Model Reduction for Dynamical Systems
–Lecture 9–
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1. Linear parametric systems

2. PMOR based on Multi-moment matching

3. A Robust Algorithm

4. IRKA based PMOR

5. Steady systems

6. Extension to nonlinearities
Example 1

A microthruster

Upper-left\(^1\): the structure of an array of pyrotechnical thrusters. Lower-right: the structure of a 2D-axisymmetric model.

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\(^1\) The picture is taken from [Rossi05], we acknowledge the author’s permission for using the picture.
Example 1

- When the PolySilicon (green) in the middle is excited by a current, the fuel below is ignited and the explosion will occur through the nozzle.

- The thermal process can be modeled by a heat transfer partial differential equation, while the heat exchange through device interfaces is modeled by convection boundary conditions with different film coefficients $h_t, h_s, h_b$.

- The film coefficients $h_t, h_s, h_b$ respectively describe the heat exchange on the top, side, and bottom of the microthruster with the outside surroundings. The values of the film coefficients can change from 1 to $10^9$. 
After finite element discretization of the 2D-axisymmetric model, a parameterized system is derived,

\[ E \dot{x} = (A - h_t A_t - h_s A_s - h_b A_b)x + B \]
\[ y = Cx. \] (1)

Here, \( h_t, h_s, h_b \) are the parameters and the dimension of the system is \( n = 4,257 \). We observe the temperature at the center of the PolySilicon heater changing with time and the film coefficient, which defines the output of the system\(^2\).

\(^2\)Detailed description of the parameterized system can be find at http://simulation.uni-freiburg.de/downloads/benchmark
Example 2

The second example is a butterfly gyroscope. The parameterized system is obtained by finite element discretization of the model for the gyroscope (The details of the model can be found in [Moosmann07]).

![Scheme of the butterfly gyroscope](image-url)
Example 2

- The paddles of the device are excited to a vibration $z(t)$, where all paddles vibrate in phase. With the external rotation $\phi$, the Coriolis force acts upon the paddles, which causes an out-of-phase movement measured as the $z$-displacement difference $\delta z$ between the two red dotted nodes.

- The interesting output of the system is $\delta z$, the difference of the displacement $z(t)$ between the two end nodes depicted as red dots on the same side of the bearing.
The system is of the following form:

\[
M(d)\ddot{x} + D(\theta, \alpha, \beta, d)\dot{x} + T(d)x = Bu(t) \\
y = Cx.
\]  
(2)

- \(M(d) = (M_1 + dM_2)\), \(D(\theta, \alpha, \beta, d) = \theta(D_1 + dD_2) + \alpha M(d) + \beta T(d)\), and \(T(d) = (T_1 + \frac{1}{d} T_2 + dT_3)\).
- Parameters \(d, \theta, \alpha, \beta\). \(d\) is the width of the bearing, and \(\theta\) is the rotation velocity along the \(x\) axis. \(\alpha, \beta\) are used to form the Rayleigh damping matrices \(\alpha M(d), \beta T(d)\) in \(D(\theta, \alpha, \beta, d)\).
- The dimension of the system is \(n = 17913\).
Example 3

The third example is a silicon-nitride membrane. This structure resembles a micro-hotplate similar to other micro-fabricated devices such as gas sensors [GrafBT04] and infrared sources [SpannSH05].

Temperature distribution over the silicon-nitride membrane.

\[^3\]Picture courtesy of T. Bechtold, IMTEK, University of Freiburg, Germany.
The model of the silicon-nitride membrane is a system with four parameters [BechtoldHRG10].

\[ (E_0 + \rho c_p E_1)\dot{x} + (K_0 + \kappa K_1 + hK_2)x = Bu(t) \]
\[ y = Cx. \]  

(3)

- The mass density \( \rho \) in kg/m\(^3\), the specific heat capacity \( c_p \) in J/kg/K, the thermal conductivity in W/m/K, and the heat transfer coefficient \( h \) in W/m\(^2\)/K.
- The dimension of the system is \( n = 60020 \).
PMOR based on Multi-moment matching

In frequency domain

Using Laplace transform, the system in time domain is transformed into

\[ E(s_1, \ldots, s_p)x = Bu(s_p), \]
\[ y = L^T x, \tag{4} \]

where the matrix \( E \in \mathbb{R}^{n \times n} \) is parametrized. The new parameter \( s_p \) is in fact the frequency parameter \( s \), which corresponds to time \( t \).

