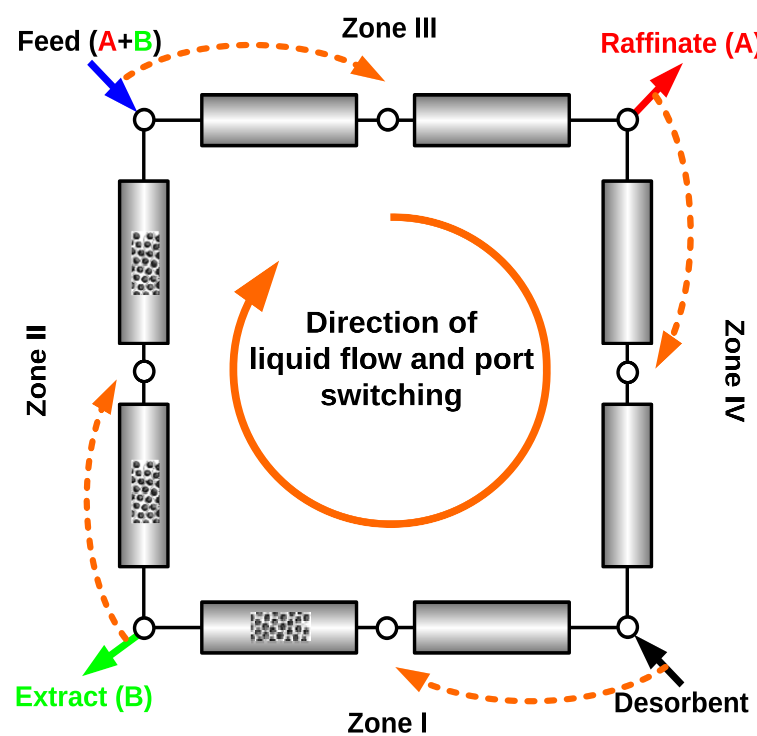
With $O(10^5)$ ordinary differential equations. $\mu=(\mu_1,\ldots,\mu_l)$ is the vector of parameters.

Motivation

Example 3: Simulated Moving Bed



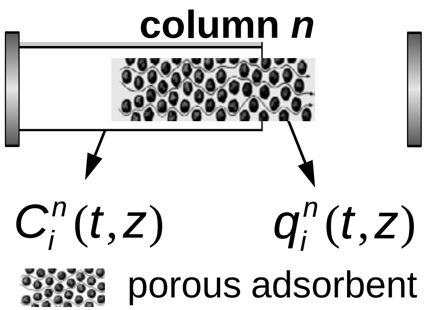
Simulated Moving Bed Process



Motivation

Mathematical modeling of SMB process

PDE system (couple N_{Col} columns)



$$\frac{\partial C_i^n(t,z)}{\partial t} + \frac{1-\varepsilon}{\varepsilon}\frac{\partial q_i^n(t,z)}{\partial t} = -u_n \frac{\partial C_i^n(t,z)}{\partial z} + D_n \frac{\partial^2 C_i^n(t,z)}{\partial z^2}, \quad i = A,B \quad n = 1,2,\ldots,N_{Col}$$

$$\frac{\partial q_i^n(t,z)}{\partial t} = Km_i(q_i^{n,Eq}(t,z) - q_i^n(t,z)), \quad i = A,B$$

$$q_i^{n,Eq} = f_i(C_A^n, C_B^n) = \frac{H_{i,1}C_i^n}{1 + K_{A,1}C_A^n + K_{B,1}C_B^n} + \frac{H_{i,2}C_i^n}{1 + K_{A,2}C_A^n + K_{B,2}C_B^n}$$

Initial and boundary conditions:

$$C_i^n(t=0,z) = 0, \quad q_i^n(t=0,z) = 0$$

$$\left.\frac{\partial C_i^n}{\partial z}\right|_{z=0} = \frac{u_n}{D_n}(C_i^n(t,0) - C_i^{n,in}(t)), \qquad \left.\frac{\partial C_i^n}{\partial z}\right|_{z=L} = 0$$

Motivation

Mathematical modeling of SMB process

PDE system (couple N_{Col} columns) —spatial discretization→ **DAE system**

A complex system of nonlinear, parametric DAEs:

$$M(\mu)\frac{dx}{dt} = f(\mu, x) + B,$$
$$y(t) = Cx(t),$$

μ : operating conditions

with proper initial conditions.

