ALAMA-GAMM-ANLA Meeting,
July 14-16, 2014, Barcelona

On an inexact Newton-ADI solver for algebraic Riccati equations related to the LQR problem for linearized Navier-Stokes equations

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Motivation (Test Scenarios)

Scenario 1: NSE on "von Kármán Vortex Street" [Bänsch/Benner/S./Weichelt 13]

PDE: NSE

Goal: $\vec{z} = \vec{v} - \vec{w} \to 0$

$\rightsquigarrow$ Linearized Navier-Stokes equations:

$$\frac{\partial \vec{z}}{\partial t} - \frac{1}{\text{Re}}\Delta\vec{z} + (\vec{z}\cdot\nabla)\vec{w} + (\vec{w}\cdot\nabla)\vec{z} + \nabla p = 0$$

$$\operatorname{div}\vec{z} = 0$$

defined on $(0,\infty)\times\Omega$

+ boundary and initial conditions

LQR

Minimize

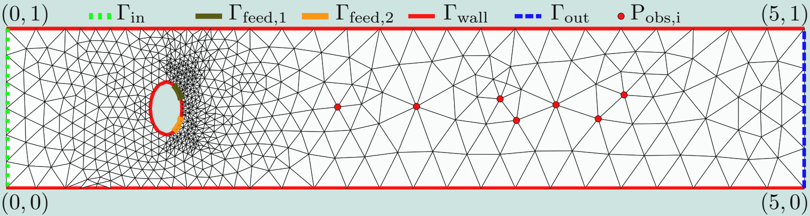
$$\mathcal{J}(\mathbf{y},\mathbf{u}) = \frac{1}{2}\int_0^\infty \lambda||\mathbf{y}||^2 + ||\mathbf{u}||^2 \, \mathrm{dt}$$

s.t.

$$\begin{bmatrix} M_v & 0 \\ 0 & 0 \end{bmatrix} \frac{d}{dt}\begin{bmatrix} \mathbf{v} \\ \mathbf{p} \end{bmatrix} = \begin{bmatrix} A_v & G \\ G^T & 0 \end{bmatrix}\begin{bmatrix} \mathbf{v} \\ \mathbf{p} \end{bmatrix} + \begin{bmatrix} B_v \\ 0 \end{bmatrix}\mathbf{u}$$

$$\mathbf{y}(t) = C_v\mathbf{v}(t)$$

Domain Ω: *von Kármán vortex street*



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Scenario 1: NSE – "von Kármán Vortex Street" [Bänsch/Benner/S./Weichelt 13]

NSE

stationary NSE

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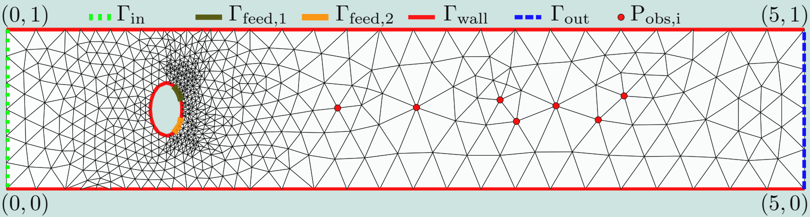

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LQR

Minimize

$$\mathcal{J}(\mathbf{y}, \mathbf{u}) = \frac{1}{2} \int_0^\infty \lambda ||\mathbf{y}||^2 + ||\mathbf{u}||^2 \, dt$$

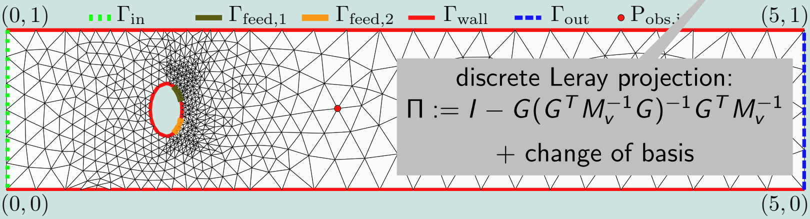
s.t.

$$\mathcal{M} \frac{d}{dt} \tilde{\mathbf{v}} = \mathcal{A} \tilde{\mathbf{v}} + \mathcal{B} \mathbf{u}$$

$$\mathbf{y}(t) = \mathcal{C} \tilde{\mathbf{v}}$$

[Heinkenschloss/Sorensen/Sun '08]

