A Goal Oriented Low Rank Dual ADI Iteration for Balanced Truncation

Jens Saak
joint work with
Peter Benner and Patrick Kürschner

Max Planck Institute for Dynamics of Complex Technical Systems
Computational Methods in Systems and Control Theory
## Outline

1. Balanced Truncation in a Nutshell
2. Solving Large Lyapunov Equations
3. Existing Approaches “Simultaneously” Treating the Dual Lyapunov Equations
4. Dual LR-BT-ADI
5. Numerical Results
6. Conclusions and Future Perspectives
Balanced Truncation in a Nutshell

Basic Idea

Idea:

- The system $\Sigma$, in realization $(A, B, C)$, is called balanced, if the solutions $P, Q$ of the Lyapunov equations

\[
AP + PA^T + BB^T = 0, \quad A^T Q + QA + C^T C = 0,
\]

satisfy: $P = Q = \text{diag}(\sigma_1, \ldots, \sigma_n)$ where $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_n > 0$. 
Balanced Truncation in a Nutshell

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satisfy: $P = Q = \text{diag}(\sigma_1, \ldots, \sigma_n)$ where $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_n > 0$.

- $\{\sigma_1, \ldots, \sigma_n\}$ are the Hankel singular values (HSV) of $\Sigma$. 
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- $\{\sigma_1, \ldots, \sigma_n\}$ are the Hankel singular values (HSV) of $\Sigma$.

- A balanced realization is computed via state space transformation

\[
\mathcal{T} : (A, B, C) \mapsto (TAT^{-1}, TB, CT^{-1})
\]

\[
= \begin{pmatrix}
A_{11} & A_{12} \\
A_{21} & A_{22}
\end{pmatrix},
\begin{pmatrix}
B_1 \\
B_2
\end{pmatrix},
\begin{pmatrix}
C_1 & C_2
\end{pmatrix}.
\]
**Balanced Truncation in a Nutshell**

**Basic Idea**

- **Idea:** The system $\Sigma$, in realization $(A, B, C)$, is called balanced, if the solutions $P, Q$ of the Lyapunov equations

  \[
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  satisfy: $P = Q = \text{diag}(\sigma_1, \ldots, \sigma_n)$ where $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_n > 0$.

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- A balanced realization is computed via state space transformation

  \[\mathcal{T} : (A, B, C) \mapsto (TAT^{-1}, TB, CT^{-1})\]

  \[= \left( \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} \right).\]

- **Truncation $\leadsto$ reduced order model:** $\left(\hat{A}, \hat{B}, \hat{C}\right) = (A_{11}, B_1, C_1)$. 
Balanced Truncation in a Nutshell

Implementation

The SR Method

1. **Compute (Cholesky)factors** of the solutions to the Lyapunov equation,

   \[ P = S^T S, \quad Q = R^T R. \]
Balanced Truncation in a Nutshell

Implementation

The SR Method

1. Compute (Cholesky) factors of the solutions to the Lyapunov equation,

\[ P = S^T S, \quad Q = R^T R. \]

2. Compute singular value decomposition

\[ SR^T = [ U_1, U_2 ] \begin{bmatrix} \Sigma_1 & \Sigma_2 \\ \Sigma_2 \end{bmatrix} \begin{bmatrix} V_1^T \\ V_2^T \end{bmatrix}. \]
Balanced Truncation in a Nutshell

Implementation

The SR Method

1. Compute (Cholesky) factors of the solutions to the Lyapunov equation,

\[ P = S^T S, \quad Q = R^T R. \]

2. Compute singular value decomposition

\[ SR^T = [U_1, U_2] \begin{bmatrix} \Sigma_1 & \ \ \ \\ \ & \Sigma_2 \end{bmatrix} \begin{bmatrix} V_1^T \\ V_2^T \end{bmatrix}. \]

3. Define

\[ W := R^T V_1 \Sigma_1^{-1/2}, \quad V := S^T U_1 \Sigma_1^{-1/2}. \]

4. Then the reduced order model is \((W^T AV, W^T B, CV)\).
Lyapunov equation

Consider

\[ FX + XF^T = -GG^T \]

with \( F \in \mathbb{R}^{n \times n} \) Hurwitz, \( G \in \mathbb{R}^{n \times m} \), \( m \ll n \).

