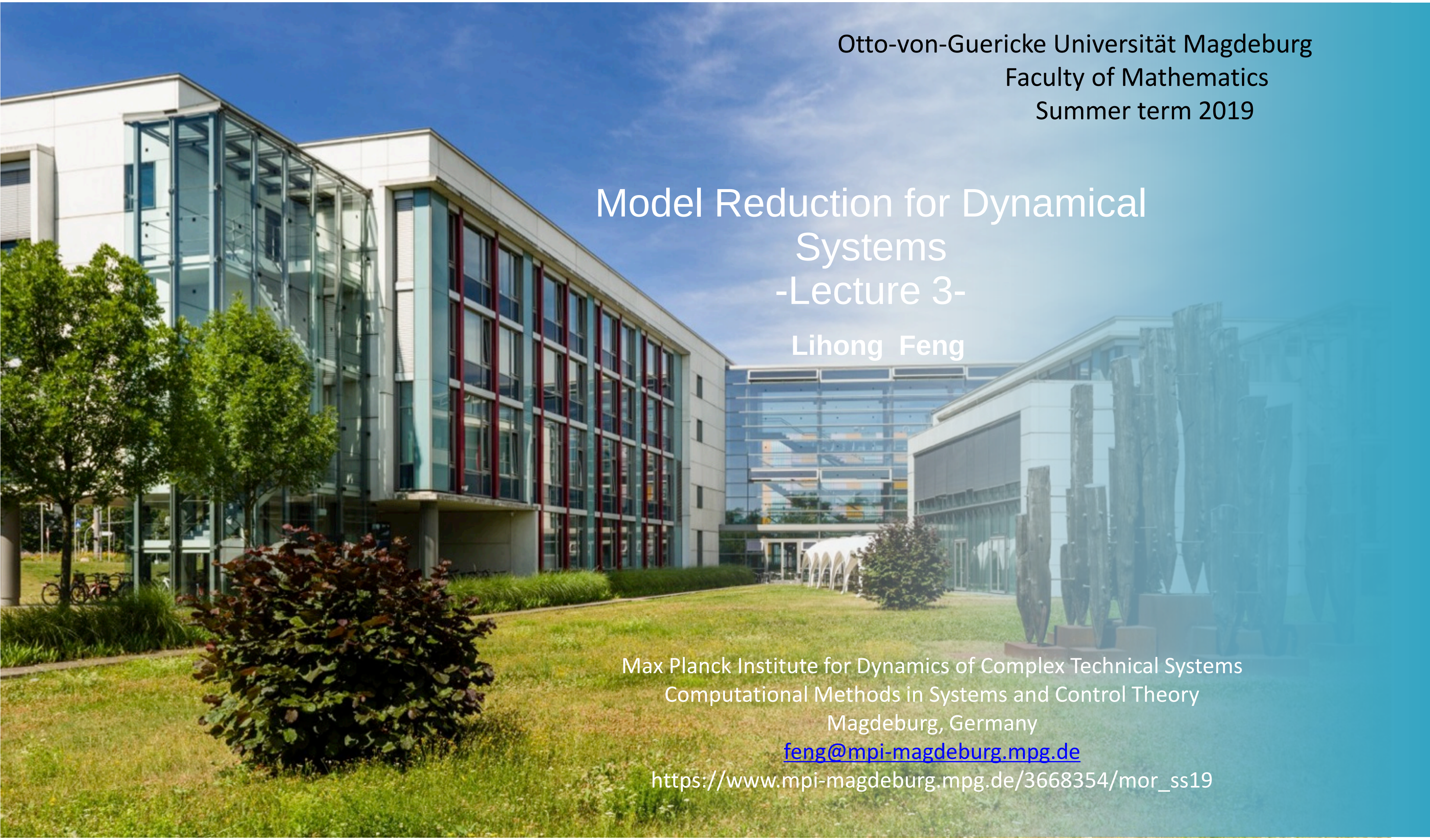
MAX PLANCK INSTITUTE FOR DYNAMICS OF COMPLEX TECHNICAL SYSTEMS MAGDEBURG

CSC COMPUTATIONAL METHODS IN SYSTEMS AND CONTROL THEORY



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Model Reduction for Dynamical Systems
-Lecture 3-

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Outline

- Mathematical basics II
  Systems and control theory

- Controllability measures

- Observability measures

- Infinite Gramians

Motivation

Balanced truncation: first balancing, then truncate.

Given a LTI system:

$$dx(t)/dt = Ax(t) + Bu(t)$$

$$y(t) = L^T x(t)$$

For convenience of discussion, we denote the system as a block form:

$$\left(\begin{array}{c|c} A & B \\ \hline L^T & \end{array}\right) \xrightarrow{\text{Balancing } T} \left(\begin{array}{c|c} \tilde{A} & \tilde{B} \\ \hline \tilde{L}^T & \end{array}\right) = \left(\begin{array}{cc|c} \tilde{A}_{11} & \tilde{A}_{12} & \tilde{B}_1 \\ \tilde{A}_{21} & \tilde{A}_{22} & \tilde{B}_2 \\ \hline \tilde{L}_1^T & \tilde{L}_2^T & \end{array}\right)$$

truncate

$$\left(\begin{array}{c|c} \tilde{A}_{11} & \tilde{B}_1 \\ \hline \tilde{L}_1^T & \end{array}\right) \rightarrow \text{reduced model}$$

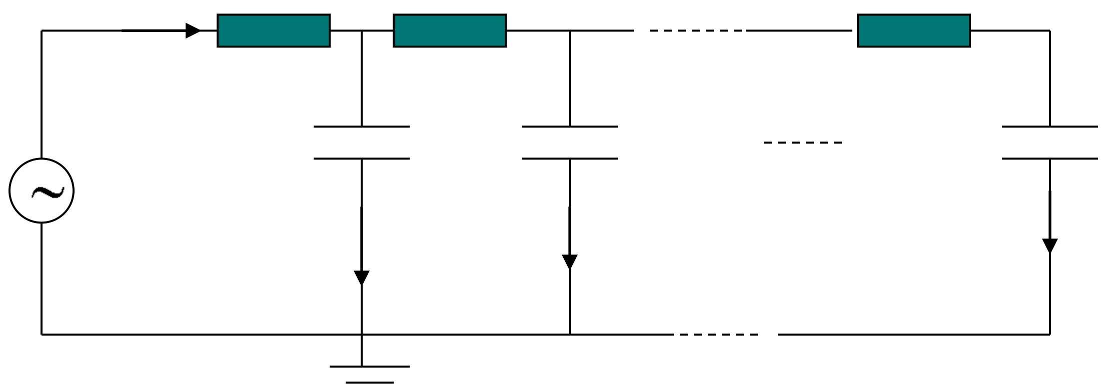
The unimportant part is truncated

Motivation

What's the unimportant part?

The states which are difficult to control and difficult to observe correspond the unimportant part.

In system theory, the unknown vector *x* is called the state (vector) of the system. Actually, the entries in *x* depict the system variables, such as branch currents, node voltages in the interconnect model, and therefore describe the state of the system.