In case of a nonlinear and/or non-affine dependence of the matrix \( E \) on the parameters, the system in (4) is first transformed to an affine form

\[ (E_0 + \tilde{s}_1 E_1 + \tilde{s}_2 E_2 + \ldots + \tilde{s}_p E_p)x = Bu(s_p), \]
\[ y = L^T x. \tag{5} \]

Here the newly defined parameters \( \tilde{s}_i, i = 1, \ldots, p \), might be some functions (rational, polynomial) of the original parameters \( s_i \) in (4).

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Model Reduction for Dynamical Systems
To obtain the projection matrix $V$ for the reduced model, the state $x$ in (5) is expanded into a Taylor series at an expansion point $\tilde{s}_0 = (\tilde{s}_1^0, \ldots, \tilde{s}_p^0)^T$ as below,

$$
x = [I - (\sigma_1 M_1 + \ldots + \sigma_p M_p)]^{-1} \tilde{E}^{-1} B u(s_p) = \sum_{m=0}^{\infty} \frac{[\sigma_1 M_1 + \ldots + \sigma_p M_p]^m \tilde{E}^{-1} B u(s_p)}{m!} \\
= \sum_{m=0}^{\infty} \sum_{k_2=0}^{m-1} \sum_{k_{p-1}=0}^{m-k_{p-1}} F_{k_2,\ldots,k_p}^m (M_1, \ldots, M_p)
$$

where $\sigma_i = \tilde{s}_i - \tilde{s}_i^0$, $\tilde{E} = E_0 + \tilde{s}_1^0 E_1 + \ldots + \tilde{s}_p^0 E_p$, $M_i = -\tilde{E}^{-1} E_i$, $i = 1, 2, \ldots p$, and $B_M = \tilde{E}^{-1} B$.  

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Model Reduction for Dynamical Systems  
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- $\sigma^0$: $L^TB_M$: the 0th order multi-moment; the columns in $B_M$: the 0th order moment vectors.
- $\sigma^1$: $L^TM_iB_M$, $i = 1, 2, \ldots, p$: the first order multi-moments; the columns in $M_iB_M$, $i = 1, 2, \ldots, p$: the first order moment vectors.
- $\sigma^2$: the columns in $M_i^2B_M$, $i = 1, 2, \ldots, p$, $(M_1M_i + M_iM_1)B_M$, $i = 2, \ldots, p$, $(M_2M_i + M_iM_2)B_M$, $i = 3, \ldots, p$, \ldots, $(M_{p-1}M_p + M_pM_{p-1})B_M$: the second order moment vectors.
- \ldots

Since the coefficients corresponding not only to $s = s_p$, but also to those associated with the other parameters $s_i$, $i = 1, \ldots, p - 1$ are, we call them as **multi-moments** of the transfer function.
For the general case, the projection matrix $V$ is constructed as

$$\text{range } \{ V \}$$

$$= \text{colspan} \left\{ \bigcup_{m=0}^{m_q} \bigcup_{k_2=0}^{m-(k_p+...+k_3)} \ldots \bigcup_{k_p-1=0}^{m-k_p} \bigcup_{k_p=0}^{m} F^m_{k_2,\ldots,k_p} (M_1, \ldots, M_p) B_M \right\}$$

$$= \text{colspan} \{ B_M, M_1 B_M, M_2 B_M, \ldots, M_p B_M, (M_1)^2 B_M, (M_1 M_2 + M_2 M_1) B_M, \ldots, (M_1 M_p + M_p M_1) B_M, (M_2)^2 B_M, (M_2 M_3 + M_3 M_2) B_M, \ldots \}.$$
By observing the power series expansion of $x$ in (6), we get the following equivalent, but different formulation,