Observation in practice:

[Penzl '99, Ant./Sor./Zhou '02, Grasedyck '04]

\[ \text{rank}(X, \tau) = r \ll n \]

\[ \Rightarrow \ \text{Compute low-rank solution factor} \]

\[ \tilde{Z} \in \mathbb{R}^{n \times r}, \ r \ll n. \]

\[ X \approx \tilde{Z} \tilde{Z}^T. \]
Solving Large Lyapunov Equations

LRCF-ADI

Consider

\[ FX + XF^T = -GG^T \]

\[ F \in \mathbb{R}^{n \times n}, \ G \in \mathbb{R}^{n \times p} \]

Task

Find \( Z \in \mathbb{C}^{n, nz} \), such that \( nz \ll n \) and \( X \approx ZZ^H \)
Solving Large Lyapunov Equations

**LRCF-ADI**

Consider

\[ FX + XF^T = -GG^T \]

\[ F \in \mathbb{R}^{n \times n}, G \in \mathbb{R}^{n \times p} \]

Task

Find \( Z \in \mathbb{C}^{n,nz} \), such that \( nz \ll n \) and \( X \approx ZZ^H \)

**Algorithm**

\[
V_1 = \sqrt{-2 \text{Re}p_1(F + p_1I)^{-1}G}, \quad Z_1 = V_1 \\
V_i = \frac{\sqrt{\text{Re}p_i}}{\sqrt{\text{Re}p_{i-1}}} \left[ I - (p_i + p_{i-1})(F + p_iI)^{-1} \right] V_{i-1}, \quad Z_i = [Z_{i-1}V_i]
\]

For certain shift parameters \( \{p_1, \ldots, p_J\} \subset \mathbb{C}_{<0} \).

Stop if

- \( \|V_iV_i^H\| \) is small, or
- \( \|FZ_iZ_i^H + Z_iZ_i^HF^T + GG^T\| \) is small.
### Solving Large Lyapunov Equations

**G-LRCF-ADI (E invertible)**

Consider

\[ FXE^T + EXF^T = -GG^T \quad E, F \in \mathbb{R}^{n \times n}, G \in \mathbb{R}^{n \times p} \]

Task

Find \( Z \in \mathbb{C}^{n \times nz} \), such that \( nz \ll n \) and \( X \approx ZZ^H \)

#### Algorithm

\[
\begin{align*}
V_1 &= \sqrt{-2 \text{Re} p_1 (F + p_1 E)^{-1} G}, \quad Z_1 = V_1 \\
V_i &= \frac{\sqrt{\text{Re} p_i}}{\sqrt{\text{Re} p_{i-1}}} \left[ I - (p_i + \frac{1}{p_{i-1}})(F + p_i E)^{-1} \right] E V_{i-1} \quad Z_i = [Z_{i-1} V_i]
\end{align*}
\]

For certain shift parameters \( \{ p_1, ..., p_J \} \subset \mathbb{C}_{<0} \).

Stop if

- \( \| V_i V_i^H \| \) is small, or
- \( \| FZ_i Z_i^H E^T + EZ_i Z_i^H F^T + GG^T \| \) is small.
Solving Large Lyapunov Equations

**S-LRCF-ADI** \(((E,F)\text{ index }1)\)

Consider \(\tilde{F}XE_{11}^T + E_{11}X\tilde{F}^T = -\tilde{G}\tilde{G}^T\) \(\quad E_{11}, \tilde{F} \in \mathbb{R}^{n \times n}, \tilde{G} \in \mathbb{R}^{n \times p}\)

Task

Find \(Z \in \mathbb{C}^{n,nz}\), such that \(nz \ll n\) and \(X \approx ZZ^H\)

**Algorithm**

\[
\begin{align*}
\begin{bmatrix} V_1^* \\ \ast \end{bmatrix} &= \sqrt{-2\text{Re} p_1} \begin{bmatrix} F_{11} + p_1 E_{11} \\ F_{21} \\ F_{22} \end{bmatrix}^{-1} \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} , \\
\begin{bmatrix} V_i^* \\ \ast \end{bmatrix} &= \frac{\sqrt{\text{Re} p_i}}{\sqrt{\text{Re} p_{i-1}}} \begin{bmatrix} I - (p_i + \overline{p_{i-1}}) \begin{bmatrix} F_{11} + p_i E_{11} \\ F_{21} \\ F_{22} \end{bmatrix}^{-1} \begin{bmatrix} E_{11} V_{i-1} \\ 0 \end{bmatrix} \end{bmatrix} , \\
Z_1 &= V_1 \\
Z_i &= [Z_{i-1} V_i]
\end{align*}
\]

For certain shift parameters \(\{p_1, \ldots, p_J\} \subset \mathbb{C}_{<0}\).

