MAX PLANCK INSTITUTE
FOR DYNAMICS OF COMPLEX
TECHNICAL SYSTEMS
MAGDEBURG

COMPUTATIONAL METHODS IN
SYSTEMS AND CONTROL THEORY

Model Reduction for Dynamical Systems

–Lecture 10–

Lihong Feng

Otto-von-Guericke Universitaet Magdeburg
Faculty of Mathematics
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Max Planck Institute for Dynamics of Complex Technical Systems
Computational Methods in Systems and Control Theory
Magdeburg, Germany

benner@mpi-magdeburg.mpg.de feng@mpi-magdeburg.mpg.de
https://www.mpi-magdeburg.mpg.de/3668354/mor_ss19

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$$\begin{aligned} E(\mu)x &= B(\mu), \\ y &= C(\mu)x, \end{aligned} \tag{1}$$

$\mu = (\mu_1, \ldots, \mu_p)$, $x = x(\mu)$.

POD based PMOR for linear parametric steady systems:

1. Specify a group of samples of μ: $\mu^i = (\mu_1^i, \ldots, \mu_p^i)$, $i = 1, \ldots, m$.
2. Solve (1) at each sample μ^i, and get the solution $x(\mu^i)$.
3. Form the snapshot matrix $X = (x(\mu^1), \ldots, x(\mu^m))$.
4. SVD: $X = U \begin{pmatrix} \Sigma & \\ & 0 \end{pmatrix} Q^T$. $\Sigma = \mathrm{diag}(\sigma_1, \ldots, \sigma_d)$, $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_d$.
5. Specify the order r of the ROM by checking, e.g., $\frac{\sum_{j=r+1}^{d} \sigma_j}{\sum_{j=1}^{d} \sigma_j} < tol$, tol is a small number, e.g., 10^{-7}.
6. Take the first r columns from U: $V = U(:, 1:r)$.

Form the ROM:

$$V^T E(\mu) V z = V^T B(\mu), \; y = C(\mu) V z.$$

POD for nonlinear parametric steady systems

Nonlinear Steady systems

$$\begin{aligned} E(\mu)x + f(\mu, x) &= B(\mu), \\ y &= C(\mu)x, \end{aligned} \quad (2)$$

$f(\mu, x)$ is a nonlinear function of the state $x = x(\mu)$.

POD based PMOR for nonlinear parametric steady systems:

- Specify a group of samples: $\mu^i = (\mu_1^i, \ldots, \mu_p^i)$, $i = 1, \ldots, m$.
- Form the snapshot matrix: $X = (x(\mu^1), \ldots, x(\mu^m))$, and $F = (f(\mu^1, x(\mu^1)), \ldots, f(\mu^m, x(\mu^m)))$.
- SVD: $X = U \begin{pmatrix} \Sigma & \\ & 0 \end{pmatrix} Q^T$. $\Sigma = diag(\sigma_1, \ldots, \sigma_d)$, $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_d$.
- Specify the order r of the ROM by, e.g., $\frac{\sum_{j=r+1}^{d} \sigma_j}{\sum_{j=1}^{d} \sigma_j} < tol$.
- Form the projection matrix V: $V = U(:, 1:r)$.

- SVD: $F = U^F \begin{pmatrix} \Sigma^F & \\ & 0 \end{pmatrix} (Q^F)^T$. $\Sigma^F = \mathrm{diag}(\sigma_1^F, \ldots, \sigma_d^F)$, $\sigma_1^F \geq \sigma_2^F \geq \ldots \geq \sigma_d^F$.
- Specify dimension r_F of the interpolation basis by, e.g.,

$$\frac{\sum\limits_{j=r_F+1}^{d} \sigma_j^F}{\sum\limits_{j=1}^{d} \sigma_j^F} < tol.$$

- Form the DEIM interpolation basis: $U_r^F = U^F(:, 1 : r_F)$.
- Use U_r^F as the input for the DEIM algorithm and generate the indecis $\wp_1, \ldots, \wp_l$, and the index matrix $P = (e_{\wp_1}, \ldots, e_{\wp_l})$.
- Form the ROM:

$$\begin{aligned} V^T E(\mu) V z &+ V^T U_r^F (P^T U_r^F)^{-1} P^T f(\mu, Vz) \\ &= V^T B(\mu), \\ y &= C(\mu) V z. \end{aligned}$$