Outline

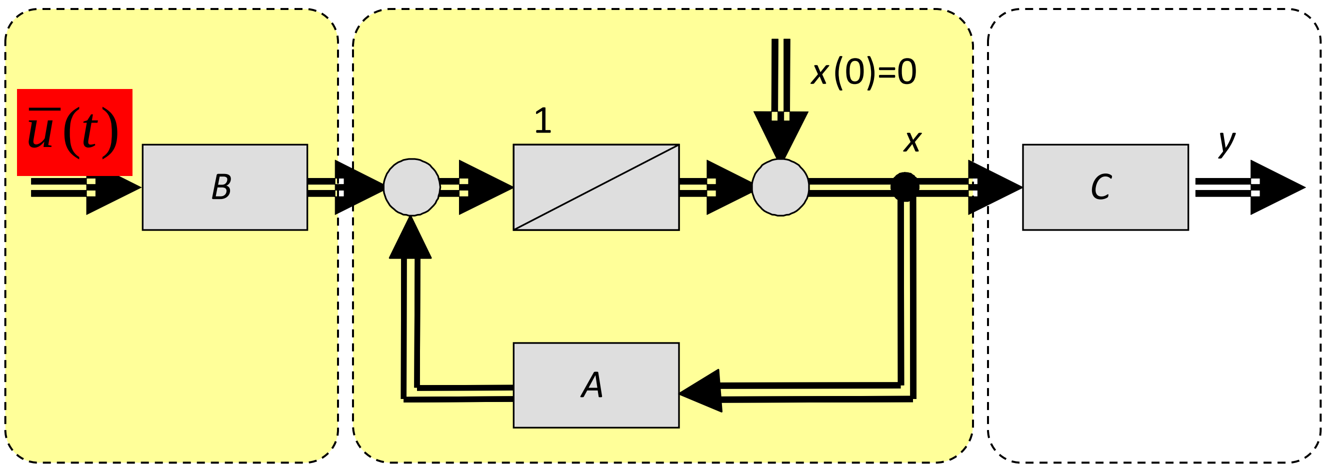
- Controllability measures
- Observability measures
- Infinite Gramians

Controllability measure

Reachability

Definition: Given a system $\left(\begin{array}{c|c} A & B \\ \hline L^T & \end{array}\right)$, a state *x* is reachable from the zero state

if there exist an input function $\overline{u}(t)$ of finite energy such that *x* can be obtain from the zero state and within a finite period of time $\bar{t} < \infty$.



$\bar{t} < \infty$

Controllability measure

Denote X^{reach} the subspace spanned by the reachable states, then

$$X^{reach} \subseteq X$$

X is the whole state space, e.g.

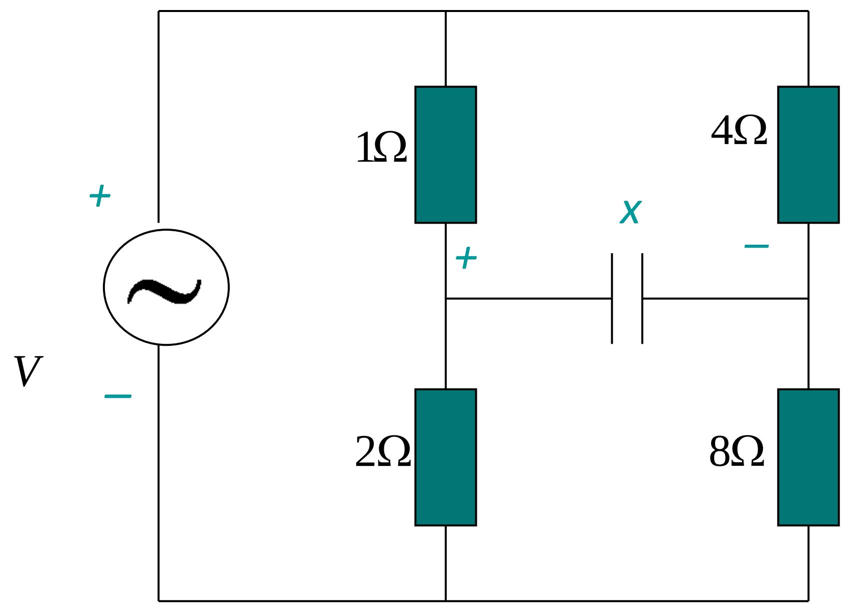
$$X = \{x(t) : R_+ \rightarrow C^n\}$$

The system is reachable $\Longleftrightarrow$ $X^{reach} = X$: every state in the state space is reachable.

Controllability measure

Example 1

Picture referred to [Chi-Tsong Chen, Linear system Theory and Design, 3rd edition, New York Oxford, Oxford University Press, 1999]



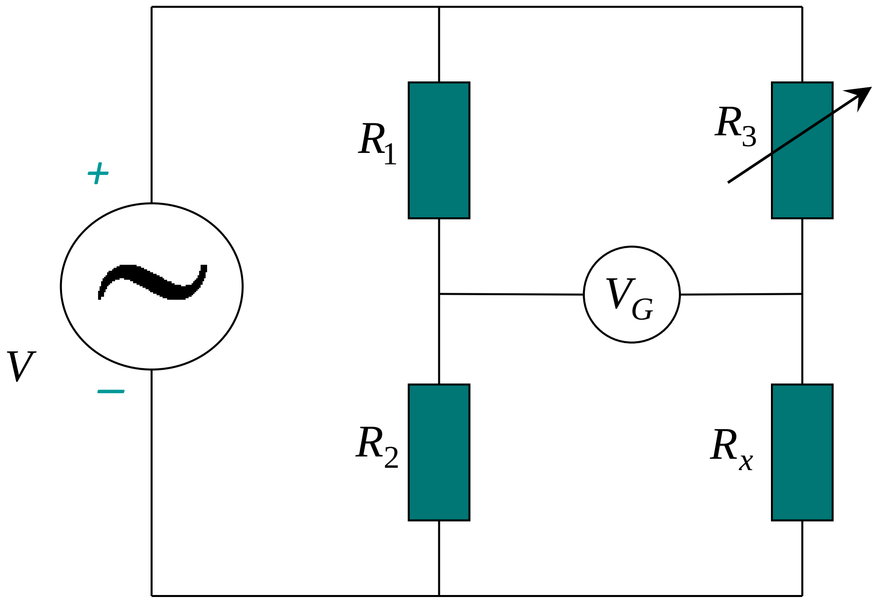
x denotes the voltage drop along the capacitor, and is the state of the system. In this circuit, x=0 at any time.

Conclusion:

In this circuit, 0 state is a reachable state, but any nonzero state is a unreachable state! Therefore the whole system is unreachable.

Controllability measure

Example 1 is actually the Wheatstone bridge.



Wheatstone bridge

R_3 is adjustable, it is adjusted till V_G becomes zero. It means there is no voltage drop through V_G .

