MAX PLANCK INSTITUTE
FOR DYNAMICS OF COMPLEX
TECHNICAL SYSTEMS
MAGDEBURG

COMPUTATIONAL METHODS IN
SYSTEMS AND CONTROL THEORY

Model Reduction for Dynamical Systems

–Lecture 5–

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Outline

1. Mathematical Basics IV
   Systems and control theory

2. A brief summary

3. A brief summary

System Norms

Definition

The $L_2^n(-\infty, +\infty)$ space is the vector-valued function space $f : \mathbb{R} \mapsto \mathbb{R}^n$, with the norm

$$||f||_{L_2^n} = \left(\int_{-\infty}^{\infty} ||f(t)||^2 dt \right)^{1/2}.$$

Here and below, $||\cdot||$ denotes the Euclidean vector or spectral matrix norm.

Definition

The frequency domain $\mathcal{L}_2(\jmath\mathbb{R})$ space is the matrix-valued function space $F : \mathbb{C} \mapsto \mathbb{C}^{p\times m}$, with the norm

$$||F||_{\mathcal{L}_2} = \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} ||F(\jmath\omega)||^2 d\omega \right)^{1/2},$$

System Norms

Definition

The $L_\infty^n(-\infty, +\infty)$ space is the vector-valued function space $f : \mathbb{R} \mapsto \mathbb{R}^n$, with the norm

$$\|f\|_{L_\infty^n} = \sup_t ||f(t)||_\infty.$$

System Norms

The maximum modulus theorem will be used repeatedly. [1]

Theorem

Let $f(z) : \mathbb{C}^n \mapsto \mathbb{C}$ be a regular analytic, or holomorphic, function of n complex variables $z = (z_1, \ldots, z_n), n \geq 1$, defined on an (open) domain $\mathbb{D}$ of the complex space $\mathbb{C}^n$, which is not a constant, $f(z) \neq$ const. Let

$$max_f = \max\{|f(z)| : z \in \mathbb{D}\}.$$

If $f(z)$ is continuous in a bounded domain $\mathbb{D}$, then max_f can only be attained on the boundary of $\mathbb{D}$, i.e. $max_f = \max\{|f(z)| : z \in \partial\mathbb{D}\}$.

[1] In mathematics, a holomorphic function is a complex-valued function of one or more complex variables that is, at every point of its domain, complex differentiable in a neighbourhood of the point. from: https://en.wikipedia.org/wiki

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Consider the transfer function $G(s) = C(sI - A)^{-1}B + D$, and input functions $u \in \mathcal{L}_2(\jmath\mathbb{R})$, with the $\mathcal{L}_2$-norm

$$||u||^2_{\mathcal{L}_2} := \frac{1}{2\pi} \int_{-\infty}^{\infty} u(\jmath\omega)^H u(\jmath\omega)\, d\omega.$$

System Norms

Assume A is (asymptotically) stable: $\Lambda(A) \subset \mathbb{C}^- := \{z \in \mathbb{C} : \mathrm{Re}(z) < 0\}$. Then G is analytic in $\mathbb{C}^+ \cup \jmath\mathbb{R}$, and following the maximal modulus theorem, $G(s)$ is bounded: $||G(s)||_F \leq M < \infty$, $\forall s \in \mathbb{C}^+ \cup \jmath\mathbb{R}$. Thus we have

$$\begin{aligned}
\|y(\jmath\omega)\|^2_{\mathcal{L}_2} &= \frac{1}{2\pi}\int_{-\infty}^{\infty} y(\jmath\omega)^H y(\jmath\omega)\, d\omega \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} u(\jmath\omega)^H G(\jmath\omega)^H G(\jmath\omega) u(\jmath\omega)\, d\omega \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} ||G(\jmath\omega)u(\jmath\omega)||^2\, d\omega \leq \frac{1}{2\pi}\int_{-\infty}^{\infty} M^2 ||u(\jmath\omega)||^2 d\omega \\
&= M^2 \frac{1}{2\pi}\int_{-\infty}^{\infty} u(\jmath\omega)^H u(\jmath\omega)\, d\omega = M^2\|u(\jmath\omega)\|^2_{\mathcal{L}_2} \; < \; \infty,
\end{aligned}$$