Domain Ω: *von Kármán vortex street*



discrete Leray projection:

$$\Pi := I - G(G^T M_v^{-1} G)^{-1} G^T M_v^{-1}$$

$+$ change of basis

Motivation

Scenario 2: NSE Coupled with DCE in Reactor Model [Bänsch/Benner/S./Weichelt 13-2]

PDE: NSE+DCE

Goal: $\vec{z} = \vec{v} - \vec{w} \to 0, c = c^{(\vec{v})} - c^{(\vec{w})} \to 0$

$\rightsquigarrow$ Linearized coupled system:

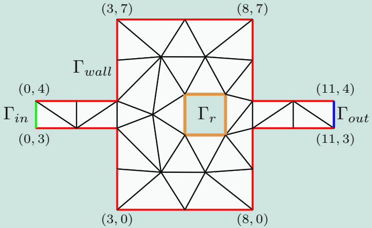
$$\frac{\partial \vec{z}}{\partial t} - \frac{1}{\mathrm{Re}} \Delta \vec{z} + (\vec{z} \cdot \nabla)\vec{w} + (\vec{w} \cdot \nabla)\vec{z} + \nabla p = 0$$

$$\frac{\partial c}{\partial t} - \frac{1}{\mathrm{Re}\,\mathrm{Sc}} \Delta c + (\vec{w} \cdot \nabla) c + (\vec{z} \cdot \nabla) c^{(\vec{w})} = 0$$

$$\operatorname{div} \vec{z} = 0$$

defined on $(0, \infty) \times \Omega$ plus BC,IC

Domain Ω: Reactor Model



LQR

Minimize

$$\mathcal{J}(\mathbf{y}, \mathbf{u}) = \frac{1}{2} \int_0^\infty \lambda ||\mathbf{y}||^2 + ||\mathbf{u}||^2 \,\mathrm{dt}$$

s.t.

$$\begin{bmatrix} M_v & 0 & 0 \\ 0 & M_c & 0 \\ 0 & 0 & 0 \end{bmatrix} \frac{d}{dt} \begin{bmatrix} \mathbf{v} \\ \mathbf{c} \\ \mathbf{p} \end{bmatrix} = \begin{bmatrix} A_v & 0 & G \\ -R & A_c & 0 \\ G^T & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{v} \\ \mathbf{c} \\ \mathbf{p} \end{bmatrix} + \begin{bmatrix} B_v \\ 0 \\ 0 \end{bmatrix} \mathbf{u}$$

$$\mathbf{y}(t) = C_c \mathbf{c}$$

Motivation

Scenario 2: NSE Coupled with DCE DCE **actor Model** [Bänsch/Benner/S./Weichelt 13-2]

stationary DCE

PDE: NSE+DCE

Goal: $\vec{z} = \vec{v} - \vec{w} \to 0$, $c = c^{(\vec{v})} - c^{(\vec{w})} \to 0$

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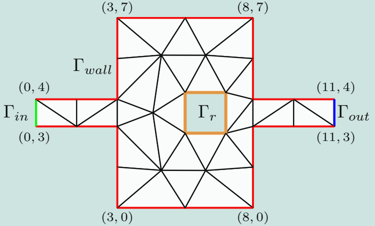

LQR

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Motivation

Scenario 2: NSE Coupled with DCE in Reactor Model [Bänsch/Benner/S./Weichelt 13-2]

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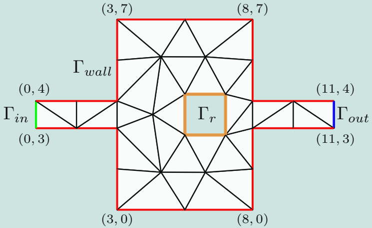

LQR

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s.t.

$$\mathcal{M} \frac{d}{dt} \begin{bmatrix} \tilde{\mathbf{v}} \\ \mathbf{c} \end{bmatrix} = \mathcal{A} \begin{bmatrix} \tilde{\mathbf{v}} \\ \mathbf{c} \end{bmatrix} + \begin{bmatrix} \mathcal{B} \\ 0 \end{bmatrix} \mathbf{u}$$

$$\mathbf{y}(t) = C_c \mathbf{c}$$

[Heinkenschloss/Sorensen/Sun '08]