POD for linear parametric dynamical systems

Linear dynamical systems

$$\begin{aligned} E(\mu)dx/dt &= A(\mu)x + B(\mu)u(t), \\ y &= C(\mu)x. \end{aligned} \quad (3)$$

POD based PMOR for linear parametric dynamical systems:

- Specify samples of μ and $u(t)$: $w^i = (\mu^i, u^i(t))$, $i = 1, \ldots, m$, $\mu^i = (\mu_1^i, \ldots, \mu_p^i)$.
- Solve (3) at each sample w^i, and get the solution $X(w^i) := [x_{t_1}(w^i), \ldots, x_{t_N}(w^i)]$.
- SVD of $X(w^i)$: $X(w^i) = U^i \begin{pmatrix} \Sigma^i & \\ & 0 \end{pmatrix} (Q^i)^T$. $\Sigma^i = diag(\sigma_1^i, \ldots, \sigma_d^i)$, $i = 1, \ldots, m$, $\sigma_1^i \geq \sigma_2^i \geq \ldots \geq \sigma_d^i$.
- Determine the order r_i for each i by e.g.

$$\left(\sum_{j=r_i+1}^{d} \sigma_j^i\right) / \left(\sum_{j=1}^{d} \sigma_j^i\right) < tol.$$

- Let: $V_i = U^i(:, 1 : r_i)$, $i = 1, \ldots, m$.

- Form $X = (V_1, \ldots, V_m) \in \mathbb{R}^{n \times \tilde{N}}$, $\tilde{N} = \sum_{i=1}^{m} r_i$.
- SVD: $X = U \begin{pmatrix} \Sigma & \\ & 0 \end{pmatrix} Q^T$, $\Sigma = \mathrm{diag}(\sigma_1, \ldots, \sigma_d)$.
- Specify the order r of the ROM by e.g. $\dfrac{\sum_{j=r+1}^{d} \sigma_j}{\sum_{j=1}^{d} \sigma_j} < tol$.
- Form the projection matrix: $V = U(:, 1:r)$.
- Form the ROM:

$$
\begin{aligned}
V^T E(\mu) V dz/dt &= V^T A(\mu) V z + V^T B(\mu) u(t), \\
\hat{y} &= C(\mu) V z.
\end{aligned}
$$

Nonlinear parametric dynamical systems

$$\begin{aligned} E(\mu)dx/dt &= A(\mu) + f(\mu, x) + B(\mu)u(t), \\ y &= C(\mu)x. \end{aligned} \qquad (4)$$

POD based PMOR for nonlinear parametric dynamical systems:

- 1. Specify samples of μ and $u(t)$: $w^i = (\mu^i, u^i(t))$, $i = 1, \ldots, m$, $\mu^i = (\mu^i_1, \ldots, \mu^i_p)$.
- 2. Solve (4) at each sample w^i, and get the solution $X(w^i) := [x_{t_1}(w^i), \ldots, x_{t_N}(w^i)]$.
- 3. Form the snapshots of f:

$$F = [f(X(w^1), \ldots, f(X(w^m))] \in \mathbb{R}^{n \times N \cdot m},$$

  where $f(X(w^i)) := [f(x_{t_1}(w^i)), \ldots, f(x_{t_N}(w^i))]$.
- 4. SVD of $X(w^i)$: $X(w^i) = U^i \begin{pmatrix} \Sigma^i & \\ & 0 \end{pmatrix} (Q^i)^T$. $\Sigma^i = \mathrm{diag}(\sigma^i_1, \ldots, \sigma^i_d)$, $i = 1, \ldots, m$.

POD for nonlinear parametric dynamical systems

- 5. Determine the order r_i for each i by, e.g., $\frac{\sum_{j=r_i+1}^{d} \sigma_j^s}{\sum_{j=1}^{d} \sigma_j^i} < tol$, let: $V_i = U^i(:, 1 : r_i)$, $i = 1, \ldots, m$.
- 6. Form $X = (V_1, \ldots, V_m) \in \mathbb{R}^{n \times \tilde{N}}$, $\tilde{N} = \sum_{i=1}^{m} r_i$.
- 7. SVD: $X = U \begin{pmatrix} \Sigma \\ & 0 \end{pmatrix} Q^T$. $\Sigma = \mathrm{diag}(\sigma_1, \ldots, \sigma_d)$.
- 8. Specify the order r of the ROM by e.g. $\frac{\sum_{j=r+1}^{d} \sigma_j}{\sum_{j=1}^{d} \sigma_j} < tol$.
- 9. Form the projection matrix: $V = U(:, 1 : r)$.