So that $y = Gu \in \mathcal{L}_2(\jmath\mathbb{R})$. $(||Ax||_\infty \leq ||Ax||_2 \leq ||A||_F ||x||_2)$
Consequently, the $\mathcal{L}_2$-induced operator norm is well defined:

$$||G||_{\mathcal{L}_\infty} := \sup_{||u||_2 \neq 0} \frac{||Gu||_{\mathcal{L}_2}}{||u||_{\mathcal{L}_2}}. \qquad (1)$$

System Norms

It can be further proved that

$$||G||_{\mathcal{L}_\infty} = \sup_{\omega\in\mathbb{R}} ||G(\jmath\omega)|| = \sup_{\omega\in\mathbb{R}} \sigma_{max}\left(G(\jmath\omega)\right).$$

With the above defined $\mathcal{L}_\infty$-norm, the frequency domain $\mathcal{L}_\infty$ space is defined as

Definition

The frequency domain $\mathcal{L}_\infty(\jmath\mathbb{R})$ space is the matrix-valued function space $F : \mathbb{C} \mapsto \mathbb{C}^{p\times m}$, with the norm

$$||F||_{\mathcal{L}_\infty} = \sup_{\omega\in\mathbb{R}} ||F(\jmath\omega)|| = \sup_{\omega\in\mathbb{R}} \sigma_{max}\left(F(\jmath\omega)\right).$$

System Norms

Error bound 1

$$||Gu||_{\mathcal{L}_2} \leq ||G||_{\mathcal{L}_\infty} ||u||_{\mathcal{L}_2}$$

Consequently,

$$||y - \hat{y}||_{\mathcal{L}_2} = ||Gu - \hat{G}u||_{\mathcal{L}_2} \leq ||G - \hat{G}||_{\mathcal{L}_\infty} ||u||_{\mathcal{L}_2}$$

System Norms

When the funtion has better property, for example analytic, then we can define $\mathcal{H}$ norms for these functions.

Definition

The Hardy space $\mathcal{H}_\infty$ is the function space of matrix-, scalar-valued functions that are analytic and bounded in $\mathbb{C}^+ := \{z \in \mathbb{C} : \mathrm{Re}\,(z) > 0\}$.

The $\mathcal{H}_\infty$-norm is defined as

$$||F||_{\mathcal{H}_\infty} := \sup_{z \in \mathbb{C}^+} ||F(z)|| = \sup_{\omega \in \mathbb{R}} ||F(\jmath\omega)|| = \sup_{\omega \in \mathbb{R}} \sigma_{max}\left(F(\jmath\omega)\right).$$

The second equality follows the maximum modulus theorem.

System Norms

Definition

The Hardy space $\mathcal{H}_2(\mathbb{C}^+)$ is the function space of matrix-, scalar-valued functions that are analytic in $\mathbb{C}^+$ and bounded w.r.t. the $\mathcal{H}_2$-norm defined as

$$\begin{aligned} ||F||_{\mathcal{H}_2} &:= \left(\frac{1}{2\pi}\sup_{\sigma>0}\int_{-\infty}^{\infty}||F(\sigma+\jmath\omega)||_F^2 d\omega\right)^{\frac{1}{2}} \\ &= \left(\frac{1}{2\pi}\int_{-\infty}^{\infty}||F(\jmath\omega)||_F^2\, d\omega\right)^{\frac{1}{2}}. \end{aligned}$$

The last equality follows maximum modulus theorem.

System Norms

Theorem [[ANTOULAS '05]](Section 5.5.1)

Practical Computation of the $\mathcal{H}_2$-norm of the transfer function

$$||G||_{\mathcal{H}_2}^2 = \mathrm{tr}(B^T Q B) = \mathrm{tr}(C P C^T),$$

where P, Q are the controllability and observability Gramians (the infinite Gramians) of the corresponding LTI system.