Analytical solution of the LTI System

LTI System: $\begin{cases} dx(t)/dt = Ax(t) + Bu(t) \\ x(0) = x_0 \end{cases}$

Multiplying e^{-At} on both sides of $dx(t)/dt = Ax(t) + Bu(t)$ yields

$$e^{-At}\frac{dx(t)}{dt} - e^{-At}Ax(t) = e^{-At}Bu(t)$$

which implies,

$$\frac{d}{dt}(e^{-At}x(t)) = e^{-At}Bu(t)$$

Its integration from 0 to t yields,

$$e^{-A\tau}x(\tau)\Big|_{\tau=0}^{t} = \int_0^t e^{-A\tau}Bu(\tau)d\tau$$

Analytical solution of the LTI System

Thus we have
$$e^{-At}x(t) - e^{0}x_0 = \int_0^t e^{-A\tau}Bu(\tau)d\tau \qquad (1)$$

Because the inverse of e^{-At} is e^{At} and $e^{A\cdot 0} = I$, (1) implies
$$x(t) = e^{At}x_0 + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau \qquad (2)$$

- It is impossible to plot the waveform of $x(t)$ by hand, we need computers to compute $x(t)$ numerically and plot $x(t)$ at many samples of time.
- It is difficult to compute $x(t)$ by following the analytical formulation in (2) if A is very large. We need to solve the LTI system numerically with some numerical methods, like backward Euler, ...etc.
- If the system is very large, then MOR is necessary!

Basic Idea of MOR

Original model (discretized)

$$\Sigma \begin{cases} M(\mu)\dfrac{dx}{dt} = f(\mu, x) + B, \\ y(t) = Cx(t), \end{cases}$$

states $x \in \mathbb{R}^{n}$,

inputs $u(t) \in \mathbb{R}^{m}$

output $y(t) \in \mathbb{R}^{q}$

$$r << n$$

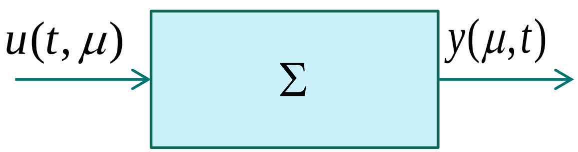
Reduced order model

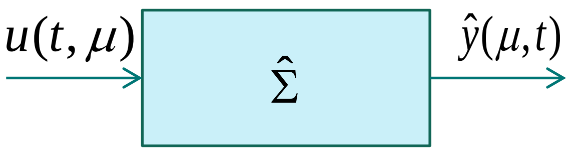
$$\hat{\Sigma} \begin{cases} \hat{M}(\mu)\dfrac{dz}{dt} = \hat{f}(\mu, z) + \hat{B}, \\ \hat{y}(t) = \hat{C}z(t), \end{cases}$$

states $z(t) \in \mathbb{R}^{r}$,

inputs $u(t) \in \mathbb{R}^{m}$

output $\hat{y}(t) \in \mathbb{R}^{q}$





$$\| y - \hat{y} \| < \text{tol} \qquad \forall u(t,\mu)$$

Basic Idea of MOR

Original model → MOR → Reduce order model (ROM)

↓

Replace the original model with the reduced model. The reduced model is then integrated with other components in the device or circuit; or is integrated into a process.

↓

The reduced model is repeatedly used in design analysis.

Basic Idea of MOR

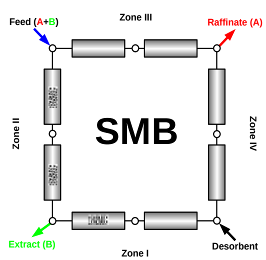
Example: Optimization for SMB

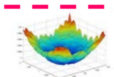
$$\max_{p \in R^5} \; Q_F$$

s.t.