- 10. SVD: : $F = U^F \begin{pmatrix} \Sigma^F & \\ & 0 \end{pmatrix} (Q^F)^T$. $\Sigma^F = \mathrm{diag}(\sigma_1^F, \dots, \sigma_d^F)$.
- 11. Specify the dimension r of the basis for interpolation by, e.g.,

$$\frac{\sum_{j=r+1}^{d} \sigma_j^F}{\sum_{j=1}^{d} \sigma_j^F} < tol.$$

- 12. Form the DEIM interpolation basis: $U_f = U^F(:, 1:r)$.
- 13. Use U_f as the input for DEIM algorithm and generate the indices $\wp_1, \dots, \wp_l$, and the index matrix $P = (e_{\wp_1}, \dots, e_{\wp_l})$.
- 14. Form the ROM:

$$\begin{aligned} V^T E(\mu) V dz/dt &= V^T A(\mu) V z \\ &\quad + V^T U_f (P^T U_f)^{-1} P^T f(\mu, Vz) \\ &\quad + V^T B(\mu) u(t), \\ \hat{y} &= C(\mu) V z. \end{aligned}$$

Issues under active research

- Adaptive (optimal) sampling of the parameters $\mu_1, \ldots, \mu_l$, and the inputs $u(t)$.
- When and how to update the DEIM interpolation?
- Error estimation/bound.

RB method for linear parametric steady systems

1. Specify a training set of samples: $\mathcal{P}_{train} := \{\mu^1, \ldots, \mu^m\}$.
2. Specify an error estimator/indicator/bound $\eta(\mu)$.
3. Use Greedy algorithm to compute the projection matrix V.
4. Form the ROM:

$$\begin{aligned} V^T E(\mu) V z &= V^T B(\mu), \\ y &= C(\mu) V z. \end{aligned}$$

Compute V for steady systems

Algorithm 1 Greedy Algorithm for steady systems

Input: $\mathcal{P}_{train}, tol_{RB}$

Output: Projection matrix V representing the reduced basis.

1: Initialization: $V = [\,]$, $\mu^\star = \mu^1$, $\eta(\mu^\star)(> tol_{RB})$
2: **while** the error $\eta(\mu^\star) > tol_{RB}$ **do**
3: Solve (1) or (2) to get the solution $x(\mu^\star)$.
4: Update: $V :=$orth $[V, x(\mu^\star)]$.
5: Find $\mu^\star := \arg \max_{\mu \in \mathcal{P}_{train}} \eta(\mu)$.
6: **end while**

RB method for nonlinear parametric steady systems

1. Specify $\mathcal{P}_{train} := \{\mu^1, \dots, \mu^m\}$.
2. Specify an error estimator/indicator/bound $\eta(\mu)$.
3. Use Greedy algorithm to compute the projection matrix V.
4. Compute the EIM basis $U_{\mathbf{f}}$ using the empirical interpolation method (EIM).
5. Form the ROM:

$$\begin{aligned} V^T E(\mu) V z + V^T U_{\mathbf{f}} \beta(\mu) &= V^T B(\mu), \\ y &= C(\mu) V z. \end{aligned}$$

Key points:

- EIM.
- $\beta(\mu)$ must be updated for each sample of μ when solving the ROM!

RB method for linear parametric dynamical systems

1. Specify $\mathcal{P}_{train} := \{w^1, \ldots, w^m\}$, $w^i = (\mu^i, u^i(t))$.
2. Specify an error estimator/indicator/bound $\eta(w)$.
3. Use POD-Greedy algorithm to compute the projection matrix V.
4. Form the ROM:

$$
\begin{aligned}
V^T E(\mu) V dz/dt &= V^T A(\mu) V z + V^T B(\mu) u(t), \\
y &= C(\mu) V z.
\end{aligned}
$$

