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Computational Methods for Systems and Control Theory

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Winterterm 2024/2025

Scientific Computing 1
Handout - Tutorial 8
October 16, 2024

Exact Flop-Count for the LU Decomposition

LU Decomposition

We consider the Gaussian Elimination in its "kij"-formulation:

Input: $A \in \mathbb{R}^{n \times n}$
Output: $L \in \mathbb{R}^{n \times n}$, $U \in \mathbb{R}^{n \times n}$
  for $k = 1 : n-1$ **do**
    $A(k+1:n,k) = A(k+1:n,k)/A(k,k)$ $\{(n-k)$ flops $\}$
    $A(k+1:n,k+1:n) = A(k+1:n,k+1:n) - A(k+1:n,k)A(k,k+1:n)$ $\{2(n-k)^2$ flops $\}$
  end for

This leads to a overall flop-count of

$$\sum_{k=1}^{n-1} \left((n-k) - 2(n-k)^2\right) = \frac{2}{3}n^3 - \frac{1}{2}n^2 - \frac{1}{6}n. \tag{1}$$

Furthermore, we know the flop-count for the following operations:

- the **forward solve** with unit-triangular $L \in \mathbb{R}^{n \times n}$ for a single right hand side: $n^2 - n$ flops,
- the **backward solve** with triangular $U \in \mathbb{R}^{n \times n}$ for a single right hand side: n^2 flops,
- and the **rank-k update** of an n-by-n matrix: $n^2 k$ flops.

Block LU Decomposition

By the knowledge of the flop-count of all building blocks we regard the Block LU decomposition with the block size r. Without loss of generality we assume $n = Nr$, $N \in \mathbb{N}$. Then the matrix A can be written as block matrix

$$A = \begin{bmatrix} A_{11} & \cdots & A_{1N} \\ \vdots & & \vdots \\ A_{N1} & \cdots & A_{NN} \end{bmatrix},$$

where each block A_{ii} is in $\mathbb{R}^{r \times r}$. Then we obtain the following formulation of the block LU decomposition:

Input: $A \in \mathbb{R}^{n\times n}$, block size r
Output: $L \in \mathbb{R}^{n\times n}$, $U \in \mathbb{R}^{n\times n}$
for $k = 1 : N$ **do**
  $\tilde{L}\tilde{U} = A_{kk}$, $L \in \mathbb{R}^{r\times r}$, $U \in \mathbb{R}^{r\times r}$ $\{\frac{2}{3}r^3 - \frac{1}{2}r^2 - \frac{1}{6}r$ flops$\}$
  Solve $\tilde{L}Z = A_{k,k+1:N}$ $\{(N-k)r \cdot (r^2 - r)$ flops$\}$
  Solve $W\tilde{U} = A_{k+1:N,k}$ $\{(N-k)r \cdot r^2$ flops$\}$
  $A(k+1:N, k+1:N) = A(k+1:N, k+1:N) - WZ$ $\{2(N-k)^2 r^2 \cdot r$ flops$\}$
end for

and so we get for the overall flop-count:

$$\sum_{k=1}^{N} \left(\frac{2}{3}r^3 - \frac{1}{2}r^2 - \frac{1}{6}r + (N-k)r \cdot (r^2 - r) + (N-k)r \cdot r^2 + 2(N-k)^2 r^2 \cdot r \right)$$
$$= \frac{2}{3}r^3N^3 - \frac{1}{2}r^2N^2 - \frac{1}{6}rN$$

and by inserting $r = \frac{n}{N}$ we obtain the same flop-count

$$\frac{2}{3}n^3 - \frac{1}{2}n^2 - \frac{1}{6}n \tag{2}$$

as for the standard LU decomposition.

$\Longrightarrow$ **The flop-count for the LU decomposition is invariant from the block size r.**