Therefore, we have

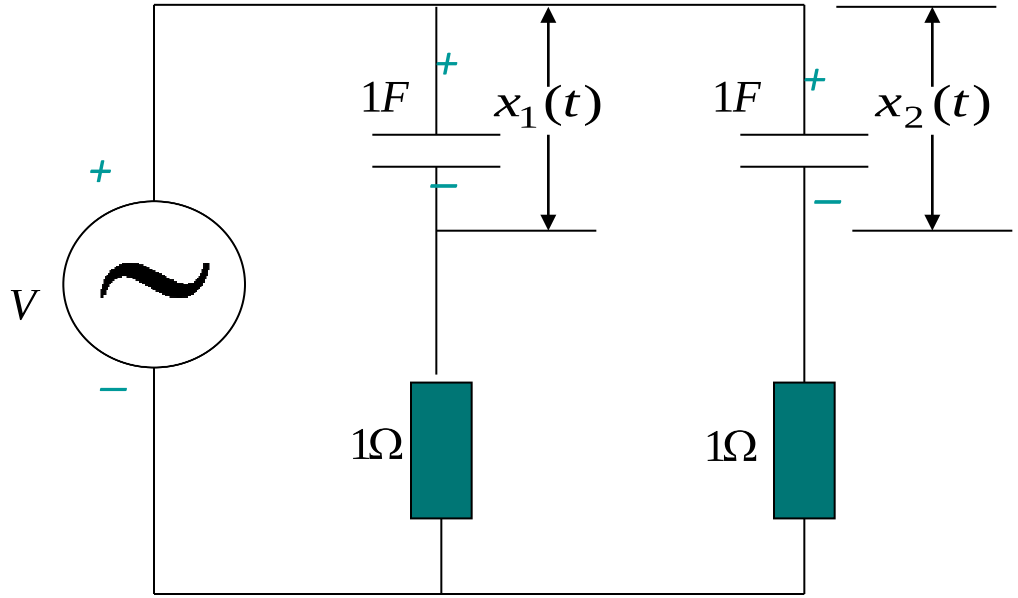
$$\frac{R_2}{R_1} = \frac{R_x}{R_3}$$

The value of R_x can be easily computed by the above relation.

A **Wheatstone bridge** is a measuring instrument invented by Samuel Hunter Christie in 1833 and improved and popularized by Sir Charles Wheatstone in 1843. (http://en.wikipedia.org/wiki/Wheatstone_bridge)

Controllability measure

Example 2



$$dx(t)/dt = Ax(t) + Bu(t)$$

$$y(t) = L^T x(t)$$

$$x(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$$

voltage drops through the two capacitors.

Those states $x(t)$ with $x_1(t) = x_2(t)$ are reachable, but those states with $x_1(t) \neq x_2(t)$ are not reachable. Because whatever the input is, the voltage drops through the two capacitors are always identical. Therefore the whole system is unreachable.

Controllability measure

Example 3

$$dx(t)/dt = Ax(t) + Bu(t)$$

$$y_1(t) = L^T x(t)$$

$$\tilde{A} = \begin{pmatrix} A & 0 \\ 0 & A_2 \end{pmatrix}, \ \tilde{x}(t) = \begin{pmatrix} x(t) \\ x_2(t) \end{pmatrix}, \ \tilde{B} = \begin{pmatrix} B \\ 0 \end{pmatrix}, \ \tilde{L} = \begin{pmatrix} L \\ L_2 \end{pmatrix}$$

The state variables in the vector $x_2(t)$ are unreachable, since no input u(t) exists to reach $x_2(t)$.

Controllability measure

For a standard LTI system, the reachability matrix is defined as:

$$R(A,B)=[B, AB, A^2 B \cdots A^{n-1} B \cdots]$$

- Why it is called reachability matrix?
- Any connection between $R(A,B)$ and reachability?

Controllability measure

Notice the analytical solution of system state equation $dx/dt = Ax + Bu$ is

$$x(u, x_0, t) = e^{At}x_0 + \int_{t_0}^{t} e^{A(t-\tau)}Bu(\tau)d\tau, t \geq t_0,$$

The reachability of a state *x* of the system is tested by the zero initial state, $x_0 = 0$, we look at the above analytical solution with $x_0 = 0$,

$$x(u,0,t) = \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau$$

Notice:

$$e^{At} = I_n + \frac{t}{1!}A + \frac{t^2}{2!}A^2 + \cdots + \frac{t^k}{k!}A^k + \cdots$$

Controllability measure

$$x(u,0,t)=\int_0^t e^{A(t-\tau)}Bu(\tau)d\tau=\int_0^t (B+(t-\tau)AB+\frac{(t-\tau)^2}{2!}A^2B+\ldots)u(\tau)d\tau$$

$$=B\int_0^t u(\tau)d\tau+AB\int_0^t (t-\tau)u(\tau)d\tau+A^2B\int_0^t \frac{(t-\tau)^2}{2!}u(\tau)d\tau$$

$$=B\alpha_0(t)+AB\alpha_1(t)+A^2B\alpha_2(t)+\cdots+A^kB\alpha_k(t)+\cdots,$$

which means a reachable state *x* is the linear combination of the terms:

$$B, AB, A^2B,\cdots, A^kB,\cdots$$

Therefore $R(A,B)=(B, AB, A^2B\cdots, A^{n-1}B,\cdots)$ is defined as the reachability Matrix.

Controllability measure

By the Cayley-Halmilton theorem, the rank of the reachability matrix and the span of its columns are determined (at most) by the first *n* terms (not the first *n* columns), i.e. $A^t B, t = 0, 1, 2, \cdots, n-1.$

Thus for computational purpose the following (finite) reachability matrix is of importance:

$$R_n(A, B) = [B, AB, A^2 B \cdots A^{n-1} B]$$

Sometimes $R_n(A, B)$ is directly defined as the reachability matrix.

Controllability measure

Actually there is a Theorem (Theorem 4.5 in Chapter 4 in [Antoulas05]):

Theorem 1 If X^{reach} is the subspace spanned by the reachable states, then

$$X^{reach} = \mathrm{im}\, R(A,B): \text{subspace spanned by the columns of } R(A,B).$$

The theorem tells us the subspace spanned by all reachable states is exactly the subspace spanned by the columns of the reachability matrix $R(A,B)$.

The finite reachability gramian at time $t < \infty$ is defined as :

$$P(t) = \int_0^t e^{A\tau} BB^T e^{A^T\tau} d\tau, \quad for \quad 0 < t < \infty$$

Controllability measure

Connection between reachability matrix and reachability gramians

Proposition 1 The finite reachability gramians have the following properties: (a) $P(t) = P^T(t) \geq 0,$ and (b) their columns span the reachability subspace, i.e., $\text{im}\, P(t) = \text{im}\, R(A, B).$ (Proposition 4.8 in [Antoulas 05])

Proof An easier way is to prove $\text{im}\, P^{\oplus}(t) = \text{im}\, R^{\oplus}(A, B),$ where

$$\text{im}\, P^{\oplus}(t) \oplus \text{im}\, P(t) = C^n \quad \text{and} \quad \text{im}\, R^{\oplus}(A, B) \oplus \text{im}\, R(A, B) = C^n$$

We first prove $\forall x \in \text{im}\, P^{\oplus}(t) \Rightarrow x \in \text{im}\, R^{\oplus}(A, B)$