- *SMB model (PDEs)*
- *Cyclic steady state (CSS) constraints*
- *Product purity constraints:* $Pur_A \geq Pur_{A,min}, \;\; Pur_B \geq Pur_{B,min}$
- *Operational constraints on p*

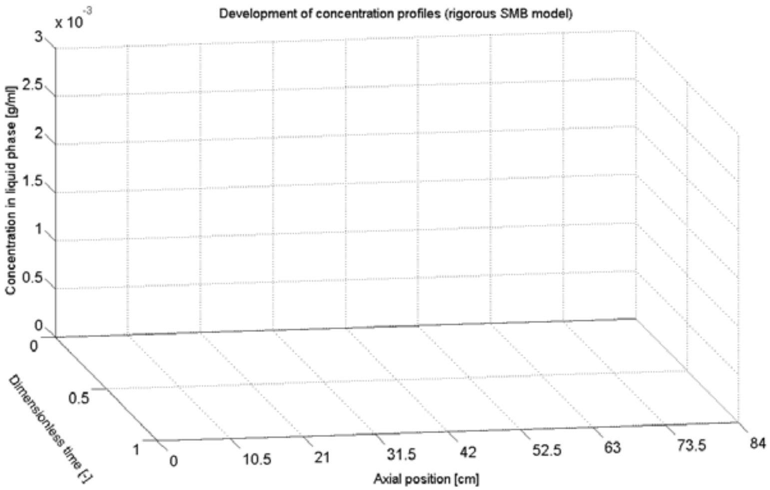
ROMs



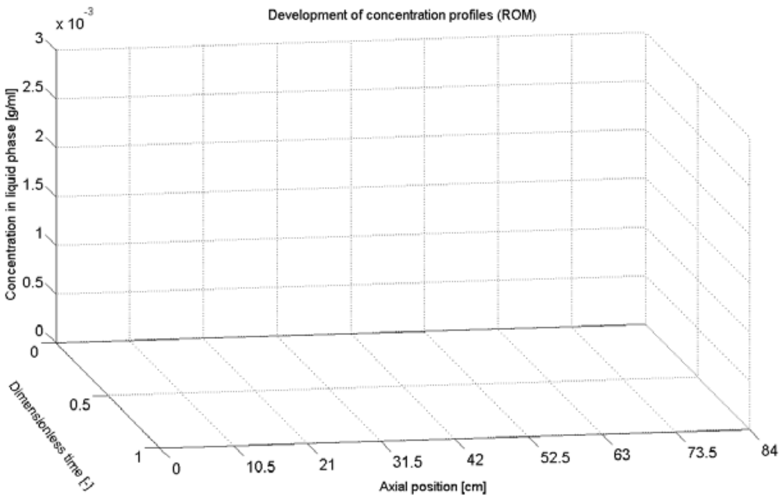


ROM-based optimization

Basic Idea of MOR