Stop if

- \(\|V_i V_i^H\|\) is small, or
- \(\|\tilde{F} Z_i Z_i^H E_{11}^T + E_{11} Z_i Z_i^H \tilde{F}^T + \tilde{G} \tilde{G}^T\|\) is small.
Solving Large Lyapunov Equations

**S-LRCF-ADI**

Consider $\tilde{F} X E_{11}^T + E_{11} X \tilde{F}^T = -\tilde{G} \tilde{G}^T$, $E_{11}, \tilde{F} \in \mathbb{R}^{n \times n}, \tilde{G} \in \mathbb{R}^{n \times p}$

Task

Find $Z \in \mathbb{C}^{n,nz}$, such that $nz \ll n$ and $X \approx ZZ^H$

**Algorithm**

\[
\begin{align*}
\begin{bmatrix} V_1 \\ * \end{bmatrix} &= \sqrt{-2 \Re p_1} \begin{bmatrix} F_{11} + p_1 E_{11} \\ F_{21} \\ F_{22} \end{bmatrix}^{-1} \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \\
\begin{bmatrix} V_i \\ * \end{bmatrix} &= \frac{\sqrt{\Re p_i}}{\sqrt{\Re p_{i-1}}} \begin{bmatrix} 1 - (p_i + p_{i-1}) \\ F_{21} \\ F_{22} \end{bmatrix}^{-1} \begin{bmatrix} E_{11} V_{i-1} \\ 0 \end{bmatrix} \\
Z_1 &= V_1 \\
Z_i &= [Z_{i-1} V_i]
\end{align*}
\]

Can ensure $Z \in \mathbb{R}^{n,nz}$ even if $\{p_1, ..., p_J\} \not\subset \mathbb{R}$

see next talk by Patrick Kürschner
Solving Large Lyapunov Equations

Drawbacks of LR-ADI Based BT

- Performing LR-ADI based BT is computationally expensive compared to simple Krylov based approaches.
- Main effort is solving the dual Lyapunov equations.

Central Question

Can we reduce the time for computing the balancing transformations?
Existing Approaches “Simultaneously” Treating the Dual Lyapunov Equations

Cross Gramian based Methods e.g. [Fernando/Nicholson ’83; Sorensen/Antoulas ’02]

Idea

- Solve $AX + XA = -BC$ for the cross gramian $X$,
- Proceed with knowledge $X^2 = PQ$, i.e., for the HSV $\sigma_i = |\lambda_j(X)|$.

E.g., repeatedly solve projected equations of the form

\[
\begin{bmatrix}
A & B \\
C & D
\end{bmatrix}
+ \begin{bmatrix}
E & F \\
G & H
\end{bmatrix}
= \begin{bmatrix}
I & J \\
K & L
\end{bmatrix}
\text{ for }
\begin{bmatrix}
M & N \\
O & P
\end{bmatrix}
\]

- Requires symmetric system or symmetrizer $\Psi$, such that
  \[
  A\Psi = \Psi A^*, \quad B = \Psi C^*,
  \]
- $\Psi$ may be expensive to compute
- number of inputs and outputs must coincide.
Existing Approaches “Simultaneously” Treating the Dual Lyapunov Equations

Approximate Implicit Subspace Iteration with Alternating Directions (AISIAD) [Zhou/Sorensen ’08]

Idea

- Solve $AP + PAT = -BB^T$ and $ATQ + QA = -C^TC$ simultaneously.
- Alternate the projection spaces.

alternatingly solve 2 equations of the form

\[
\begin{align*}
\text{Red square} + \text{Red bar} &= \text{Blue bar} \\
\text{Blue square} + \text{Blue bar} &= \text{Brown bar}
\end{align*}
\]
Existing Approaches “Simultaneously” Treating the Dual Lyapunov Equations

Approximate Implicit Subspace Iteration with Alternating Directions (AISIAD) [Zhou/Sorensen ’08]

Idea

- Solve \( AP + PA^T = -BB^T \) and \( A^T Q + QA = -C^T C \) simultaneously.
- Alternate the projection spaces.

alternatingly solve 2 equations of the form
Existing Approaches “Simultaneously” Treating the Dual Lyapunov Equations

Approximate Implicit Subspace Iteration with Alternating Directions (AISIAD) [Zhou/Sorensen ’08]

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- Solve $AP + PA^T = -BB^T$ and $A^TQ + QA = -C^TC$ simultaneously.
- Alternate the projection spaces.

alternatingly solve 2 equations of the form

![Diagram showing the alternating solution process for the equations.]
Existing Approaches “Simultaneously” Treating the Dual Lyapunov Equations

Approximate Implicit Subspace Iteration with Alternating Directions (AISIAD) [Zhou/Sorensen ’08]

Idea

- Solve $AP + PA^T = -BB^T$ and $A^TQ + QA = -C^TC$ simultaneously.
- Alternate the projection spaces.