$\forall x \in \text{im}\, P^{\oplus}$ we have

$$x^T P(t) x = \int_0^t \| B^T e^{A^T \tau} x \|^2 \, d\tau = 0,$$

$$\Leftrightarrow B^T e^{A^T t} x = 0, \; for\ all \quad t \geq 0$$

Controllability measure

$$B^T e^{A^T t} x = 0,\ \forall t \Rightarrow \frac{d^i}{dt^i}\left(B^T e^{A^T t} x\right) = 0, \forall t \Rightarrow \frac{d^i}{dt^i}\left(B^T e^{A^T t} x\right)\Big|_{t=0} = 0,\ \forall i \geq 0$$

$$\Rightarrow \boxed{B^T (A^T)^i x = 0,} \forall i \geq 0.$$

$$\Downarrow$$

$$x \perp A^{i-1} B$$

$$\Downarrow$$

$$x \perp \operatorname{im} R(A, B)$$

$$\Updownarrow$$

$$x \in \operatorname{im} R^{\oplus}(A, B)$$

We have proved: $\forall x \in \operatorname{im} P^{\oplus}(t) \Rightarrow x \in \operatorname{im} R^{\oplus}(A, B)$

Controllability measure

Next we prove: $\forall x \in \operatorname{im} R^{\oplus}(A,B) \Rightarrow x \in \operatorname{im} P^{\oplus}$

$$x \in \operatorname{im} R^{\oplus}(A,B) \Longrightarrow x \perp \operatorname{im} R(A,B) \Longrightarrow x \perp A^{i-1}B, \text{ for all } i > 0$$

$$\Downarrow$$

$$B^T (A^T)^{i-1} x = 0, \text{ for all } i > 0.$$

$$\Downarrow$$

$$B^T e^{A^T t} x = 0, \text{ for all } t \geq 0$$

$$\Downarrow$$

$$\| B^T e^{A^T t} x \|_2^2 = x^T e^{At} B B^T e^{A^T t} x = 0$$

$$\Downarrow$$

$$x^T P(t) x = \int_0^t x^T e^{A\tau} B B^T e^{A^T \tau} x \, d\tau = 0, \Longrightarrow x \perp \operatorname{im}(P)$$

$$x \perp \operatorname{im}(P) \Longleftrightarrow x \in \operatorname{im} P^{\oplus}$$

$\blacksquare$

Controllability measure

The relation $\operatorname{im} P(t) = \operatorname{im} R(A, B)$ provides a way to derive the minimal energy which is needed to reach a state *x.*

The states needing large minimal energy to reach will be truncated during MOR based on balanced truncation.

Therefore, the minimal energy for reaching a state *x* is a key concept for model order reduction based on balanced truncation.

Next, we will derive the **minimal energy** for reaching a state *x.*

Controllability measure

From the analytical solution, if a state *x* is reached at time $\bar{T}$, then $\exists u(t)$ with finite energy, such that

$$x = \int_0^{\bar{T}} e^{A(\bar{T}-\tau)} Bu(\tau) d\tau$$

How much energy must the input *u*(*t*) have?

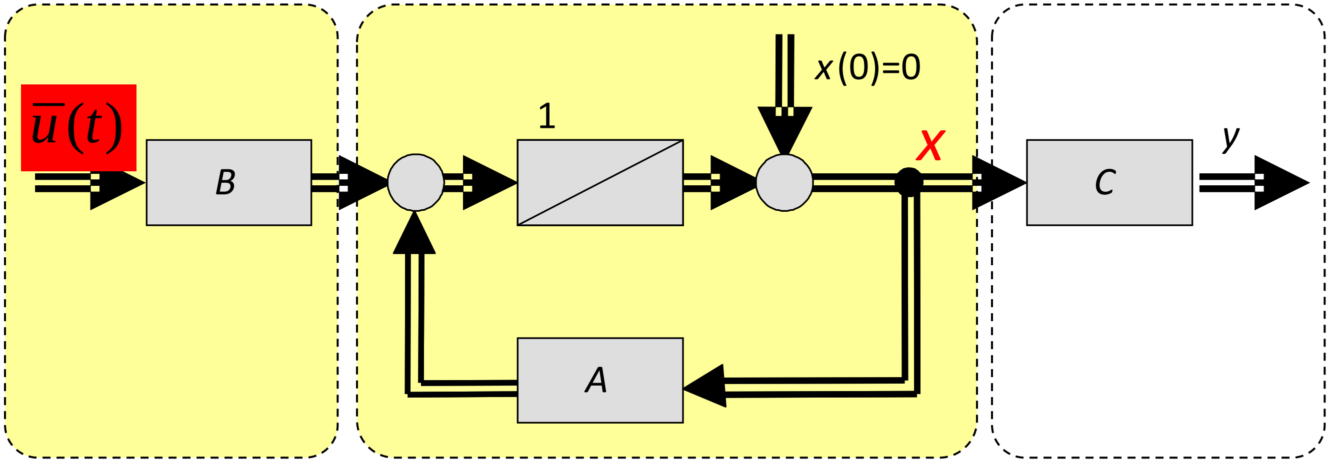
From Theorem 1 and propositon 1, we know $X^{reach} = \text{im}\, P(t)$, therefore, if *x* is reachable, then $x \in \text{im}\,(P(t))$, i.e. $\exists \xi, \bar{T}$,

$$x = P(\bar{T})\xi \Rightarrow x = \int_0^{\bar{T}} e^{At} BB^T e^{A^T t} \xi dt = \int_0^{\bar{T}} e^{A(\bar{T}-\tau)} BB^T e^{A^T(\bar{T}-\tau)} \xi d(-\tau)$$

$$= \int_0^{\bar{T}} e^{A(\bar{T}-\tau)} B\bar{u} d\tau \qquad and \qquad \bar{u}(\tau) = -B^T e^{A^T(\bar{T}-\tau)} \xi$$

This means *x* can be reached at time $\bar{T}$ with input $\bar{u}$

Controllability measure



$\bar{t} < \infty$

The input *u*(t) is the excitation of the system, its energy is the energy required to reach the state *x* .

Energy of a function is defined as: $$\| u \|^2 = \int_0^{\overline{T}} u^*(t) u(t) dt$$

Controllability measure

We see from above analysis, if x is reachable at time $\bar{t}$, *x* can be represented as:

$$x = \int_0^{\bar{t}} e^{A(\bar{t}-\tau)} B\bar{u}\, d\tau \qquad (\bar{u} = -B^T e^{A^T(\bar{t}-\tau)}\xi)$$

Any other input $\| u(t) \|^2 > \| \bar{u}(t) \|^2$ can also reach *x*. However if $\| u(t) \|^2 < \| \bar{u}(t) \|^2$, it cannot reach *x* at time $\bar{t}$, but needs longer time.

Actually the energy of $\bar{u}$ is the minimal energy to reach the state *x* at the given time period $\bar{t}$. (Proposition 4.10 in [Antoulas 05])

Energy of $\bar{u}$:

$$\| \bar{u} \|^2 = \int_0^{\bar{t}} \bar{u}^T(t)\bar{u}(t)dt = \int_0^{\bar{t}} \xi^T e^{A(\bar{t}-t)} BB^T e^{A^T(\bar{t}-t)} \xi dt = \xi^T P(\bar{t})\xi$$

relation to *x*?

$\updownarrow$

x

Controllability measure

A system is reachable means every state x in the whole state space is reachable.

From theorem 1: $X^{reach} = \text{im}\, R(A,B) = \text{im}\, R_n(A,B)$

Therefore the system is reachable $\Longleftrightarrow$ $\text{rank}\,(R_n(A,B)) = n$

From Proposition 1: $\text{im}\, P(t) = \text{im}\, R(A,B)$

Therefore the system is reachable $\Longleftrightarrow$ $\text{rank}\,(P(t)) = n, \forall t > 0$

Therefore, $P(t)$ is nonsingular for any *t*, if the system is reachable.