System Norms

Following [Antoulas, Beattie, Gugercin '10][2] (pp. 15-16), the $\mathcal{H}_2$ approximation error gives the following bound

$$
\begin{aligned}
&\max_{t>0} \|y(t) - \hat{y}(t)\|_\infty \\
&= \max_{t>0} \|\tfrac{1}{2\pi}\int_{-\infty}^{\infty}(y(\jmath\omega) - \hat{y}(\jmath\omega))e^{\jmath\omega t}d\omega\|_\infty \quad (\text{inverse Fourier transform}) \\
&\leq \max_{t>0} \tfrac{1}{2\pi}\int_{-\infty}^{\infty} \|(y(\jmath\omega) - \hat{y}(\jmath\omega))\|_\infty |\cos\omega t + \jmath\sin\omega t| d\omega \\
&\leq \tfrac{1}{2\pi}\int_{-\infty}^{\infty} \|(G(\jmath\omega) - \hat{G}(\jmath\omega))\|_F \|u(\jmath\omega)\|_2 d\omega \quad (*) \\
&\leq (\tfrac{1}{2\pi}\int_{-\infty}^{\infty} \|(G(\jmath\omega) - \hat{G}(\jmath\omega))\|_F^2)^{1/2} (\tfrac{1}{2\pi}\int_{-\infty}^{\infty} \|u(\jmath\omega)\|_2^2)^{1/2} d\omega \\
&\qquad\qquad (\text{ Cauchy-Schwarz inequality}) \\
&= \|G - \hat{G}\|_{\mathcal{H}_2} \|u\|_{\mathcal{L}_2},
\end{aligned}
$$

where (*) uses the facts $y(\jmath\omega) = G(\jmath\omega)u(\jmath\omega)$ and $\|Ax\|_\infty \leq \|Ax\|_2 \leq \|A\|_F\|x\|_2$ (http://de.wikipedia.org/wiki/Frobeniusnorm). G and $\hat{G}$ are original and reduced transfer functions. $\|\cdot\|_\infty$ is the vector norm in Euclidean space for any fixed t.

[2] A. C. Antoulas, C. A. Beattie, S. Gugercin. Interpolatory Model Reduction of Large-scale Dynamical Systems.

System Norms

Then

Error bound 2

$$\|y-\hat{y}\|_\infty := \max_{t>0} \|y(t)-\hat{y}(t)\|_\infty \leq \|G-\hat{G}\|_{\mathcal{H}_2}\|u\|_{\mathcal{L}_2}.$$

System Norms

(Plancherel Theorem)

The Fourier transform of $f \in L_2^n(-\infty, \infty)$:

$$F(\jmath\omega) = \int_{-\infty}^{\infty} f(t)e^{-\jmath\omega t}dt$$

is a Hilbert space isomorphism between $L_2^n(-\infty, \infty)$ and $\mathcal{L}_2(\jmath\mathbb{R})$. Furthermore, the Fourier transform maps $L_2^n(0, \infty)$ onto $\mathcal{H}_2(\mathbb{C}^+)$. In addition it is an isometry, that is, it preserves distances:

$$L_2^n(-\infty, \infty) \cong \mathcal{L}_2(\jmath\mathbb{R}),$$

Consequently, L_2^n-norm in time domain and $\mathcal{L}_2$-norm, $\mathcal{H}_2$-norm in frequency domain coincide.

Approximation Problems

Therefore the Error bound 1,

$$||y - \hat{y}||_2 = ||Gu - \hat{G}u||_2 \leq ||G - \hat{G}||_{\mathcal{L}_\infty} ||u||_2, \tag{2}$$

holds in time and frequency domain due to Plancherel theorem, i.e. the $||\cdot||_2$ in (2) can be the L_2^n-norm in time domain, or the $\mathcal{L}_2$-norm in frequency domain.

The transfer function is analytic, therefore $||G||_{\mathcal{H}_\infty}$ is defined. Futhermore, from their definitions, we have

$$||G||_{\mathcal{L}_\infty} = ||G||_{\mathcal{H}_\infty},$$

so that,

$$||y - \hat{y}||_2 \leq ||G - \hat{G}||_{\mathcal{H}_\infty} ||u||_2.$$

Approximation Problems

Similarly, the Error bound 2 holds as

$$||y - \hat{y}||_\infty \leq ||G - \hat{G}||_{\mathcal{H}_2} ||u||_2,$$

where $||\cdot||_2$ can be the L_2^n-norm in time domain, or the $\mathcal{L}_2$-norm in frequency domain.
Finally, we get two error bounds,

Output errors bounds

$$\begin{aligned} ||y - \hat{y}||_2 &\leq ||G - \hat{G}||_{\mathcal{H}_\infty} ||u||_2 && \Longrightarrow \left|\left| G - \hat{G} \right|\right|_\infty < \mathrm{tol} \\ ||y - \hat{y}||_\infty &\leq ||G - \hat{G}||_{\mathcal{H}_2} ||u||_2 && \Longrightarrow ||G - \hat{G}||_{\mathcal{H}_2} < \mathrm{tol} \end{aligned}$$

Goal of MOR: $||G - \hat{G}||_\infty < tol$ or $||G - \hat{G}||_{\mathcal{H}_2} < tol$.