$$x = [I - (\sigma_1 M_1 + \ldots + \sigma_p M_p)]^{-1} \tilde{E}^{-1} Bu$$

$$= \sum_{m=0}^{\infty} [\sigma_1 M_1 + \ldots + \sigma_p M_p]^m B_M u$$

$$= B_M u + [\sigma_1 M_1 + \ldots + \sigma_p M_p] B_M u$$

$$+ [\sigma_1 M_1 + \ldots + \sigma_p M_p]^2 B_M u + \ldots$$

$$+ [\sigma_1 M_1 + \ldots + \sigma_p M_p]^j B_M u + \ldots$$

(8)

By defining

$$x_0 = B_M,$$

$$x_1 = [\sigma_1 M_1 + \ldots + \sigma_p M_p] B_M,$$

$$x_2 = [\sigma_1 M_1 + \ldots + \sigma_p M_p]^2 B_M, \ldots,$$

$$x_j = [\sigma_1 M_1 + \ldots + \sigma_p M_p]^j B_M, \ldots,$$

we have $x = (x_0 + x_1 + x_2 + \ldots + x_j + \ldots) u$ and obtain the recursive relations
A Robust Algorithm

\[ x_0 = B_M, \]
\[ x_1 = [\sigma_1 M_1 + \ldots + \sigma_p M_p] x_0, \]
\[ x_2 = [\sigma_1 M_1 + \ldots + \sigma_p M_p] x_1, \ldots \]
\[ x_j = [\sigma_1 M_1 + \ldots + \sigma_p M_p] x_{j-1}, \ldots \]

If we define a vector sequence based on the coefficient matrices of \( x_j, \ j = 0, 1, \ldots \) as below,

\[ R_0 = B_M, \]
\[ R_1 = [M_1 R_0, M_2 R_0, \ldots, M_p R_0], \]
\[ R_2 = [M_1 R_1, M_2 R_1, \ldots, M_p R_1], \]
\[ \vdots \]
\[ R_j = [M_1 R_{j-1}, M_2 R_{j-1}, \ldots, M_p R_{j-1}], \]
\[ \vdots \]
and let $R$ be the subspace spanned by the vectors in $R_j$, $j = 0, 1, \ldots, m$:

$$R = \text{colspan}\{R_0, \ldots, R_j, \ldots, R_m\},$$

then there exists $z \in \mathbb{R}^q$, such that $x \approx Vz$. Here the columns in $V \in \mathbb{R}^{n \times q}$ is a basis of $R$. We see that the terms in $R_j$, $j = 0, 1, \ldots, m$ are the coefficients of the parameters in the series expansion (8). They are also the $j$-th order moment vectors.

How to compute an orthonormal basis $V$?
A Robust Algorithm

Algorithm 1: Compute $V = [v_1, v_2, \ldots, v_q]$ [Benner, Feng’14]

1. Initialize $a_1 = 0$, $a_2 = 0$, $sum = 0$.
2. Compute $R_0 = \tilde{E}^{-1}B$.
3. **if** multiple input **then**
   - Orthogonalize the columns in $R_0$ using MGS: $[v_1, v_2, \ldots, v_q] = \text{orth}\{R_0\}$ with respect to a user given tolerance $\varepsilon > 0$ specifying the deflation criterion for numerically linearly dependent vectors.
   - $sum = q_1$
4. **else**
   - $v_1 = R_0/||R_0||_2$
   - $sum = 1$
5. **end if**

Compute the orthonormal columns in $R_1, R_2, \ldots, R_m$ iteratively as below:
A Robust Algorithm

continued

for $i = 1, 2, \ldots, m$ do
    $a_2 = \text{sum}$;
    for $t = 1, 2, \ldots, p$ do
        if $a_1 = a_2$ then
            stop
        else
            for $j = a_1 + 1, \ldots, a_2$ do
                $w = \tilde{E}^{-1}E_t v_j$; $\text{col} = \text{sum} + 1$;
                for $k = 1, 2, \ldots, \text{col} - 1$ do
                    $h = v_k^T w$; $w = w - hv_k$
                end for
                if $\|w\|_2 > \varepsilon$ then
                    $v_{\text{col}} = \frac{w}{\|w\|_2}$; $\text{sum} = \text{col}$;
                end if
            end for
            $a_1 = a_2$;
        end if
    end for
end for
Orthogonalize the columns in $V$ by MGS w.r.t. $\varepsilon$. 
Let $\mu = (\bar{s}_1, \ldots, \bar{s}_p)$, $\Delta(\mu)$ is an error estimation, or error bound for $\hat{x}/\hat{y}$, the state/output of the system computed from ROM.

