Otto-von-Guericke-Universität Magdeburg
Max Planck Institute for Dynamics of Complex Technical Systems
Prof. Dr. Peter Benner, Dr. Ion Victor Gosea, Jennifer Przybilla

winter semester 2021/2022

Chemnitz University of Technology
Faculty of Mathematics
Dr. Sara Grundel, Dr. Jan Blechta

Please send your solutions (including a MATLAB® implementation if applicable) by **Nov 28, 11:59 PM (Magdeburg students), Nov 29, 11:59 PM (Chemnitz students)** to przybilla@mpi-magdeburg.mpg.de (Magdeburg students) or jan.blechta@math.tu-chemnitz.de (Chemnitz students) with subject `NLA-HW06`. Late submissions are only possible if requested by email before the due date for a valid reason.

Numerical Linear Algebra – homework #06

Problem 1 (CG)

Write the conjugate gradients (CG) algorithm that solves the linear system of equations

$$M^{-1}Ax = M^{-1}b.$$

Use the following notation at iteration j, α_j is the step, p_j is the search direction, x_j is the approximate solution, z_j is the residual j orthogonality condition parameter.
Hint. Replace the Euclidean inner product with the M-inner product.
At iteration j of your algorithm, let $r_j = b - Ax_j$.

a) What is the relation between r_j and z_j?

b) Expand the M-inner products in the algorithm (i.e., rewrite $\langle v, q\rangle_M$ as $\langle Mv, q\rangle = q^{\mathrm{T}}Mv$) and reformulate the algorithm such that no operation involving multiplication by M occurs.

Hint. Use the auxiliary vector r_j.

Define the auxiliary vectors and matrix:

$$\hat{p}_j = Lp_j$$
$$\hat{u}_j = Lx_j$$
$$\hat{r}_j = Lz_j$$
$$\hat{A} = L^{-1}AL^{-\mathrm{T}}.$$

Rewrite the algorithm by using the previous notation and compare it to the CG algorithm for solving (2) from the previous homework.

Problem 2 (Lanczos-Recursion)

Realize that the Lanczos recursion

$$AQ_k = Q_kT_k + v_{k+1}e_k^T$$

follows directly from the definition of the algorithm.

Problem 3 (CG & Lanczos)

We consider the connection between the CG method and the Lanzcos method from the lecture. Show that $\tilde{T}_k = \Delta_k S_k \Delta_k^{-1}$ is symmetric with

$$S_k = \mathrm{tridiag}\left(\frac{-1}{\alpha_{j-1}}, \frac{1}{\alpha_j} + \frac{\beta_{j-1}}{\alpha_{j-1}}, -\frac{\beta_j}{\alpha_j}\right), \qquad \Delta_k = \mathrm{diag}\left(\|r_0\|, \dots, \|r_k\|\right).$$