Alternatingly solve 2 equations of the form:

$$\begin{align*}
\text{Max Planck Institute Magdeburg} & \quad \text{Jens Saak, Dual ADI for Balanced Truncation}
\end{align*}$$
Existing Approaches “Simultaneously” Treating the Dual Lyapunov Equations

Approximate Implicit Subspace Iteration with Alternating Directions (AISIAD) [Zhou/Sorensen ’08]

Idea

- Solve \( AP + PA^T = -BB^T \) and \( A^T Q + QA = -C^T C \) simultaneously.
- Alternate the projection spaces.

alternatingly solve 2 equations of the form

\[
\begin{align*}
\text{After convergence use} & \quad \begin{bmatrix}1 & 1 & + & 1 & = & 1\end{bmatrix} \\
\text{and} & \quad \begin{bmatrix}1 & 1 & + & 1 & = & 1\end{bmatrix}
\end{align*}
\]

to compute balancing transformations.
**Existing Approaches “Simultaneously” Treating the Dual Lyapunov Equations**

Approximate Implicit Subspace Iteration with Alternating Directions (AISIAD)  
[Zhou/Sorensen ’08]

### Idea

- Solve $AP + P A^T = -BB^T$ and $A^T Q + QA = -C^T C$ simultaneously.
- Alternate the projection spaces.

- Initial projection basis required.
- 2 growingly expensive linear equations per step.
- Reuse of solvers for the step equations?
Dual LR-BT-ADI

Simultaneous Solution of the Dual Lyapunov Equations

Key Target

Reuse LU-factorizations in a sparse direct solver framework.

\[ V_i = \frac{\sqrt{\text{Re} p_i}}{\sqrt{\text{Re} p_{i-1}}} \left[ I - (p_i + \overline{p_{i-1}})(F + p_i E)^{-1} \right] E V_{i-1}, \]
**Key Target**

Reuse LU-factorizations in a sparse direct solver framework.

\[
V_i = \frac{\sqrt{\text{Re} p_i}}{\sqrt{\text{Re} p_{i-1}}} \left[ I - (p_i + \overline{p}_{i-1})(F + p_i E)^{-1} \right] E V_{i-1},
\]

\[
LU := (F + p_i E) \quad \Rightarrow \quad U^H L^H = (F^T + \overline{p_i} E^T)
\]
Dual LR-BT-ADI
Simultaneous Solution of the Dual Lyapunov Equations

Key Target
Reuse LU-factorizations in a sparse direct solver framework.

\[ V_i = \frac{\sqrt{\text{Re } p_i}}{\sqrt{\text{Re } p_{i-1}}} \left[ I - (p_i + \overline{p}_{i-1})U^{-1}L^{-1} \right] EV_{i-1}, \]

- complex shifts come in conjugate pair,
- they are used one after another,
- reverse order of complex conjugate shifts for the second equation.
Dual LR-BT-ADI

Simultaneous Solution of the Dual Lyapunov Equations

Key Target

Reuse LU-factorizations in a sparse direct solver framework.

\[ V_i = \frac{\sqrt{\text{Re} p_i}}{\sqrt{\text{Re} p_{i-1}}} \left[ I - (p_i + \overline{p_{i-1}})U^{-1}L^{-1} \right] EV_{i-1}, \]

\[ W_i = \frac{\sqrt{\text{Re} p_i}}{\sqrt{\text{Re} p_{i-1}}} \left[ I - (\overline{p_i} + p_{i-1})L^{-H}U^{-H} \right] E^T W_{i-1}. \]

- complex shifts come in conjugate pair,
- they are used one after another,

\[ \rightarrow \text{reverse order of complex conjugate shifts for the second equation.} \]
Dual LR-BT-ADI

Stopping the Iteration

Problem

- Number of columns in LRCFs limits ROM dimension.
- Lyapunov residuals usually totally unrelated to ROM quality.

Idea

Use goal oriented stopping criteria.
Dual LR-BT-ADI

Stopping the Iteration

**Problem**
- Number of columns in LRCFs limits ROM dimension.
- Lyapunov residuals usually totally unrelated to ROM quality.

**Idea**

*Use goal oriented stopping criteria.*

Identify and monitor the property of interest to underlying application.
**Dual LR-BT-ADI**

**Stopping the Iteration**

**Problem**
- Number of columns in LRCFs limits ROM dimension.
- Lyapunov residuals usually totally unrelated to ROM quality.

**Idea**

**Use goal oriented stopping criteria.**
Identify and monitor the property of interest to underlying application.

In Balanced Truncation MOR:

Assume we are interested in an order $k$ ROM.
Dual LR-BT-ADI

Stopping the Iteration

Problem

- Number of columns in LRCFs limits ROM dimension.
- Lyapunov residuals usually totally unrelated to ROM quality.

Idea

Use goal oriented stopping criteria.
Identify and monitor the property of interest to underlying application.