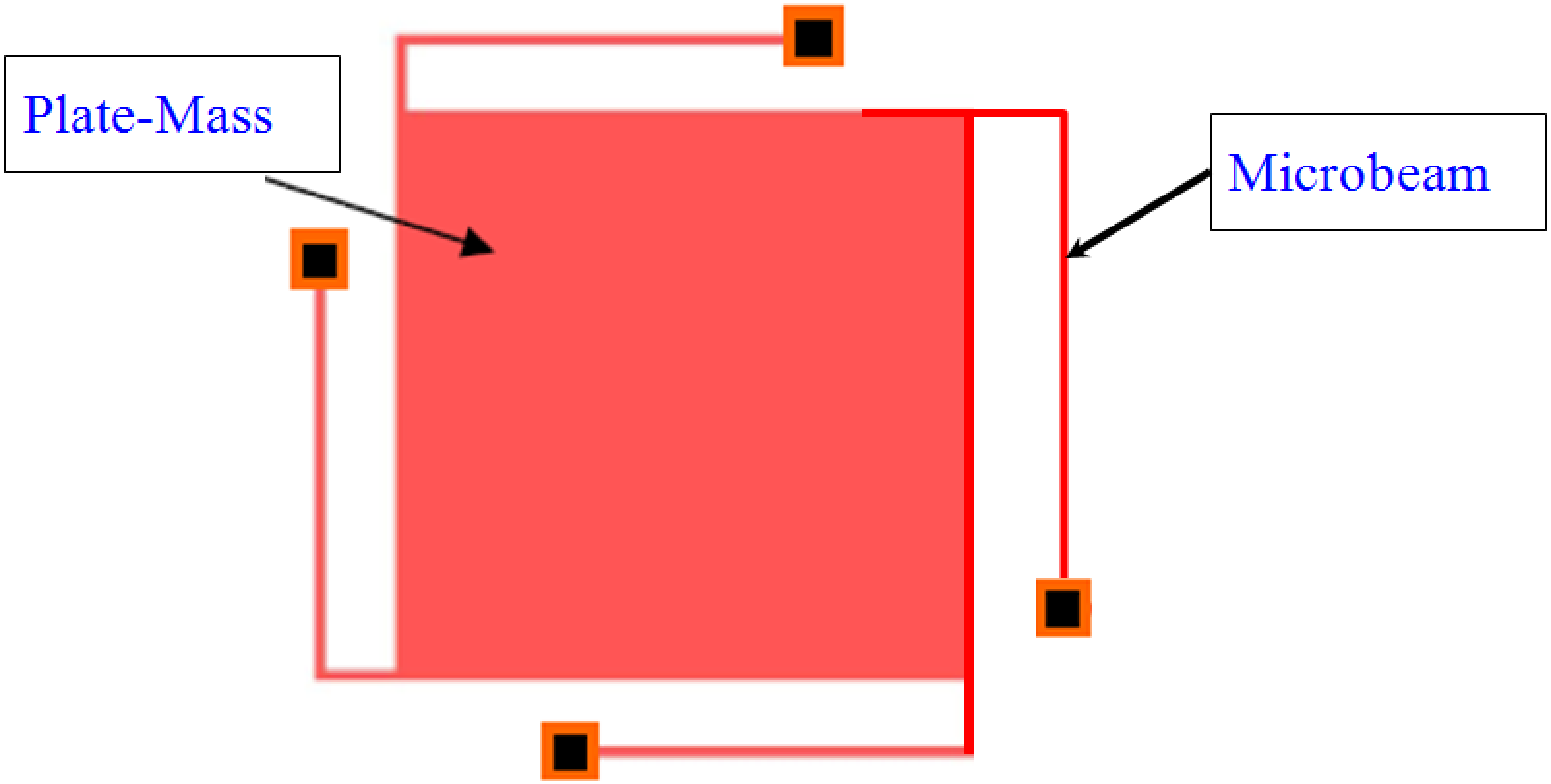
Axial concentration profiles of the full-order DAE model with order of 672.



Axial concentration profiles reproduced by POD-based ROM (with reduced order of only 2).

Basic Idea of MOR

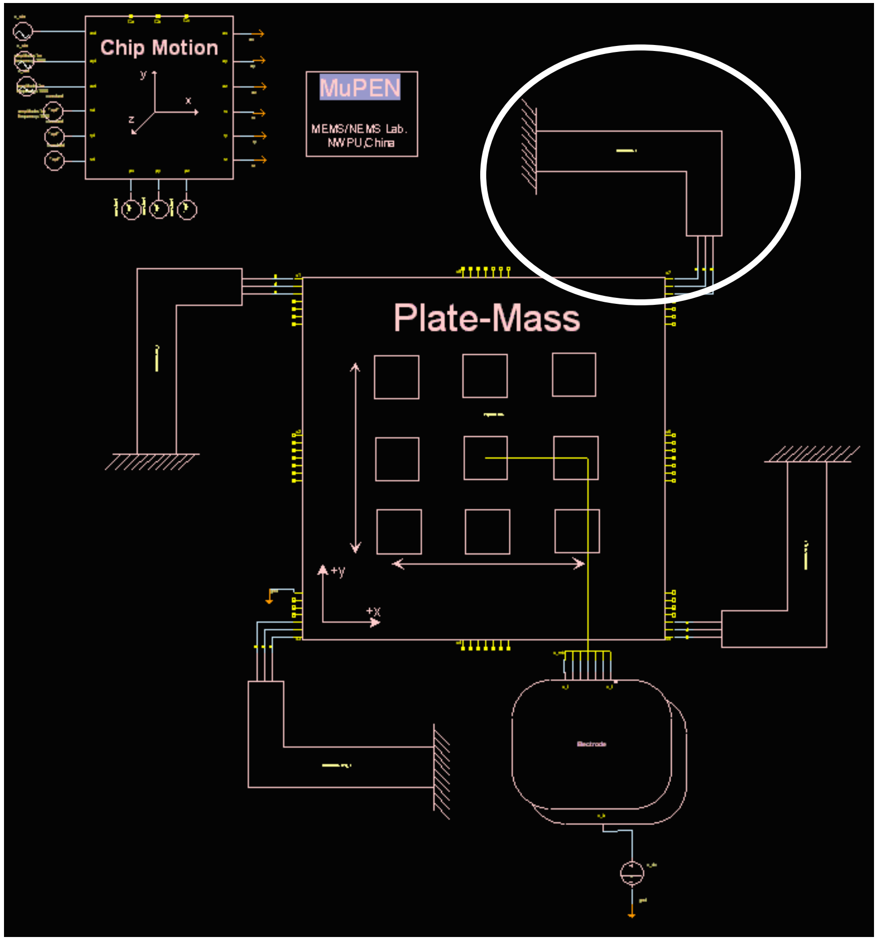
Example: Layout of a switch with four microbeams