Controllability measure

Energy of $\bar{u} = B^T e^{A^T(\bar{t}-\tau)}\xi$ (notice $x = P(\bar{t})\xi$) :

$$\|\bar{u}\|^2 = \xi^T P(\bar{t})\xi = (P^{-1}(\bar{t})x)^T P(\bar{t})(P^{-1}(\bar{t})x) = x^T P^{-1}(\bar{t})x$$

$$\|\bar{u}\|^2 = x^T P^{-1}(\bar{t})x$$

Only for reachable systems.

Controllability measure

Remark 1:

Reachability is a generic property for LTI systems with the form:

$$dx/dt = Ax + Bu$$

This means, intuitively, that almost every LTI system with the standard form is reachable. If there are any unreachable systems, they are very rare. The unreachable LTI systems like examples 1,2 are rare.

Remark 2:

The reachability of the system can be more easily checked by the criteria:

The system is reachable $\Longleftrightarrow \quad rank(R_n(A,B)) = n$

Controllability measure

A concept which is closely related to reachability is that of controllability.

Here, instead of driving the zero state to a desired state, a given non-zero state is steered to the zero state. More precisely we have:

Definition of controllability: Given a LTI system as above, a non-zero state x is controllable if there exist an input $u(t)$ with finite energy such that the state of the system goes to zero from x within a finite time: $\bar{t} < \infty$.

Controllability measure

It has been proved that for standard **time continuous** LTI systems

$$dx(t)/dt = Ax(t) + Bu(t)$$

$$y(t) = L^T x(t)$$

the concepts of reachability and controllability are equivalent.

Theorem 2 For time continuous systems $X^{reach} = X^{contr}$. (Theorem 4.16 in [Antoulas 05])

Similarly, X^{contr} is the subspace spanned by the controllable states.

From the property of reachable system, we have

The system is controllable $\Longleftrightarrow$ $\text{rank}\,(R_n(A,B)) = n$

Controllability measure

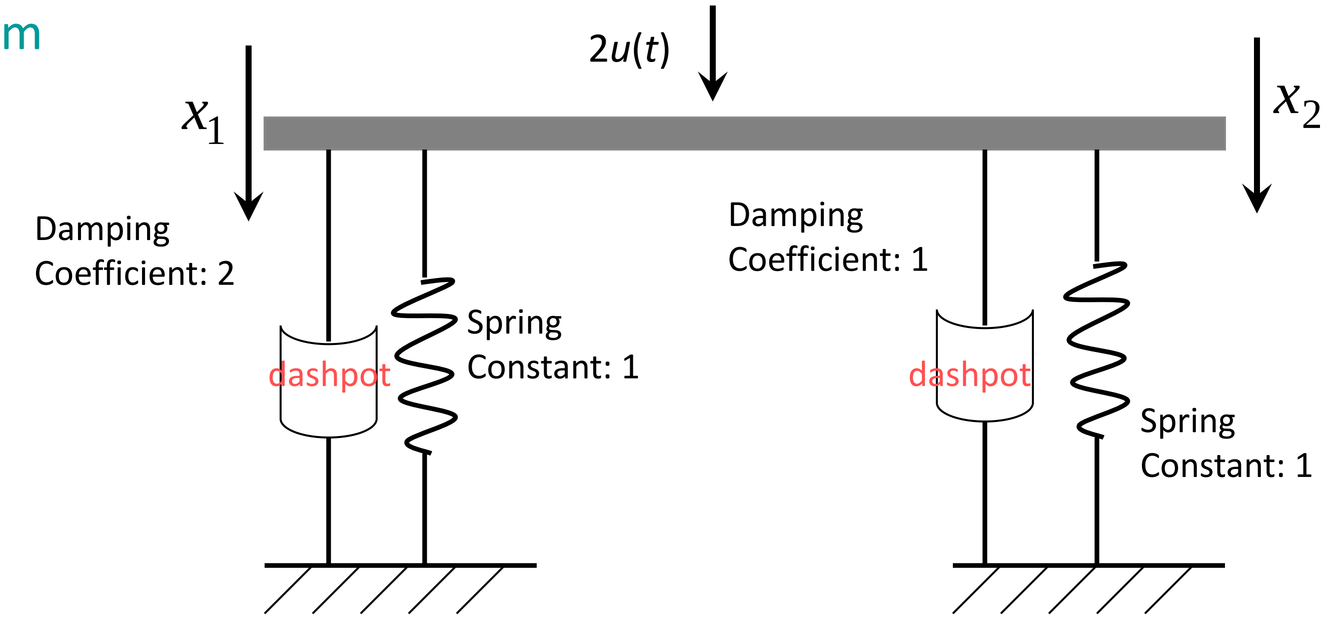
$$\|\bar{u}\|^2 = x^T P^{-1}(\bar{t})x$$

Controllability measure!

Only for reachable systems.

Controllability measure

Example: Platform system



The system is described by the following linear time invariant (LTI) system:

assume mass of the platform is zero, then from Newton's law: $F - \eta v - kx = ma$

$$u - 2\dot{x}_1 - x_1 = 0$$
$$u - \dot{x}_2 - x_2 = 0$$

$$\Longrightarrow \quad dx(t)/dt = \underbrace{\begin{pmatrix} -0.5 & 0 \\ 0 & -1 \end{pmatrix}}_{A} x(t) + \underbrace{\begin{pmatrix} 0.5 \\ 1 \end{pmatrix}}_{B} u(t)$$

Controllability measure

Is the platform system controllable?

The system is controllable $\Longleftrightarrow$ $rank(R_n(A,B)) = n$

$$R_n(A,B) = [B, AB,]$$

$$B = \begin{pmatrix} 0.5 \\ 1 \end{pmatrix} \qquad AB = \begin{pmatrix} -0.5 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0.5 \\ 1 \end{pmatrix} = \begin{pmatrix} -0.25 \\ -1 \end{pmatrix}$$

B, AB are linearly independent!

$$rank(R_n(A,B)) = 2 = n$$

Therefore, the platform system is controllable.

Controllability measure

Associated with controllability, there is the concept of observability.

Controllability: input $u(t) \longrightarrow$ state $x(t)$.

Possibility of steering the state using the input.

Observability: output $y(t) \longrightarrow$ state $x(t)$.

Possibility of estimating the state from the output.