In Balanced Truncation MOR:

Assume we are interested in an order $k$ ROM.
- Monitor the relative change of $k$ leading HSVs.
- Stop when leading HSVs do no longer change.
Numerical Results
Test Examples and Hardware

Rail Model

- Oberwolfach Collection Rail Cooling\(^a\)
  - \(n = 1357\)
  - inputs 7, outputs 6

\(^a\)http://portal.uni-freiburg.de/imteksimulation/downloads/benchmark/Steel\Profiles\(38881)\n
BIPS07

- index 1 descriptor model of a power network\(^b\)
  - \(n = 13275\), \(n_{E11} = 1693\)
  - inputs 4, outputs 4
  - \(\text{condest}(A_{22}) = 8.275 \cdot 10^{26}\)

\(^b\)http://sites.google.com/site/rommes/software/bips07_1693.mat
## Numerical Results

### Test Examples and Hardware

### Hardware
- Dual Intel® Xeon® W3503 @ 2.40GHz
- Cache Size: 4MB
- RAM: 6GB

### Software
- OS: Ubuntu Linux 10.04LTS
- 32bit kernel 2.6.32-25-generic-pae
- MATLAB® 7.11.0 (R2010b)
Numerical Results

Rail Model

Leading Hankel Singular Values \( \left( \frac{\sigma_i}{\sigma_{\text{max}}} \geq \text{eps} \right) \)

Iteration: 28
Leading Hankel Singular Values ($\frac{\sigma_i}{\sigma_{\text{max}}} \geq \epsilon$)
Numerical Results

Rail Model

Leading Hankel Singular Values \( \left( \frac{\sigma_i}{\sigma_{\text{max}}} \geq \text{eps} \right) \)

- Original
- dualADI

Iteration: 30
Numerical Results

Rail Model

Leading Hankel Singular Values \( \left( \frac{\sigma_i}{\sigma_{\text{max}}} \geq \text{eps} \right) \)

- Original
- dualADI

Iteration: 31
Numerical Results

Rail Model

Leading Hankel Singular Values \( \left( \frac{\sigma_i}{\sigma_{\text{max}}} \geq \epsilon \right) \)

- Original
- dualADI

Iteration: 32
Numerical Results

Rail Model

Leading Hankel Singular Values \( \left( \frac{\sigma_i}{\sigma_{\text{max}}} \geq \text{eps} \right) \)

![Graph showing Leading Hankel Singular Values](image)

- Original
- dualADI

Iteration: 33
Numerical Results

Rail Model

Leading Hankel Singular Values ($\frac{\sigma_i}{\sigma_{\text{max}}} \geq \text{eps}$)

- Original
- dualADI

Iteration: 34
Numerical Results

Rail Model

Leading Hankel Singular Values ($\frac{\sigma_i}{\sigma_{\text{max}}} \geq \text{eps}$)

- Original
- dualADI

Iteration: 35
Numerical Results

Rail Model

Leading Hankel Singular Values \( \left( \frac{\sigma_i}{\sigma_{\max}} \geq \text{eps} \right) \)

![Graph showing leading Hankel singular values](image)

- Original
- dualADI

Iteration: 36
Numerical Results

Rail Model

Leading Hankel Singular Values ($\frac{\sigma_i}{\sigma_{\text{max}}} \geq \text{eps}$)

Original
\times dualADI

\begin{figure}
\centering
\includegraphics[width=\textwidth]{LeadingHankelSingularValues.png}
\caption{Leading Hankel Singular Values ($\frac{\sigma_i}{\sigma_{\text{max}}} \geq \text{eps}$) for Original and dualADI methods.}
\end{figure}
Numerical Results

Rail Model

Leading Hankel Singular Values \( \left( \frac{\sigma_i}{\sigma_{\text{max}}} \geq \text{eps} \right) \)

- Original
- \text{dualADI}

Iteration: 38
Numerical Results

Rail Model

Leading Hankel Singular Values ($\frac{\sigma_i}{\sigma_{\text{max}}} \geq \text{eps}$)

Iteration: 39
Numerical Results

Rail Model

Leading Hankel Singular Values \( \left( \frac{\sigma_i}{\sigma_{\text{max}}} \geq \epsilon \right) \)

- Original
- dualADI

Iteration: 40
Numerical Results

Rail Model

Leading Hankel Singular Values \( \left( \frac{\sigma_i}{\sigma_{\text{max}}} \geq \text{eps} \right) \)

- Original
- dualADI

Iteration: 41
Numerical Results

Rail Model

Leading Hankel Singular Values \( \left( \frac{\sigma_i}{\sigma_{\text{max}}} \geq \text{eps} \right) \)

![Graph showing Leading Hankel Singular Values](image)

Original

Dual ADI
Numerical Results

Rail Model

Leading Hankel Singular Values \( \left( \frac{\sigma_i}{\sigma_{\text{max}}} \geq \text{eps} \right) \)

- Original
- dualADI

Iteration: 43
**Numerical Results**

**Rail Model**

Leading Hankel Singular Values ($\frac{\sigma_i}{\sigma_{max}} \geq \text{eps}$)

![Graph showing Leading Hankel Singular Values](image)

Original

Dual ADI

Iteration: 44
Numerical Results

Rail Model

Leading Hankel Singular Values ($\frac{\sigma_i}{\sigma_{\text{max}}} \geq \text{eps}$)

![Graph showing Leading Hankel Singular Values](image)