Compute V for dynamical systems

Algorithm 2 POD-Greedy algorithm for dynamical systems

Input: $\mathcal{P}_{train}$, $tol_{RB}(<1)$
Output: Projection matrix: $V = [V_1, \dots, V_m]$
1: Initialization: $s = 0$, $V = [\,]$, $w^\star = (\mu^1, u^1(t))$, $\eta_s(w^\star) = 1$.
2: **while** the error $\eta(w^\star) > tol_{RB}$ **do**
3: Compute the trajectory $X = [x(w^\star, t_1), \dots, x(w^\star, t_N)]$.
4: $\mathcal{W} = \mathrm{colspan}\{V\}$.
5: IF $V \neq [\,]$ do
For each t_k, compute $x(w^\star, t_k) := x(w^\star, t_k) - \mathrm{Proj}_{\mathcal{W}}[x(w^\star, t_k)]$. (Orthorgonalize the vector $x(w^\star, t_k)$ with the orthogonal vectors in V.)
ENDIF
6: Update X and do SVD: $X = U\Sigma V^T$, $V_{s+1} := U(:, 1)$ (only take the first POD basis).
7: Update: $V = [V, V_{s+1}]$,
8: $s = s + 1$.
9: Find $w^\star := \arg\max_{w \in \mathcal{P}_{train}} \eta(w)$.
10: **end while**

RB method for nonlinear parametric dynamical systems

1. Specify $\mathcal{P}_{train} := \{w^1, \ldots, w^m\}$, $w^i = (\mu^i, u^i(t))$.
2. Specify an error estimator/indicator/bound $\eta(w)$.
3. Use POD-Greedy algorithm to compute the projection matrix V.
4. Compute the EIM basis $U_{\mathbf{f}}$ using the empirical interpolation method (EIM).
5. Form the ROM:

$$\begin{aligned} V^T E(\mu) V dz/dt &= V^T A(\mu) V z + V^T U_{\mathbf{f}} \beta(w,t) + V^T B(\mu) u(t,) \\ y &= C(\mu) V z. \end{aligned}$$

Key points:

- EIM.
- $\beta(w,t)$ must be updated for each value of μ, and each $u(t)$, at each time step t_i, when solving the ROM!
- Efficient error estimation.

Algorithm 3 Algorithm 3: EIM for steady systems

Input: Snapshots of $f(\mu, x(\mu))$:

$$F = \{f(\mu^1, x(\mu^1)), \dots, f(\mu^m, x(\mu^m))\}, f^k := f(\mu^k, x(\mu^k)),$$

$P = [\ ], U = [\ ]$.

Output: EI basis: $U = [u_1, \dots, u_M]$ and

Indices: $\{\wp_1, \dots, \wp_M\}$, index matrix: $P = [e_{\wp_1}, \dots, e_{\wp_M}]$.

1: $s = 1$, $\xi_1 := \arg\max_{f^k \in F} \|f^k\|_2$, $\wp_1 := \arg\max_{j \in \{1,\dots,n\}} |\xi_{1,j}|$, $u_1 := \xi_1 / \xi_{1,\wp_1}$. $P = [P, e_{\wp_1}], U = [U, u_1]$.

Empirical Interpolation Method (EIM)

(continued) Algorithm 3: EIM for steady systems

1: **while** $\|\xi_s\|_2 > \epsilon$ **do**

2: $s := s + 1$. For each f^k, $k = 1, \dots, m$: $f^k \approx \sum_{i=1}^{s-1} \beta_{ki} u_i$, i.e.

$$f^k \approx U\beta_k, \beta_k = (\beta_{k1}, \dots, \beta_{ks-1})^T$$

β_k is determined by interpolation:

$$P^T f^k = P^T U \beta_k,$$

so that $\beta_k = (P^T U)^{-1} P^T f^k$.

3: Let $f^{k^\star} := \arg\max_{f^k \in F} \|f^k - U\beta_k\|_2$, $\xi_s = f^{k^\star} - U\beta_{k^\star}$

4: **if** $\|\xi_s\|_2 \leq \epsilon$ **then**

5: Stop and set $M = s - 1$.

6: **else**

7: Define $\wp_s := \arg \max_{j \in \{1,\dots,n\}} |\xi_{s,j}|$, $u_s := \xi_s / \xi_{s,\wp_s}$.

8: $P = [P, e_{\wp_s}]$, $U = [U, u_s]$.