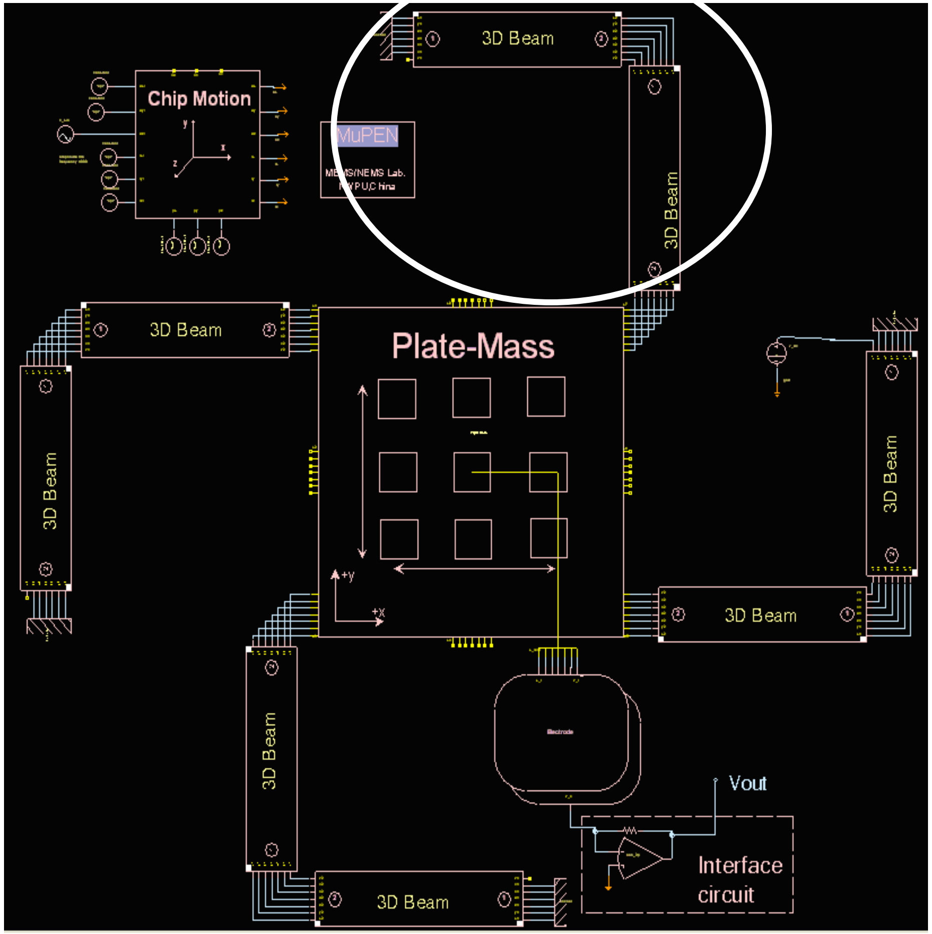


Basic Idea of MOR

The microbeam is replaced by the ROM

The schematic switch

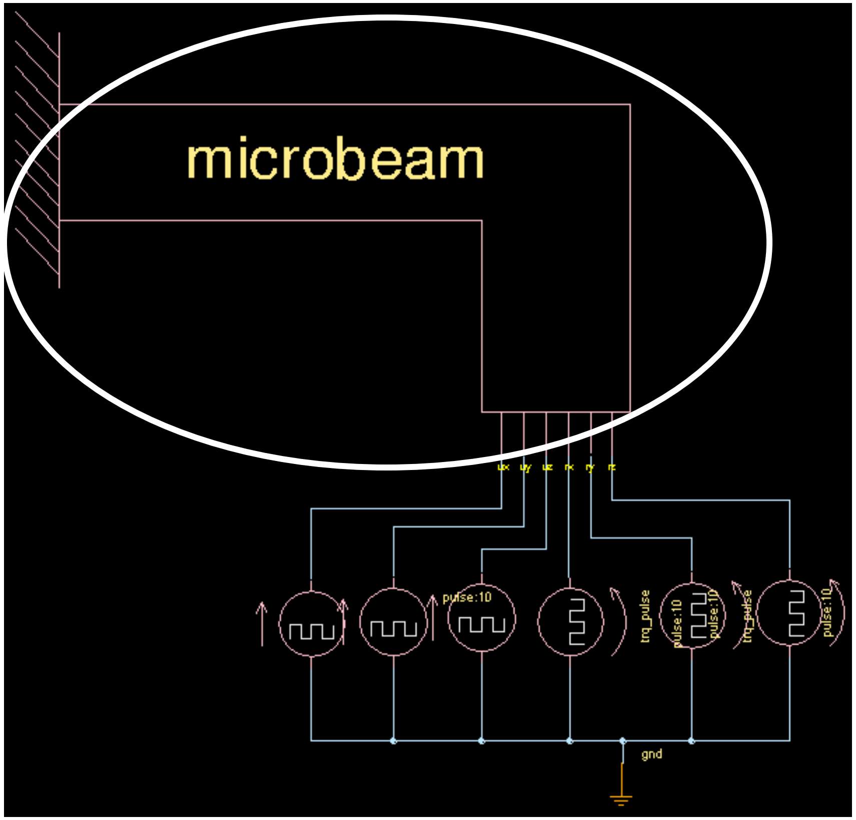


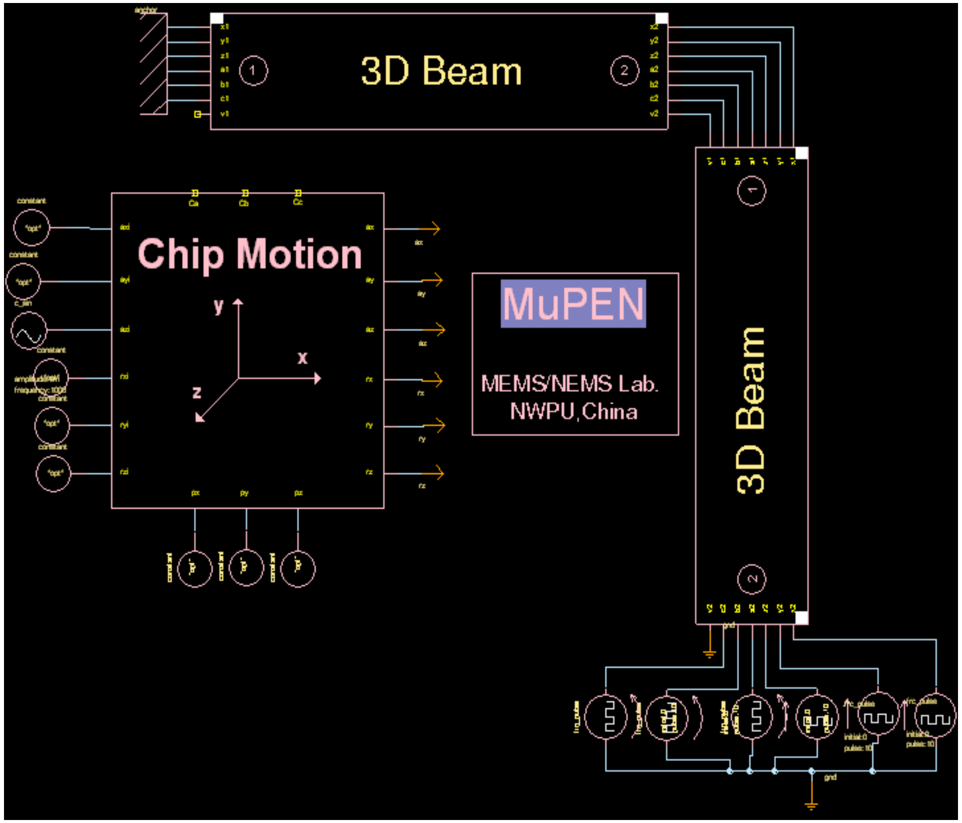


Basic Idea of MOR

ROM of the microbeam

Enlarged microbeam Part





Projection based MOR

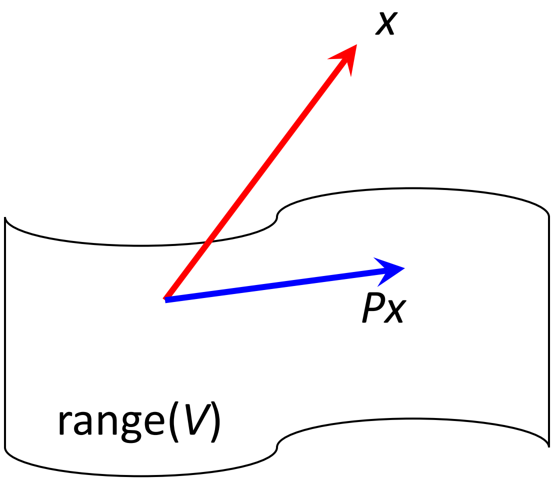
Original model (discretized)

$$\Sigma \begin{cases} M(\mu)\dfrac{dx}{dt} = A(\mu, x) + Bu(t), \\ y(t) = Cx(t), \end{cases}$$

Reduced order model

$$\hat{\Sigma} \begin{cases} \hat{M}(\mu)\dfrac{dz}{dt} = \hat{A}(\mu, z) + \hat{B}u(t), \\ \hat{y}(t) = \hat{C}z(t), \end{cases}$$