- Original
- dualADI

Iteration: 45
Numerical Results

Rail Model

Leading Hankel Singular Values ($\frac{\sigma_i}{\sigma_{\text{max}}} \geq \text{eps}$)

- Original
- dualADI

Iteration: 46
Numerical Results

Rail Model

Leading Hankel Singular Values ($\frac{\sigma_i}{\sigma_{\text{max}}} \geq \text{eps}$)

Iteration: 47
Numerical Results

Rail Model

Leading Hankel Singular Values ($\frac{\sigma_i}{\sigma_{\text{max}}} \geq \text{eps}$)

![Graph showing leading Hankel singular values for Original and dualADI methods. The graph includes a log-log scale with iteration values on the x-axis and singular values on the y-axis. The iteration number is 48.]
Numerical Results

Rail Model

Leading Hankel Singular Values \( \left( \frac{\sigma_i}{\sigma_{\text{max}}} \geq \text{eps} \right) \)

![Graph showing leading Hankel singular values](image)

- Original
- dualADI

Iteration: 49
Numerical Results

Rail Model

Leading Hankel Singular Values \( \left( \frac{\sigma_i}{\sigma_{\max}} \geq \epsilon \right) \)

![Graph showing Leading Hankel Singular Values](image)

- Original
- dualADI

Iteration: 50
Numerical Results

Rail Model

Leading Hankel Singular Values \( \left( \frac{\sigma_i}{\sigma_{\text{max}}} \geq \text{eps} \right) \)

- Original
- dualADi

Iteration: 51
**Numerical Results**

**Rail Model**

Leading Hankel Singular Values ($\frac{\sigma_i}{\sigma_{\text{max}}} \geq \text{eps}$)

![Graph showing Leading Hankel Singular Values](image)

- **Original**
- **dualADI**

Iteration: 52
Numerical Results

Rail Model

Leading Hankel Singular Values ($\frac{\sigma_i}{\sigma_{\text{max}}} \geq \epsilon$)

![Graph showing Leading Hankel Singular Values](image)

Iteration: 53
Numerical Results

Rail Model

Leading Hankel Singular Values \( \left( \frac{\sigma_i}{\sigma_{\text{max}}} \geq \text{eps} \right) \)

![Graph showing Leading Hankel Singular Values](image)

- Original
- dualADI

Iteration: 54
Numerical Results

Rail Model

Leading Hankel Singular Values ($\frac{\sigma_i}{\sigma_{max}} \geq \text{eps}$)

Iteration: 55
Numerical Results

Rail Model

Leading Hankel Singular Values \( \frac{\sigma_i}{\sigma_{\text{max}}} \geq \text{eps} \)

- Original
- dualADI

Iteration: 56
Leading Hankel Singular Values ( $\frac{\sigma_i}{\sigma_{\text{max}}} \geq \text{eps}$)

- Original
- dualADI

Iteration: 57

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Jens Saak, *Dual ADI for Balanced Truncation*
Leading Hankel Singular Values \( \left( \frac{\sigma_i}{\sigma_{\text{max}}} \geq \text{eps} \right) \)

- Original
- dualADi

Iteration: 58
Numerical Results

Rail Model

Leading Hankel Singular Values \( \left( \frac{\sigma_i}{\sigma_{max}} \geq \text{eps} \right) \)

![Graph showing leading Hankel singular values for Original and dualADI with Iteration: 59]
Numerical Results

Rail Model

Leading Hankel Singular Values ($\frac{\sigma_i}{\sigma_{\text{max}}} \geq \text{eps}$)

Iteration: 60
Numerical Results

Rail Model

relative change of HSVs

\[ \left| \sigma - \sigma_{\text{max}}^{\text{old}} \right| \]

\[ \sigma_{\text{max}} \]

\[ 10^{-6} \]

\[ 10^{-12} \]

\[ 10^{-18} \]

\[ 10^{-24} \]

\[ 10^0 \]

\[ 10^{-6} \]

\[ 10^{-12} \]

\[ 10^{-18} \]

\[ 10^{-24} \]

\[ 20 \]

\[ 40 \]

\[ 60 \]

\[ 80 \]

\[ 100 \]

\[ 120 \]

\[ 140 \]

\[ 160 \]

Iteration: 29
Numerical Results

Rail Model

relative change of HSVs

$\|\sigma - \sigma^{\text{old}}\|$ vs $\sigma_{\text{max}}$

Iteration: 30
Numerical Results

Rail Model

relative change of HSVs

![Graph showing relative change of HSVs](image_url)

Iteration: 31
**Numerical Results**

**Rail Model**

![Graph showing relative change of HSVs](image)

Relative change of HSVs

\[
\frac{\sigma - \sigma^{(\text{old})}}{\sigma_{\text{max}}} 
\]