9: **end if**

10: **end while**

Application of EI to ROM simulation

Simulating the ROM:

$$\begin{aligned} V^T E(\mu) V z + V^T U_{\mathbf{f}} \beta(\mu) &= V^T B(\mu), \\ y &= C(\mu) V z. \end{aligned}$$

- $U_{\mathbf{f}} = U$, U is from Algorithm 3 (EIM). $f(\mu, Vz(\mu)) \approx U\beta(\mu)$, $\forall \mu$.
- Using the index matrix P from Algorithm 1,

$$P^T f(\mu, Vz(\mu)) = P^T U \beta(\mu), \forall \mu. (\text{Interpolation})$$

$$\beta(\mu) = (P^T U)^{-1} P^T f(\mu, Vz(\mu)), \forall \mu.$$

For each value of $\mu = (p_1, \ldots, p_l)$, using Newton's method to solve

$$\hat{g}(\mu, Vz(\mu)) = \hat{B}(\mu),$$

where $\hat{g}(\mu, Vz(\mu)) = V^T E(\mu) V z + \underbrace{V^T U (P^T U)^{-1}}_{\text{precomputed}} P^T f(\mu, Vz(\mu))$, $\hat{B}(\mu) = V^T B(\mu)$.

Algorithm 4: EIM for dynamical systems

Input: Snapshots of $f(w, x(t, w))$ (note that f depends not only on μ, but also on the input $u(t)$.):

$$F = [f(w^1, x(t_1, w^1)), \ldots, f(w^1, x(t_N, w^1)) \ldots, f(w^m, x(t_N, w^m))],$$

$w^i = (\mu^i, u^i(t))$, $f^k := F(:, k), P = [\ ], U = [\ ]$.

Output: EI basis: $U = [u_1, \ldots, u_M]$ and

Indices: $\{\wp_1, \ldots, \wp_M\}$, index matrix P: $P = [e_{\wp_1}, \ldots, \wp_M]$.

1. $s = 1$, $\xi_1 := \arg \max\limits_{f^k, 1 \le k \le N \cdot m} \|f^k\|_2$,

$\wp_1 := \arg \max\limits_{j \in \{1, \ldots, n\}} |\xi_{1,j}|$, $u_1 := \xi_1 / \xi_{1,\wp_1}$. $P = [P, e_{\wp_1}], U = [U, u_1]$.

Empirical Interpolation Method (EIM)

(Continued) Algorithm 4: EIM for dynamical systems

while $\|\xi_m\|_2 > \epsilon$ **do**

$s := s + 1$. For each $\mathbf{f}^k, k = 1, \dots, N \cdot m : f^k \approx \sum_{i=1}^{s-1} c_{ki} u_i$, i.e.

$$f^k \approx U c_k, c_k = (c_{k1}, \dots, c_{ks-1})^T$$

c_k is determined by interpolation:

$$P^T f^k = P^T U c_k,$$

so that $c_k = (P^T U)^{-1} P^T f^k$.

Let $f^{k^\star} := \arg \max_{f^k, 1 \le k \le N \cdot m} \|f^k - U c_k\|$. $\xi_s = f^{k^\star} - U c_{k^\star}$

if $\|\xi_s\| \le \epsilon$ **then**

Stop and set $M = s - 1$.

else

Define $\wp_s := \arg \max_{j \in \{1, \dots, n\}} |\xi_{s,j}|, u_s := \xi_s / \xi_{s, \wp_s}$.

end if

end while

Simulating the ROM:

$$\begin{aligned} V^T E(\mu) V dz/dt &= V^T A(\mu) V z + V^T U_{\mathsf{f}} \beta(w,t) \\ &+ V^T B(\mu), \\ y &= C(\mu) V z. \end{aligned}$$

- $U_{\mathsf{f}} = U$, U is from Algorithm 4. $f(\mu, Vz(\mu,t)) \approx U\beta(w,t)$, $\forall \mu$.
- Using the index matrix P from Algorithm 4,

$$P^T f(\mu, Vz(\mu,t)) = P^T U \beta(w,t), \forall \mu, \forall t. (\text{Interpolation})$$

$$\beta(w,t) = (P^T U)^{-1} P^T f(\mu, Vz(\mu,t)), \forall \mu.$$

For each value of μ, each value of input $u(t)$, and at each time step t_i, compute:

$$\underbrace{V^T U (P^T U)^{-1}}_{\text{precomputed}} P^T f(\mu, Vz(\mu, t_i)),$$

to get $V^T U_{\mathsf{f}} \beta(w, t_i)$ in the ROM.

Issues under ative research

- Optimal/adaptive defining/updating the trainning set: $\mathcal{P}_{train}$.
- Adaptive adjustment of the EIM basis and the reduced basis.
- More efficient error estimation/bound.

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And many more ....