Find a subspace which includes the trajectory of *x,* use the projection of x in the subspace to approximate x.



Let: $x \approx Vz$

$$M(\mu)V\frac{dz}{dt} = A(\mu, Vz) + Bu(t) + e,$$
$$\hat{y}(t) = CVz(t),$$

Projection based MOR

Petrov-Galerkin projection: $W := [w_1, \ldots, w_r]$

$$e = 0 \text{ in range}(W) \Leftrightarrow w_i^T e = 0 \text{ for all } i = 1, \ldots, n, \text{ i.e. } W^T e = 0$$

$$\Updownarrow$$

$$W^T M(\mu) V \frac{dz}{dt} = W^T A(\mu, Vz) + W^T Bu(t),$$
$$\hat{y}(t) = CVz(t),$$

$$\hat{M} = W^T MV, \hat{f} = W^T f(\mu, Vz), \hat{B} = W^T B, \hat{C} = CV.$$

Conclusion

The question:

How to construct the ROMs (W, V) for large-scale complex systems?

Answer:

The lecture will provide many solutions.

Outline of the Lecture

Lecture 1: Introduction

Lecture 2-5: Mathematical basics

Lecture 6: Balanced truncation method for linear time invariant systems.

Lecture 7: Moment-matching and rational interpolation methods for linear time invariant systems.

Lecture 8: Krylov subspace based method for nonlinear systems and POD method for nonlinear systems.

Lecture 9: Krylov subspace based method for linear parametric systems.

Lecture 10: POD and reduced basis method for nonlinear parametric systems.

Notice: Lecture slides, excercises, time and location changes can be found at:

https://www.mpi-magdeburg.mpg.de/3668354/mor_ss19