- Iteration: 32
Numerical Results

Rail Model

relative change of HSVs

$|\sigma - \sigma_{\text{old}}| / \sigma_{\text{max}}$

Iteration: 33
Numerical Results

Rail Model

![Graph showing the relative change of HSVs over iterations]

- **Iter.**: 34
Numerical Results

Rail Model

relative change of HSVs

\[ \sigma - \sigma \text{ (old)} \]

\[ \sigma_{\text{max}} \]

Iteration: 35
Numerical Results

Rail Model

relative change of HSVs

\[ \frac{\sigma - \sigma_{\text{old}}}{\sigma_{\max}} \]

Iteration: 36
Numerical Results

Rail Model

relative change of HSVs

\[ \left| \sigma - \sigma_{\text{old}} \right| / \sigma_{\text{max}} \]

\[ \log_{10} \left( \left| \sigma - \sigma_{\text{old}} \right| / \sigma_{\text{max}} \right) \]

Iteration: 37
Numerical Results

Rail Model

relative change of HSVs

\[\frac{|\sigma_i - \sigma_{i, \text{old}}|}{\sigma_{i, \text{max}}}\]

Iteration: 38
Numerical Results

Rail Model

relative change of HSVs

\[ \frac{\sigma_i - \sigma_i^{\text{old}}}{\sigma_{\text{max}}} \]

Iteration: 39
Numerical Results

Rail Model

relative change of HSVs

$|\sigma - \sigma_{\text{old}}| / \sigma_{\text{max}}$

$10^{0}$

$10^{-6}$

$10^{-12}$

$10^{-18}$

$10^{-24}$

20 40 60 80 100 120 140 160

$i$

Iteration: 40
Numerical Results

Rail Model

relative change of HSVs

\[ \frac{|\sigma_i - \sigma_i^{\text{old}}|}{\sigma_{\text{max}}} \]

- $10^{0}$
- $10^{-6}$
- $10^{-12}$
- $10^{-18}$
- $10^{-24}$

- $i$

Iteration: 41
Numerical Results

Rail Model

relative change of HSVs

\[ |\sigma - \sigma_{\text{old}}| / \sigma_{\text{max}} \]

Iteration: 42
Numerical Results

Rail Model

![Graph showing the relative change of HSVs](image)

Relative change of HSVs

Iteration: 43
Numerical Results

Rail Model

relative change of HSVs

\[ \frac{|\sigma - \sigma^{(\text{old})}|}{\sigma_{\text{max}}} \]

Iteration: 44
Numerical Results

Rail Model

relative change of HSVs

\[ \frac{\sigma - \sigma_{\text{old}}}{\sigma_{\max}} \]

Iteration: 45
Numerical Results

Rail Model

relative change of HSVs

![Graph showing the relative change of HSVs over iterations. The x-axis represents the iteration number (i) ranging from 20 to 160, while the y-axis represents the relative change on a logarithmic scale ranging from $10^{-24}$ to $10^0$. The graph indicates a decreasing trend in the relative change with increasing iterations. The iteration number at the lowest point is 46.](image-url)
Numerical Results

Rail Model

relative change of HSVs

\[ \frac{\sigma - \sigma_{\text{old}}}{\sigma_{\text{max}}} \]

\( i \)  Iteration: 47
**Numerical Results**

**Rail Model**

![Graph showing the relative change of HSVs](image)

- **Iteration:** 48
- **Relative change of HSVs**
  - $|\sigma - \sigma_{\text{old}}| / \sigma_{\text{max}}$ vs. iteration $i$
Numerical Results

Rail Model

relative change of HSVs

\[ \frac{\sigma - \sigma_{\text{old}}}{\sigma_{\text{max}}} \]

Graph showing the relative change of HSVs over iterations. The x-axis represents iteration number, and the y-axis shows the relative change on a logarithmic scale. The graph indicates a decreasing trend with iteration, approaching a value close to zero. Iteration: 49
Numerical Results

Rail Model

Relative change of HSVs

\[ \frac{\sigma_i - \sigma_{\text{old}}}{\sigma_{\text{max}}} \]

Iteration: 50
Numerical Results

Rail Model

relative change of HSVs

\[ \frac{\sigma - \sigma_{\text{old}}}{\sigma_{\text{max}}} \]

\[ i \]

Iteration: 51
Numerical Results

Rail Model

relative change of HSVs

\[ \frac{\sigma - \sigma^{\text{old}}}{\sigma_{\text{max}}} \]

Iteration: 52

Max Planck Institute Magdeburg

Jens Saak, *Dual ADI for Balanced Truncation*
Numerical Results

Rail Model

relative change of HSVs

$\frac{|\sigma - \sigma_{\text{old}}|}{\sigma_{\text{max}}}$

Iteration: 53
Numerical Results

Rail Model

Relative change of HSVs

\[ \frac{|\sigma - \sigma_{\text{old}}|}{\sigma_{\text{max}}} \]