Outline

Discretized Control Systems

Finite Element Discretization

- Standard FE discretization linearized (coupled) flow problems yields

$$M\frac{d}{dt}\mathbf{x}(t) = A\mathbf{x}(t) + \tilde{G}\mathbf{p}(t) + B\mathbf{u}(t) \tag{1a}$$

$$0 = G^T\mathbf{v}(t), \tag{1b}$$

$$\mathbf{y}(t) = C\mathbf{x}(t). \tag{1c}$$

Scenario 1	**Scenario 2**
$\mathbf{x}(t) = \mathbf{v}(t)$	$\mathbf{x}(t) = \begin{bmatrix} \mathbf{v}(t) \\ \mathbf{c}(t) \end{bmatrix}$
$M = M_v$	$M = \begin{bmatrix} M_v & 0 \\ 0 & M_c \end{bmatrix}$
$A = A_v$	$A = \begin{bmatrix} A_v & 0 \\ -R & A_c \end{bmatrix}$
$\tilde{G} = G$	$\tilde{G} = \begin{bmatrix} G \\ 0 \end{bmatrix}$

Discretized Control Systems

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$$\mathbf{y}(t) = C\mathbf{x}(t). \tag{1c}$$

Properties

- Differential algebraic system (DAE) of D-index 2 (iff $\tilde{G}$ has full rank).
- Matrix pencil:

$$\left(\begin{bmatrix} A & \tilde{G} \\ \tilde{G}^T & 0 \end{bmatrix}, \begin{bmatrix} M & 0 \\ 0 & 0 \end{bmatrix}\right).$$

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$\mathbf{x}(t) = \mathbf{v}(t)$	$\mathbf{x}(t) = \begin{bmatrix} \mathbf{v}(t) \\ \mathbf{c}(t) \end{bmatrix}$
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Discretized Control Systems

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$$
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0 &= G^T\mathbf{v}(t), && \text{(1b)}\\
\mathbf{y}(t) &= C\mathbf{x}(t). && \text{(1c)}
\end{aligned}
$$

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- Descriptor system with multiple inputs and outputs (MIMO).

Discretized Control Systems

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- Differential algebraic system (DAE) of D-index 2 (iff $\tilde{G}$ has full rank).
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- Descriptor system with multiple inputs and outputs (MIMO).
- Implicit index reduction to apply standard LQR approach [Heinkenschloss/Sorensen/Sun '08].

Discretized Control Systems

LQR Approach for Projected System [Bänsch/Benner/S./Weichelt 13]

Minimize

$$\mathcal{J}(\mathbf{y},\mathbf{u}) = \frac{1}{2}\int_0^\infty \lambda||\mathbf{y}||^2 + ||\mathbf{u}||^2 \,\mathrm{dt}$$

subject to

$$\begin{aligned} \mathcal{M}\frac{d}{dt}\tilde{\mathbf{x}}(t) &= \mathcal{A}\tilde{\mathbf{x}}(t) + \mathcal{B}\mathbf{u}(t), \\ \mathbf{y}(t) &= \mathcal{C}\tilde{\mathbf{x}}(t). \end{aligned} \tag{2}$$

Discretized Control Systems

LQR Approach for Projected System

[Bänsch/Benner/S./Weichelt 13]

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Riccati Based Feedback Approach (e.g.:[Locatelli '01])

- Optimal control: $\mathbf{u}(t) = -\mathcal{K}\tilde{\mathbf{x}}(t)$.
- Feedback: $\mathcal{K} = \mathcal{B}^T X \mathcal{M}$,

  where X is the solution of the generalized algebraic Riccati equation

$$\mathcal{R}(X) = \mathcal{C}^T\mathcal{C} + \mathcal{A}^T X\mathcal{M} + \mathcal{M}^T X\mathcal{A} - \mathcal{M}^T X\mathcal{B}\mathcal{B}^T X\mathcal{M} = 0.$$

Nested Iteration – Overview

Compute feedback matrix $\mathcal{K} = \mathcal{B}^T X \mathcal{M}$ with X solves:

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Kleinman-Newton method

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Step $(\mathbf{m}+\mathbf{1})$: solve Lyapunov equation [Kleinman '68]

$$(\mathcal{A} - \mathcal{B}\mathcal{K}^{(m)})^T X^{(m+1)} \mathcal{M} + \mathcal{M}^T X^{(m+1)} (\mathcal{A} - \mathcal{B}\mathcal{K}^{(m)}) = -(\mathcal{W}^{(m)})^T \mathcal{W}^{(m)}$$