### Greedy algorithm: Adaptive selection of the expansion points $\mu^i$

$V = []$; $\epsilon = 1$;

Initial expansion point: $\mu^0$; $i = -1$;

$\Xi_{\text{train}}$: a large set of the samples of $\mu$

**WHILE** $\epsilon > \epsilon_{\text{tol}}$

- $i = i + 1$;
- $\mu^i = \hat{\mu}$;
- Use Algorithm 1 to compute $V_i = \text{span}\{R_0, \ldots, R_q\}_{\mu^i}$;
- $V = [V, V_i]$;
- $\hat{\mu} = \arg\max_{\mu \in \Xi_{\text{train}}} \Delta(\mu)$;
- $\epsilon = \Delta(\hat{\mu})$;

**END WHILE.**
Example 1: A MEMS model with 4 parameters (benchmark available at http://modlereduction.org),

\[
M(d)\ddot{x} + D(\theta, \alpha, \beta, d)\dot{x} + T(d)x = Bu(t),
\]
\[
y = Cx.
\]

Here, \(M(d) = (M_1 + dM_2),\ T(d) = (T_1 + \frac{1}{d}T_2 + dT_3),\ 
D(\theta, \alpha, \beta, d) = \theta(D_1 + dD_2) + \alpha M(d) + \beta T(d) \in R^{n \times n}, \ n=17,913.\) Parameters, \(d, \theta, \alpha, \beta.\)
θ ∈ [10^{-7}, 10^{-5}], s ∈ 2\pi\sqrt{-1} \times [0.05, 0.25], d ∈ [1, 2].

Σ_{train}: 3 random θ, 10 random s, 5 random d, α = 0, β = 0 [Salimbahrami et al.’ 06]. Totally 150 samples of µ.

V_{\mu, i} = \text{span}\{B_M, R_1, R_2\}_{\mu, i}, i = 1, \ldots, 33. \epsilon_{tol} = 10^{-7}, \epsilon_{max} = \max_{\mu \in \Xi_{train}} |H(\mu) - \hat{H}(\mu)|, \text{ROM size}=804.
\[ V_{\mu^i} = \text{span}\{B_M, R_1\}_{\mu^i}, \ i = 1, \ldots, 36. \ \epsilon_{\text{tol}} = 10^{-7}, \ \text{ROM size}=210. \]
Example 2: a silicon nitride membrane

\[
(E_0 + \rho c_p E_1) \frac{dx}{dt} + (K_0 + \kappa K_1 + h K_2)x = bu(t) \\
y = Cx.
\]

Here, the parameters \( \rho \in [3000, 3200] \), \( c_p \in [400, 750] \), \( \kappa \in [2.5, 4] \), \( h \in [10, 12] \), \( f \in [0, 25] \) Hz

\( \Xi_{\text{train}} \): 2250 random samples have been taken for the four parameters and the frequency.

\[
\varepsilon_{\text{true}}^{\text{re}} = \max_{\mu \in \Xi_{\text{train}}} |H(\mu) - \hat{H}(\mu)|/|H(\mu)|, \quad \hat{\Delta}^{\text{re}}(\mu) = \hat{\Delta}(\mu)/|\hat{H}(\mu)|
\]

\( V_{\mu^i = \text{span}\{B_M, R_1\}}, \varepsilon_{\text{tol}}^{\text{re}} = 10^{-2}, n = 60,020, r = 8, \)