Iteration: 54
Numerical Results

Rail Model

relative change of HSVs

\[ \frac{\sigma - \sigma_{\text{old}}}{\sigma_{\text{max}}} \]

Iteration: 55
Numerical Results

Rail Model

relative change of HSVs

\[
\frac{\sigma - \sigma_{\text{old}}}{\sigma_{\text{max}}} \times 10^i
\]

Iteration: 56
Numerical Results

Rail Model

relative change of HSVs

\[
\frac{\sigma_i - \sigma_{i \text{ (old)}}}{\sigma_{\max}}
\]

Iteration: 57
Numerical Results

Rail Model

relative change of HSVs

\[ \frac{|\sigma - \sigma_{\text{old}}|}{\sigma_{\text{max}}} \]

- \[ 10^{-24} \]
- \[ 10^{-20} \]
- \[ 10^{-18} \]
- \[ 10^{-16} \]
- \[ 10^{-14} \]
- \[ 10^{-12} \]
- \[ 10^{-10} \]
- \[ 10^{-8} \]
- \[ 10^{-6} \]
- \[ 10^{-4} \]
- \[ 10^{-2} \]
- \[ 1 \]

Iteration: 58
Numerical Results

Rail Model

Relative change of HSVs

\[ \frac{\sigma_i - \sigma_i^{\text{old}}}{\sigma_{\text{max}}} \]

Iteration: 59
Numerical Results

Rail Model

relative change of HSVs

\[ \frac{\sigma_i - \sigma_i^{(\text{old})}}{\sigma_{\text{max}}} \]

Iteration: 60
Numerical Results

Rail Model

Observation
Leading HSVs converge from below.

Qualitative explanation:
- LRCF-ADI is a shifted rational Arnoldi process.
- HSVs are thus approximated via Ritz values.
- Ritz values converge to the eigenvalues from the inside of the spectrum.

Quantitative analysis: work in progress.
Numerical Results

BIPS07

Test Setup

We take the same settings as in [Freitas/Rommes/Martins ’08], i.e.:

- shift parameters
  - 50 Penzl shifts
  - from 80 Ritz values and 80 harmonic Ritz values

- reduced order model dimension 32

- maximum iteration number 80

- $\alpha$-shifted S-LRCF-ADI with $\alpha = 0.05$
Numerical Results
BIPS07

Leading 32 Hankel Singular Values

- Original
- S-LRCF-ADI
- dual-S-LRCF-ADI

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Numerical Results

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Transfer functions of original and reduced systems

\[ \sigma_{\text{max}}(G(j\omega)) \]

- **original system**
- **reduced system**
- **dual reduced**
Numerical Results

BIPS07

absolute model reduction error

\[ \sigma_{\text{max}}(G(j\omega) - G_r(j\omega)) \]

- **2 S-LRCF-ADIs**
- **dual-S-LRCF-ADI**
Numerical Results

BIPS07

relative model reduction error

$\sigma_{\text{max}} \left( G(j\omega) - G_r(j\omega) \right)$

$\sigma_{\text{max}}(G(j\omega))$

$2$ S-LRCF-ADIs

dual-S-LRCF-ADI
Numerical Results

BIPS07

Residual History for S-LRCF-ADI
Numerical Results

BIPS07

Leading 32 Hankel Singular Values

Do not trust residuals!
Numerical Results

BIPS07

Leading 32 Hankel Singular Values

\[ \sigma_i \]

- Original
- S-LRCF-ADI(80)
- dual-S-LRCF-ADI(80)
- S-LRCF-ADI(15)
- dual-S-LRCF-ADI(250)

\[ i \]

Do not trust residuals!
## Numerical Results

### BIPS07

<table>
<thead>
<tr>
<th>maxiter</th>
<th>50</th>
<th>75</th>
<th>100</th>
<th>125</th>
<th>150</th>
</tr>
</thead>
<tbody>
<tr>
<td>$t_{\text{dual}} \text{ in s}$</td>
<td>2.97</td>
<td>5.22</td>
<td>7.31</td>
<td>10.33</td>
<td>14.72</td>
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<tr>
<td>$t_{\text{S-LRCF-ADI}} \text{ in s}$</td>
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Column compression via RRQR can reduce the SVD cost in the dual method:

$$\text{maxiter} = 250, \text{CC-frequency} = 20, \text{RRQR-tolerance} = \varepsilon \quad \Rightarrow \quad t_{\text{dual}} \approx 37s$$
Conclusions and Future Perspectives

Conclusions

- Simultaneous handling of the dual Lyapunov equations can drastically improve the performance.
- Goal oriented stopping criteria enhance the reliability of the results.

Future Work

- Quantify convergence results.
- Investigate reliable adaptive ROM dimension control.
- Adapt stopping criterion to finite arithmetic.
Conclusions and Future Perspectives

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Thank you very much for listening.