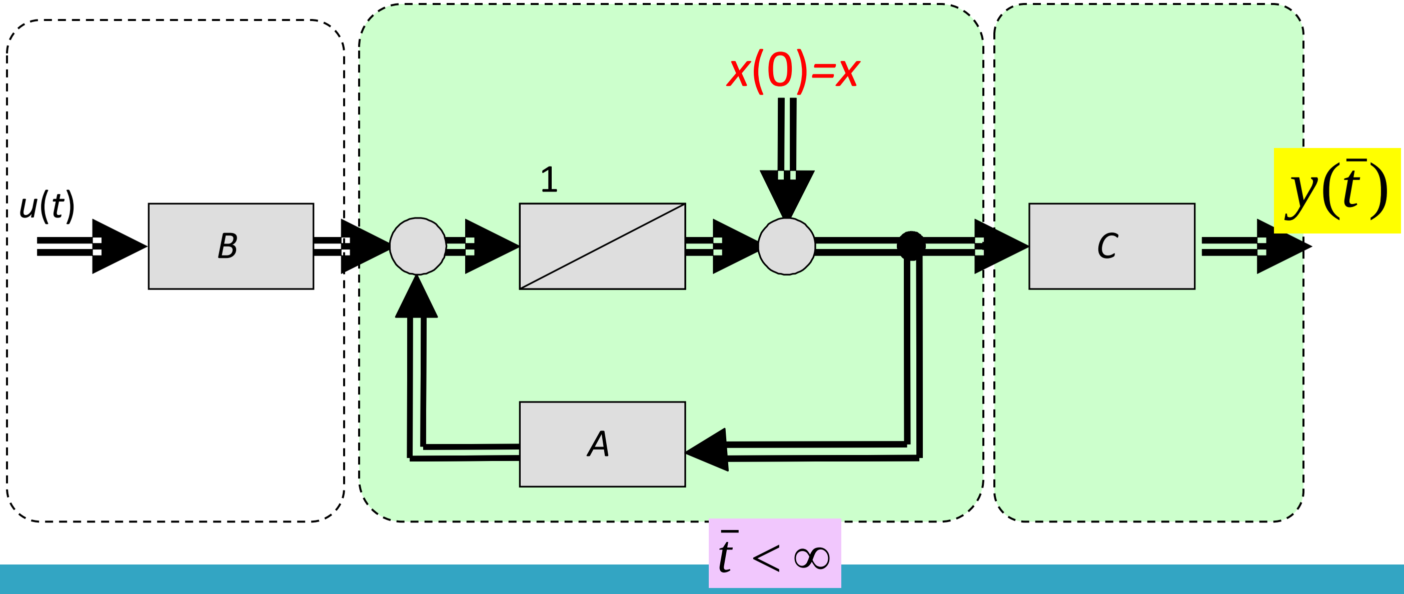
Outline

- Observability measures
- Infinite Gramians

Observability measure

Observability is a measure for how well internal states of a system can be estimated by knowledge of its external outputs.

Definition of Observability: Given any input $u(t)$, a state x of the system is observable, if starting with the state x ($x(0)=x$), and after a finite period of time $\bar{t} < \infty$, x can be uniquely determined by the output $y(\bar{t})$.



Observability measure

Observability matrix?

Observability Gramian?

Output energy?

$$O(L,A)=\begin{pmatrix} L^T \\ L^T A \\ L^T A^2 \\ \vdots \end{pmatrix}$$

From the analytical solution to $dx/dt = Ax + Bu$, we see that after time $\bar{t} < \infty$:

$$\tilde{x}(\bar{t}) = e^{A\bar{t}} x_0 + \int_0^{\bar{t}} e^{A(\bar{t}-\tau)} Bu(\tau) d\tau$$

The system starting with *x*(0)=*x*, therefore

$$\tilde{x}(\bar{t}) = e^{A\bar{t}} x + \int_0^{\bar{t}} e^{A(\bar{t}-\tau)} Bu(\tau) d\tau$$

And the output corresponding to $\tilde{x}(\bar{t})$ is:

$$y(\bar{t}) = L^T \tilde{x}(\bar{t}) = L^T e^{A\bar{t}} x + L^T \int_0^{\bar{t}} e^{A(\bar{t}-\tau)} Bu(\tau) d\tau$$

$$= L^T e^{A\bar{t}} x + L^T e^{A\bar{t}} \int_0^{\bar{t}} e^{-A\tau} Bu(\tau) d\tau$$

$$= L^T e^{A\bar{t}} \bar{x} \quad and \quad \bar{x} = x + \int_0^{\bar{t}} e^{-A\tau} Bu(\tau) d\tau$$

Observability measure

Derivation of Observability matrix

If x is observable, then for any $u(t)$, x can be uniquely determined by the corresponding y :

$$y(\bar{t}) = L^T e^{A\bar{t}} \bar{x} \qquad and \qquad \bar{x} = x + \int_0^{\bar{t}} e^{-A\tau} Bu(\tau) d\tau$$

Since x can be uniquely determined by $\bar{x}$, it is sufficient to prove that $\bar{x}$ can be uniquely determined by $y(\bar{t})$.

Let us see under what condition can $\bar{x}$ be uniquely determined by $y(\bar{t})$?

Observability measure

Derivation of Observability matrix

$$y(\bar{t}) = L^T e^{A\bar{t}} \bar{x}$$

Differentiate the above equation on both sides and get the derivatives at *t*=0:

$$y(0) = L^T \bar{x}$$
$$y'(0) = L^T A\bar{x}$$
$$y''(0) = L^T A^2 \bar{x}$$
$$\vdots$$
$$y^{(k)}(0) = L^T A^k \bar{x}$$

$$\Longleftrightarrow \quad \begin{pmatrix} L^T \\ L^T A \\ \vdots \\ L^T A^k \end{pmatrix} \bar{x} = \begin{pmatrix} y(0) \\ y'(0) \\ \vdots \\ y^{(k)}(0) \end{pmatrix} \qquad (\#)$$

(#) has a unique solution $\bar{x}$ if $\begin{pmatrix} L^T \\ L^T A \\ \vdots \\ L^T A^k \end{pmatrix}$ is square and has full rank *n*.

Observability measure

Derivation of Observability matrix

Denote:

$$Q_k = \begin{pmatrix} L^T \\ L^T A \\ \vdots \\ L^T A^k \end{pmatrix} \qquad \bar{y} = \begin{pmatrix} y(0) \\ y'(0) \\ \vdots \\ y^{(k)}(0) \end{pmatrix} \Longrightarrow \bar{x} = Q_k^{-1} \bar{y}$$

$\bar{x}$ can be uniquely determined, with *k* being at most *n-1.*

$L^T \in R^{m \times n}$ if *m*>1, then *k*<n-1, if *m*=1, *k*=n-1.

Observability measure

Derivation of Observability matrix

For standard LTI systems, the observability matrix is defined as:

$$O(L,A)=\begin{pmatrix} L^T \\ L^T A \\ L^T A^2 \\ \vdots \end{pmatrix}$$

From above analysis, actually the finite Observability matrix is enough to determine observability:

$$O_n(L,A)=\begin{pmatrix} L^T \\ L^T A \\ \vdots \\ L^T A^{n-1} \end{pmatrix}$$

Therefore:

