On ADI parameters for solving PDE control-related matrix equations

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joint work with
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Outline

Problem background in control of parabolic PDEs

Abstract Cauchy problems in Hilbert spaces

We examine optimal control problems for

convection-diffusion-reaction equations

$$\frac{\partial}{\partial t}\mathbf{x} + \nabla \cdot (\mathbf{C}(\mathbf{x}) - \mathbf{K}(\nabla \mathbf{x})) - \mathbf{Q}(\mathbf{x}) = \mathcal{B}\mathbf{u}(t), \qquad t \in [0, T_f], \qquad (1)$$

on $\Omega \in \mathbb{R}^d$, $d = 1, 2, 3$, with appropriate initial and boundary data. Here **C** is the convective part, **K** is the diffusive part and **Q** is the reactive part.

After variational formulation and linearization these can be written as

abstract Cauchy problems

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t). \qquad (2)$$

The state space $\mathcal{X}$ and the control space $\mathcal{U}$ are considered Hilbert spaces (e.g. $\mathcal{X} = H^2(\Omega)$ and $\mathcal{U} = L^2(\Omega)$ for the Dirichlet problem for the heat equation with distributed/point control on the unit square).

Problem background in control of parabolic PDEs

Abstract Cauchy problems in Hilbert spaces

Only certain measurements of $\mathbf{x}$ available as outputs $\mathbf{y} \in \mathcal{Y} \Rightarrow$

output equation

$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t). \tag{3}$$

The linear-quadratic regulator (LQR) problem for (1) can then be expressed as

LQR problem for the abstract Cauchy equation

Minimize the quadratic cost function

$$J(\mathbf{u}) = \int_0^\infty < \mathbf{y}, \mathbf{Q}\mathbf{y} >_{\mathcal{Y}} + < \mathbf{u}, \mathbf{R}\mathbf{u} >_{\mathcal{U}} \, dt, \tag{4}$$

with respect to the linear constraints (2), (3).

Problem background in control of parabolic PDEs

Abstract Cauchy problems in Hilbert spaces

[GIBSON '78, BALAKRISHNAN '77, LASIECKA/TRIGGIANI '00] discuss that under suitable conditions on $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{Q}$ and $\mathbf{R}$, $\mathbf{u}$ is given as the

optimal feedback

$$\mathbf{u} = -\mathbf{R}^{-1}\mathbf{B}^*\mathbf{P}_\infty \mathbf{x}(t), \tag{5}$$

where $\mathbf{P}_\infty = \mathbf{P}_\infty^*$ is the stabilizing solution of the

algebraic operator Riccati equation

$$0 = \mathbf{A}^*\mathbf{P} + \mathbf{P}\mathbf{A} - \mathbf{P}\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^*\mathbf{P} + \mathbf{C}\mathbf{Q}\mathbf{C} =: \mathfrak{R}(\mathbf{P}). \tag{6}$$

Note: $0 = \mathfrak{R}(\mathbf{P}) \;\Leftrightarrow\; < \mathbf{v}, \mathfrak{R}(\mathbf{P})\mathbf{w} >= 0 \quad \forall \mathbf{v}, \mathbf{w} \in \operatorname{dom} \mathbf{A}$

Problem background in control of parabolic PDEs

Semidiscretization and finite dimensional systems

Spatial semidiscretization of (1): $\mathcal{X} \rightsquigarrow \mathcal{X}_h$ in (2) and $\mathcal{Y} \rightsquigarrow \mathcal{Y}_h$ in (3) $\Rightarrow$ large scale sparse ODE system

$$\dot{x}_h = A_h x_h + B_h u, \qquad y_h = C_h x_h$$

with cost function

$$J_h(u) = \int_0^\infty < y_h, Q_h y_h > + < u, Ru > dt,$$

and u is given in feedback form as

$$u = -R^{-1} B_h^T P_h x_h$$

where P_h is the minimal nonnegative selfadjoint solution of the algebraic matrix Riccati equation (ARE)

$$0 = A_h^T P + P A_h - P B_h R^{-1} B_h^T P + C_h Q_h C_h =: \mathfrak{R}_h(P). \qquad (7)$$

Problem background in control of parabolic PDEs

Semidiscretization and finite dimensional systems

Spatial semidiscretization of (1): $\mathcal{X} \rightsquigarrow \mathcal{X}_h$ in (2) and $\mathcal{Y} \rightsquigarrow \mathcal{Y}_h$ in (3) $\Rightarrow$ large scale sparse ODE system

$$M\dot{x}_h = A_h x_h + B_h u, \qquad y_h = C_h x_h$$

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$$J_h(u) = \int_0^\infty < y_h, Q_h y_h > + < u, Ru > dt,$$

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Problem background in control of parabolic PDEs

Approximation of the ∞-dim. system

Under some natural conditions on the discretization (e.g. Galerkin scheme) it can be shown

Theorem [Banks/Kunisch'84, Lasiecka/Triggiani '00, Curtain '03]

The n-dim. problems approximate the abstract Cauchy problem:

- $P_h \Pi_h \mathbf{v} \rightarrow P\mathbf{v}$ as $h \rightarrow 0$ for all $\mathbf{v} \in \mathcal{X}$,
- $S_h(t)\Pi_h \mathbf{v} \rightarrow S(t)\mathbf{v}$ as $h \rightarrow 0$ for all $\mathbf{v} \in \mathcal{X}$,

where $S_h(t)$ and $S(t)$ are the solution semigroups generated by $(A_h - B_h R^{-1} B_h^T P_h)$, $(A - BR^{-1}B^T P)$ respectively.

Remark

Note $\mathbf{u}$ for n-dim. and ∞-dim. problems from the same function space.