Approximation Problems

Similarly, the Error bound 2 holds as

$$||y - \hat{y}||_\infty \leq ||G - \hat{G}||_{\mathcal{H}_2} ||u||_2,$$

where $||\cdot||_2$ can be the L_2^n-norm in time domain, or the $\mathcal{L}_2$-norm in frequency domain.
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Goal of MOR: $||G - \hat{G}||_\infty < tol$ or $||G - \hat{G}||_{\mathcal{H}_2} < tol$.

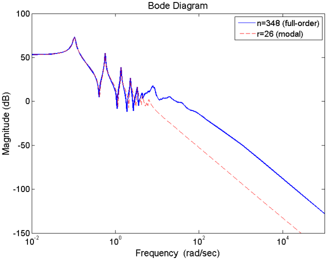
Computable error measures

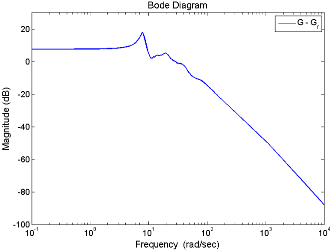
Evaluating system norms is computationally very (sometimes too) expensive.

Other measures

- absolute errors $\left|\left|G(\jmath\omega_j) - \hat{G}(\jmath\omega_j)\right|\right|_2$, $\left|\left|G(\jmath\omega_j) - \hat{G}(\jmath\omega_j)\right|\right|_\infty$ $(j = 1, \dots, N_\omega)$;
- relative errors $\frac{\left|\left|G(\jmath\omega_j) - \hat{G}(\jmath\omega_j)\right|\right|_2}{||G(\jmath\omega_j)||_2}$, $\frac{\left|\left|G(\jmath\omega_j) - \hat{G}(\jmath\omega_j)\right|\right|_\infty}{||G(\jmath\omega_j)||_\infty}$;
- "eyeball norm", i.e. look at frequency response/Bode (magnitude) plot: for SISO system, log-log plot frequency vs. $|G(\jmath\omega)|$ (or $|G(\jmath\omega) - \hat{G}(\jmath\omega)|$) in decibels, 1 dB $\simeq 20\log_{10}$(value).
- For MIMO systems, $G \in \mathbb{C}^{q\times m}$, plot G_{ij}.

Computable error measures





References

1. A.C. Antoulas.
Approximation of Large-Scale Dynamical Systems.
SIAM Publications, Philadelphia, PA, 2005.

A brief summary

Questions you should be able to answer after learning Lecture 1-5:

- 1. What is a linear time invariant (LTI) system? Standard form? Generalized form?
- 2. What is the transfer function of an LTI system?
- 3. What is the SVD of a matrix? truncated SVD?
- 4. What is a stable LTI system?
- 5. What is a realization of an LTI system? Is the realization unique?
- 6. What is the state-space transformation of an LTI system?
- 7. What is the sufficient and necessary condition of “a system is minimal”?
- 8. What is the controllability (reachability) matrix? observability matrix?
- 9. What is the controllability Gramian? Observability Gramian?
- 10. What is a sufficient condition of “a system is controllable (reachable)”?

A brief summary

- 11. What is a sufficient condition of “a system is observable”?
- 12. What condition must the reachability matrix satisfy if a system is controllable?
- 13. What condition must the observability matrix must satisfy if a system is observable?
- 14. What condition must the reachability Gramian satisfy if a system is controllable?
- 15. What condition must the observability Gramian must satisfy if a system is observable?
- 16. Under what condition do the infinite Gramians exist?
- 17. What equations do the infinite Gramians satisfy?
- 18. What is the definition of HSVs of an LTI system?
- 19. What is the relation between a state that is difficult to reach/observe and the eigenvectors of the infinite Gramians?
- 20. What is a balanced system?