The system is observable $\Longleftrightarrow$ $rank(O_n(L,A))=n$

Observability measure

Output energy

The output energy associated with the initial state *x* is:

$$\| y(\bar{t}) \|^2 = \int_0^{\bar{t}} y(t)^T y(t) dt = \int_0^{\bar{t}} \bar{x}^T e^{A^T t} L L^T e^{At} \bar{x} dt$$

$$= \bar{x}^T \int_0^{\bar{t}} e^{A^T t} L L^T e^{At} dt \bar{x}$$

$$= \bar{x}^T Q(\bar{t}) \bar{x}$$

1. Energy of observation produced by an observable state *x*.
2. Observability measure!

Finite Observability Gramian at time $t < \infty$ is defined as:

$$Q(t) = \int_0^t e^{A^T \tau} L L^T e^{A\tau} d\tau, \quad 0 < t < \infty$$

Observability measure

———————Output energy

The system is observable: $\Longleftrightarrow$ $rank(O_n(L,A)) = n$

Finite Observability Gramian at time t: $Q(t) = \int_0^t e^{A^T\tau} L L^T e^{A\tau} d\tau, \quad 0 < t < \infty$

$$rank(O_n(L,A)) = n \Leftrightarrow \ker(O_n) = 0$$

$$\ker(O_n) = \{x : L^T A^i x = 0, i \geq 0\} = \ker(Q(t))$$

$$\ker(O_n) = 0 \Rightarrow \ker(Q(t)) = 0 \Rightarrow rank(Q(t)) = n$$

The system is observable: $\Longleftrightarrow$ $rank(Q(t)) = n : Q(t)$ is nonsingular

Observability measure

Observability Gramian

Recall the minimal energy to reach a state *x* at time $\bar{t}$ is

$$\|\bar{u}\|^2 = x^T P^{-1}(\bar{t}) x$$

Notice both energies are related to time.

$$\|\bar{u}\|^2 = x^T P^{-1}(\bar{t}) x \qquad \|y(\bar{t})\|^2 = \bar{x}^T Q(\bar{t}) \bar{x}$$

$$P(t) = \int_0^t e^{A\tau} B B^T e^{A^T \tau} d\tau, \quad 0 < t < \infty \qquad Q(t) = \int_0^t e^{A^T \tau} L L^T e^{A\tau} d\tau, \quad 0 < t < \infty$$

Finite (reachability) controllability Gramian and observability Gramian will be used to derive the infinite Gramians which

1. Make the two measures computable.
2. will be directly used for truncation in MOR.

Outline

- Infinite Gramians

Infinite Gramians

——— make the two measures computable

Under which condition, $Q(t)$ and $P(t)$ are bounded when time goes to infinity: $t \to \infty$?

$$P(t) = \int_0^t e^{A\tau} B B^T e^{A^T \tau} d\tau, \quad 0 < t < \infty$$

$$Q(t) = \int_0^t e^{A^T \tau} L L^T e^{A\tau} d\tau, \quad 0 < t < \infty$$

Roughly speaking, $Q(t)$ and $P(t)$ can be bounded when $t \to \infty$, if e^{At} is bounded when $t \to \infty$.

Infinite Gramians

——— make the two measures computable

e^{At} is bounded if the real parts of all the eigenvalues of *A* are negative.

Why? Let $A = S^{-1}\Lambda S$ be the eigen-decomposition of *A*,

$$e^{At} = e^{S^{-1}\Lambda St} = S^{-1}e^{\Lambda t}S = S^{-1}e^{\Lambda_{re}t+\Lambda_{im}t}S = S^{-1}e^{\Lambda_{re}t}e^{\Lambda_{im}t}S$$

$$\Lambda_{re} = \begin{pmatrix} \lambda_1^{re} & & & \\ & \lambda_2^{re} & & \\ & & \ddots & \\ & & & \lambda_n^{re} \end{pmatrix} \qquad \Lambda_{im} = \begin{pmatrix} j\lambda_1^{im} & & & \\ & j\lambda_2^{im} & & \\ & & \ddots & \\ & & & j\lambda_n^{im} \end{pmatrix}$$

$\lambda_i = \lambda_i^{re} + j\lambda_i^{im}, \qquad i = 1,2,\cdots n$ are eigenvalues of *A*.

Infinite Gramians

make the two measures computable

$$e^{At} = e^{S^{-1}\Lambda S t} = S^{-1} e^{\Lambda t} S = S^{-1} e^{t\Lambda_{re}} e^{t\Lambda_{im}} S$$

$$e^{t\Lambda_{re}} = \begin{pmatrix} e^{t\lambda_1^{re}} & & & \\ & e^{t\lambda_2^{re}} & & \\ & & \ddots & \\ & & & e^{t\lambda_n^{re}} \end{pmatrix} \xrightarrow[\lambda_i^{re} < 0]{t \to \infty} \begin{pmatrix} 0 & & & \\ & 0 & & \\ & & \ddots & \\ & & & 0 \end{pmatrix}$$

$$e^{t\Lambda_{im}} = \begin{pmatrix} e^{tj\lambda_1^{im}} & & & \\ & e^{tj\lambda_2^{im}} & & \\ & & \ddots & \\ & & & e^{tj\lambda_n^{im}} \end{pmatrix} \xrightarrow[e^{tj\lambda_i^{im}} = \cos(t\lambda_i^{im}) + j\sin(\lambda_i^{im})]{t \to \infty} \text{bounded}$$

Infinite Gramians

——make the two measures computable

Therefore, $e^{At} = e^{S^{-1}\Lambda S} = S^{-1}e^{\Lambda}S = S^{-1}e^{t\Lambda_{re}}e^{t\Lambda_{im}}S \to 0$

if the real parts of all the eigenvalues of *A* are negative.

Therefore the follow limits exists if all the eigenvalues of *A* are negative, i.e. if the system is stable:

$$P = \lim_{t\to\infty} P(t) = \lim_{t\to\infty}\int_0^t e^{A\tau}BB^T e^{A^T\tau}d\tau = \int_0^\infty e^{At}BB^T e^{A^T t}dt$$

$$Q = \lim_{t\to\infty} Q(t) = \lim_{t\to\infty}\int_0^t e^{A^T\tau}LL^T e^{A\tau}d\tau = \int_0^\infty e^{A^T t}LL^T e^{At}dt$$

where *P* and *Q* are the infinite Gramians (only for stable systems).

Infinite Gramians

——make the two measures computable

Recall:

If a system is controllable: $P(t)$ nonsingular

If a system is observable: $Q(t)$ nonsingular

Then: (**Exercise**)

If a system is controllable: P nonsingular

If a system is observable: Q nonsingular

Infinite Gramians

—— make the two measures computable

The infinite Gramians:

$$P = \lim_{t\to\infty} P(t) = \lim_{t\to\infty} \int_0^t e^{A\tau} BB^T e^{A^T\tau} d\tau = \int_0^\infty e^{At} BB^T e^{A^T t} dt$$

$$Q = \lim_{t\to\infty} Q(t) = \lim_{t\to\infty} \int_0^t e^{A^T\tau} LL^T e^{A\tau} d\tau = \int_0^\infty e^{A^T t} LL^T e^{At} dt$$

From the property of integral, we have

$$P \ge P(t), \quad \forall t \qquad\qquad Q \ge Q(t), \quad \forall t$$

$$\Downarrow$$

In the meaning of inner product: $P \ge P(t) \Leftrightarrow (Px, x) \ge (P(t)x, x)$

Infinite Gramians

make the two measures computable

The minimal energy necessary for reaching a reachable state *x* at time *t* is:

$$\| \bar{u} \|^2 = x^T P^{-1}(t) x$$

Exercise.