<table>
<thead>
<tr>
<th>iteration</th>
<th>( \varepsilon_{\text{true}}^{\text{re}} )</th>
<th>( \hat{\Delta}^{\text{re}}(\mu^i) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( 1 \times 10^{-3} )</td>
<td>3.44</td>
</tr>
<tr>
<td>2</td>
<td>( 1 \times 10^{-4} )</td>
<td>( 4.59 \times 10^{-2} )</td>
</tr>
<tr>
<td>3</td>
<td>( 2.80 \times 10^{-5} )</td>
<td>( 4.07 \times 10^{-2} )</td>
</tr>
<tr>
<td>4</td>
<td>( 2.58 \times 10^{-6} )</td>
<td>( 2.62 \times 10^{-5} )</td>
</tr>
</tbody>
</table>
- $\Xi_{\text{train}}$: 3 samples for $\kappa$, 10 samples for the frequency.
- $\Xi_{\text{var}}$: 16 samples for $\kappa$, 51 samples for the frequency.

Relative error of the final ROM over $\Xi_{\text{var}}$. 

![Graph showing relative error over frequency and $\kappa$]
Consider a linear parametric system

\[ C(p_1, p_2, \cdots, p_l) \frac{dx}{dt} = G(p_1, p_2, \cdots, p_l)x + B(p_1, p_2, \cdots, p_l)u(t), \]
\[ y(t) = L(p_1, p_2, \cdots, p_l)^T x, \]

where the system matrices \( C(p_1, p_2, \cdots, p_l) \), \( G(p_1, p_2, \cdots, p_l) \), \( B(p_1, p_2, \cdots, p_l) \), \( L(p_1, p_2, \cdots, p_l)^T \), are (maybe, nonlinear, non-affine) functions of the parameters \( p_1, p_2, p_l \).

A straight forward way is [Baur, et.al’11]:
Set a group of samples of \( \mu = (p_1, \ldots, p_l): \mu^0, \ldots, \mu^l \).
For each sample \( \mu^i = (p^i_1, \ldots, p^i_l), i = 1 \ldots, l \), implement IRKA to get the projection matrices \( W_i, V_i \).
The final projection matrices:

- \( \text{range}(V) = \text{orth}(V_1, \ldots, V_l) \),
- \( \text{range}(W) = \text{orth}(W_1, \ldots, W_l) \),
- \( W = W(V^TW)^{-1} \).
The reduced parametric model is:

**Parametric ROM**

\[
W^T C(p_1, p_2, \cdots, p_l) V \frac{dx}{dt} = W^T G(p_1, p_2, \cdots, p_l)Vx \\
+ W^T B(p_1, p_2, \cdots, p_l)u(t), \\
y(t) = L(p_1, p_2, \cdots, p_l)^T Vx,
\]

**Question:** How to select the samples of \( \mu \)?
Nonaffine matrices are those matrices that cannot be written as:

$$E(p_1, \ldots, p_l) = E_0 + p_1 E_1 + \ldots, p_l E_l.$$ 

- PMOR based on multi-moment-matching cannot directly deal with nonaffine case. We must first approximate with affine matrices.
- IRKA can deal with nonaffine matrices directly.
Why and How MOR for Steady systems?

Steady parametric systems

\[ E(p_1, \ldots, p_l)x = B(p_1, \ldots, p_l) \]

- Solving steady systems for multi-query tasks is also time-consuming.
- Application of PMOR based on multi-moment-matching to steady systems is straightforward.
- IRKA ?.
Applicable to nonlinear parametric systems?

Nonlinear parametric systems:

\[ f(\mu, x) = b(\mu), \]

or

\[
\begin{align*}
E(\mu) \frac{dx}{dt} & = A(\mu)x + f(\mu, x) = B(\mu)u(t), \\
y(t) & = L(\mu)^T x,
\end{align*}
\]

\[ \mu = (p_1, \ldots, p_m), \ x = x(\mu, t). \]

- PMOR based on multi-moment matching or IRKA could deal with weakly nonlinear parametric systems.
- Good candidates for MOR of general nonlinear parametric systems are POD and reduced basis methods.
- **To be introduced**: POD and reduced basis method for linear and nonlinear parametric systems.


   And many more...