- no discretization of the controls required
- simulated controls can directly be applied to the ∞-dim. system

LRCF Newton Method for the ARE

Newton's method for solving the ARE

Newton's iteration for the ARE

$$\mathfrak{R}_h'|_P(N_l) = -\mathfrak{R}_h(P_l), \qquad P_{l+1} = P_l + N_l,$$

where the Frechét derivative of $\mathfrak{R}_h$ at P is the Lyapunov operator

$$\mathfrak{R}_h'|_P : \quad Q \mapsto (A_h - B_h R^{-1} B_h^T P)^T Q + Q(A_h - B_h R^{-1} B_h^T P),$$

can be rewritten as

one iteration step

$$(A_h - B_h R^{-1} B_h^T P_l)^T P_{l+1} + P_{l+1}(A_h - B_h R^{-1} B_h^T P_l) = -C_h^T Q_h C_h - P_l B_h R^{-1} B_h^T P_l$$

i.e. in every Newton step we have to solve a

Lyapunov equation

$$F^T P + PF = -GG^T. \qquad (8)$$

LRCF Newton Method for the ARE

Cholesky factor ADI for Lyapunov equations

Peaceman Rachford ADI:

Consider $Au = s$ where $A \in \mathbb{R}^{n \times n}$ spd, $s \in \mathbb{R}^n$. ADI Iteration Idea:
Decompose $A = H + V$ with $H, V \in \mathbb{R}^{n \times n}$ such that

$$\begin{aligned}(H + pI)v &= r \\ (V + pI)w &= t\end{aligned}$$

can be solved easily.

ADI Iteration

If H, V spd $\Rightarrow \exists p_j$, $j = 1, 2, \ldots J$ such that

$$\begin{array}{rcl} u_0 &=& 0 \\ (H + p_j I)u_{j-\frac{1}{2}} &=& (p_j I - V)u_{j-1} + s \\ (V + p_j I)u_j &=& (p_j I - H)u_{j-\frac{1}{2}} + s \end{array} \tag{9}$$

converges to $u \in \mathbb{R}^n$ solving $Au = s$.

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Cholesky factor ADI for Lyapunov equations

The Lyapunov operator

$$\mathcal{L}: \quad P \quad \mapsto \quad F^T P + PF$$

can be decomposed into the linear operators

$$\mathcal{L}_H : P \mapsto F^T P \qquad \mathcal{L}_V : P \mapsto PF.$$

Such that in analogy to (4) we find the

ADI iteration for the Lyapunov equation (8)

$$\begin{aligned} P_0 &= 0 \\ (F^T + p_j I)P_{j-\frac{1}{2}} &= -GG^T - P_{j-1}(F - p_j I) \\ (F^T + p_j I)P_j^T &= -GG^T - P_{j-\frac{1}{2}}^T(F - p_j I) \end{aligned}$$

- Can be rewritten to iterate on the low rank Cholesky factors Z_j of P_j to exploit $\mathrm{rk}(P_j) \ll n$. [Li/White 2002; Penzl 1999; Benner/Li/Penzl 2000]

ADI shift parameter computation

The ADI Min-Max-problem

Optimal parameters solve the

min-max-problem

$$\min_{\{p_j\in\mathbb{R}|j=1,\dots,J\}\subset\mathbb{R}} \quad \max_{\lambda\in\sigma(H),\gamma\in\sigma(V)} \quad \left|\prod_{j=1}^{J}\frac{(p_j-\lambda)(p_j-\gamma)}{(p_j+\lambda)(p_j+\gamma)}\right|.$$

Remark

- Also known as rational Zolotarev problem since he solved it first on real intervals enclosing the spectra in 1877.
- Another solution for the real case was presented by Wachspress/Jordan in 1963.
- Wachspress and Starke presented different computational methods for the complex case around 1990.

ADI shift parameter computation

The ADI Min-Max-problem

Optimal parameters solve the

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$$\min_{\{p_j \in \mathbb{R} | j=1,\ldots,J\} \subset \mathbb{R}} \quad \max_{\lambda \in \sigma(F),} \quad \left| \prod_{j=1}^{J} \frac{(p_j - \lambda)}{(p_j + \lambda)} \right| .$$

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ADI shift parameter computation

Parameter choice

We will discuss three possible parameter choices here:

1. optimal parameters [WACHSPRESS: ADI model problem '95] :
   - Solve the min-max-problem on an elliptic functions region.
   - Computation needs knowledge of the complete spectrum of F.
2. heuristic parameters [PENZL: Lyapack '99]
   - use approximated eigenvalues as shifts
   - suboptimal $\Rightarrow$ convergence might be weak
3. semi-optimal parameters: Idea: combine the advantages of these methods
   - use Arnoldi's method to approximate the outer spectrum
   - compute the optimal parameters for this approximation

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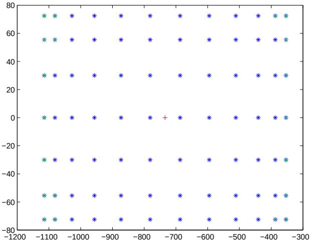
Numerical results
Parameter comparison

$$x_t = \Delta x + \begin{pmatrix} 20 \\ 0 \end{pmatrix} . \nabla x - 180x + Bu$$

on $\Omega = \{(0,1) \times (0,1)\} \times (0,\infty)$. Finite difference semidiscretization leads to a system of the form

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned}$$

where A is nonsymmetric and its spectrum is complex.



Source: Slicot Working Note 2002-2: A collection of Benchmark examples for model reduction of linear time invariant dynamical systems [Chahlaoui/Van Dooren 2002]

Numerical results
Parameter comparison

Sizes of the low rank Cholesky factors:

heuristic: m=262

optimal/ semi-optimal: m=76

all of them are real in this case.