For stable systems, lower bound of the minimal energy necessary for reaching a reachable state *x* is:

$$\| \bar{u} \|^2 = x^T P(t)^{-1} x \geq x^T P^{-1} x \quad \text{because} \quad P \geq P(t), \quad \forall t$$

For stable systems, the upper bound of the energy produced by the observable state *x* is:

$$\| y(t) \|^2 = \bar{x}^T Q(t) \bar{x} \leq \bar{x}^T Q \bar{x} \quad \text{because} \quad Q \geq Q(t), \quad \forall t$$

Computable measures!

Only suitable for stable systems!

Infinite Gramians

make the two measures computable

For stable systems, the minimal energy necessary for reaching any state is:

$$\min \| \overline{u} \|^2 = x^T P^{-1} x$$

For stable systems, the maximal energy produced by any state *x* is:

$$\max \| y(t) \|^2 = \overline{x}^T Q \overline{x}$$

Infinite Gramians

——— make the two measures computable

Because the MOR method we will introduce uses P and Q to derive the reduced-order model, and therefore is only suitable for stable systems.

$$\min \| \bar{u} \|^2 = x^T P^{-1} x \qquad \max \| y(t) \|^2 = \bar{x}^T Q \bar{x}$$

$$\Downarrow$$

The eigenspaces of P and Q make the two measurements practically computable!

Eigenspaces of P and Q

—— make the two measures parctically computable

The states which are difficult to reach are included in the subspace spanned by those eigenvectors of *P* that corresponds to small eigenvalues.

The states which are difficult to observe are included in the subspace spanned by those eigenvectors of *Q* that corresponds to small eigenvalues.

why and how?

Eigenspaces of P and Q

—— make the two measures practically computable

Denote $\xi_1, \xi_2, \cdots, \xi_n$ as the n eigenvectors of *P*, the corresponding eigenvalues are $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ (*P* is symmetric positive definite, it has positive eigenvalues.)

$\xi_1, \xi_2, \cdots, \xi_n$ are linearly independent, therefore they constitute a basis of the whole space C^n.

The state *x* can therefore be represented by $\xi_1, \xi_2, \cdots, \xi_n$

$$x = \alpha_1 \xi_1 + \alpha_2 \xi_2 + \cdots + \alpha_n \xi_n$$

$$\min \| \bar{u} \|^2 = x^T P^{-1} x$$

If a matrix is nonsingular, then its inverse has the same eigenvectors, but the eigenvalues are the reciprocals:

$$P\xi = \lambda\xi \Rightarrow P^{-1}P\xi = \lambda P^{-1}\xi \Rightarrow \xi / \lambda = P^{-1}\xi$$

Eigenspaces of P and Q

——make the two measures practically computable

$$\min \|\bar{u}\|^2 = x^T P^{-1} x$$

$$x = \alpha_1 \xi_1 + \alpha_2 \xi_2 + \cdots + \alpha_n \xi_n$$

⇓

$$P^{-1} x = \alpha_1 \frac{1}{\lambda_1} \xi_1 + \alpha_2 \frac{1}{\lambda_2} \xi_2 + \cdots + \alpha_n \frac{1}{\lambda_n} \xi_n$$

⇓

$$x^T P^{-1} x = \alpha_1^2 \frac{1}{\lambda_1} \xi_1^T \xi_1 + \alpha_2^2 \frac{1}{\lambda_2} \xi_2^T \xi_2 + \cdots + \alpha_n^2 \frac{1}{\lambda_n} \xi_n^T \xi_n$$

P is symmetric, ⇓ Therefore $\tilde{Q} = [\xi_1, \cdots, \xi_n]$ is orthogonal.

$$\min \|\bar{u}\| = \alpha_1^2 \frac{1}{\lambda_1} + \alpha_2^2 \frac{1}{\lambda_2} + \cdots + \alpha_n^2 \frac{1}{\lambda_n}$$

$\min \|\bar{u}\|^2$ indicates the minimal energy needed to reach the state *x*, therefore the larger $\min \|\bar{u}\|^2$ is, the more difficult the state *x* to reach.

Eigenspaces of P and Q

——— make the two measures practically computable

$$\begin{cases} \min \| \bar{u} \| = \alpha_1^2 \frac{1}{\lambda_1} + \alpha_2^2 \frac{1}{\lambda_2} + \cdots + \alpha_n^2 \frac{1}{\lambda_n} \\ \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \Rightarrow \frac{1}{\lambda_1} \leq \frac{1}{\lambda_2} \leq \cdots \leq \frac{1}{\lambda_n} \end{cases}$$

$\min \| \bar{u} \|^2$ is larger if $\lambda_1 \geq \lambda_2 \geq \cdots >> \lambda_k \geq \lambda_{k+1} \geq \cdots \geq \lambda_n$ and $\alpha_1, \alpha_2, \cdots << \alpha_k, \alpha_{k+1}, \cdots, \alpha_n$ than if

$\lambda_1 \geq \lambda_2 \geq \cdots >> \lambda_k \geq \lambda_{k+1} \geq \cdots \geq \lambda_n$ and

$\alpha_1, \alpha_2, \cdots >> \alpha_k, \alpha_{k+1}, \cdots, \alpha_n$

$$x = \alpha_1 \xi_1 + \alpha_2 \xi_2 + \cdots + \alpha_n \xi_n$$

This means if *x* is difficult to reach ($\| \bar{u} \|^2$ is large), *x* should have large components in the subspace spanned by the eigenvectors corresponding to the small eigenvalues of *P*. Or *x* should almost locates in the subspace spanned by the eigenvectors corresponding to the small eigenvalues.

Eigenspaces of P and Q

—— make the two measures practically computable

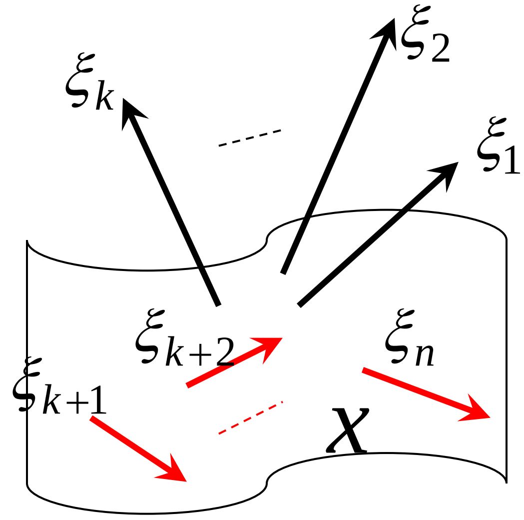
Similarly, if *x* is difficult to observe ($\| y(t) \|^2 = \bar{x}^T Q \bar{x}$ is small) *x* should have large components in the subspace spanned by the eigenvectors corresponding to the small eigenvalues of *Q*. Or *x* should almost locates in the subspace spanned by the eigenvectors corresponding to the small eigenvalues.

$$\lambda_1 \geq \lambda_2 \geq \cdots >> \lambda_k \geq \lambda_{k+1} \geq \cdots \geq \lambda_n$$

$$P\xi_i = \lambda_i \xi_i, i = 1,2,\cdots n$$

$$\tilde{\lambda}_1 \geq \tilde{\lambda}_2 \geq \cdots >> \tilde{\lambda}_k \geq \tilde{\lambda}_{k+1} \geq \cdots \geq \tilde{\lambda}_n$$

$$Q\tilde{\xi}_i = \tilde{\lambda}_i \tilde{\xi}_i, i = 1,2,\cdots n$$



References

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