Kleinman-Newton method

low rank ADI method

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low rank ADI method

Step i: solve the projected linear system [Benner/Kürschner/S. '13]

$$(\mathcal{A} - \mathcal{B}\mathcal{K}^{(m)} + q_i\mathcal{M})^T \mathcal{V}_i = \mathcal{Y}_{i-1} \qquad (3)$$

Krylov solver

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Avoid explicit projection using [Heinkenschloss/Sorensen/Sun '08]:

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low rank ADI method

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Krylov solver

Avoid explicit projection using [Heinkenschloss/Sorensen/Sun '08]:
Replace (3) and **solve instead** the saddle point system (SPS)

$$\begin{bmatrix} A^T - (K^{(m)})^T B^T + q_i M^T & \tilde{G} \\ \tilde{G}^T & 0 \end{bmatrix} \begin{bmatrix} V_i \\ * \end{bmatrix} = \begin{bmatrix} Y \\ 0 \end{bmatrix}$$

for different ADI shifts $q_i \in \mathbb{C}^-$ for a couple of rhs Y.

Nested Iteration – Overview

Compute feedback matrix $\mathcal{K} = \mathcal{B}^T X \mathcal{M}$ with X solves:

$$\mathcal{R}(X) = \mathcal{C}^T\mathcal{C} + \mathcal{A}^T X \mathcal{M} + \mathcal{M}^T X \mathcal{A} - \mathcal{M}^T X \mathcal{B}\mathcal{B}^T X \mathcal{M} = 0$$

Kleinman-Newton method

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$$(\mathcal{A} - \mathcal{B}\mathcal{K}^{(m)})^T X^{(m+1)} \mathcal{M} + \mathcal{M}^T X^{(m+1)} (\mathcal{A} - \mathcal{B}\mathcal{K}^{(m)}) = -(\mathcal{W}^{(m)})^T \mathcal{W}^{(m)}$$

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Krylov solver

Avoid explicit projection using [Heinkenschloss/Sorensen/Sun '08]:
Replace (3) and **solve instead** the saddle point system (SPS)
(using Sherman Morrison Woodbury formula)

$$\begin{bmatrix} A^T + q_i M^T & \tilde{G} \\ \tilde{G}^T & 0 \end{bmatrix} \begin{bmatrix} V_i \\ * \end{bmatrix} = \begin{bmatrix} \tilde{Y} \\ 0 \end{bmatrix}$$

for different ADI shifts $q_i \in \mathbb{C}^-$ for a couple of rhs $\tilde{Y}$.

Nested Iteration

Numerical Issues

Nested iteration depends on various parameters:

- Reynolds and Schmidt number (physical)
- ADI shifts q_i and refinement level (physical, FEM)
- regularization parameter λ (design)
- accuracy for Newton, ADI, and SPS iteration (experiences, nested influence)

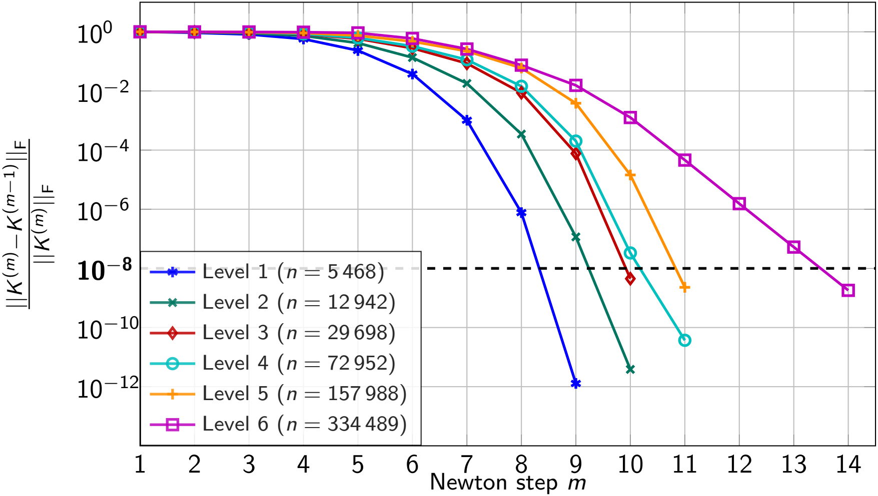
Selected Convergence Problems

- Newton-ADI vs. mesh refinement
- Newton-ADI vs. λ
- ADI vs. SPS solver

Nested Iteration

Newton-ADI vs. Mesh Refinement: Scenario 1

[Bänsch/Benner/S./Weichelt '13]

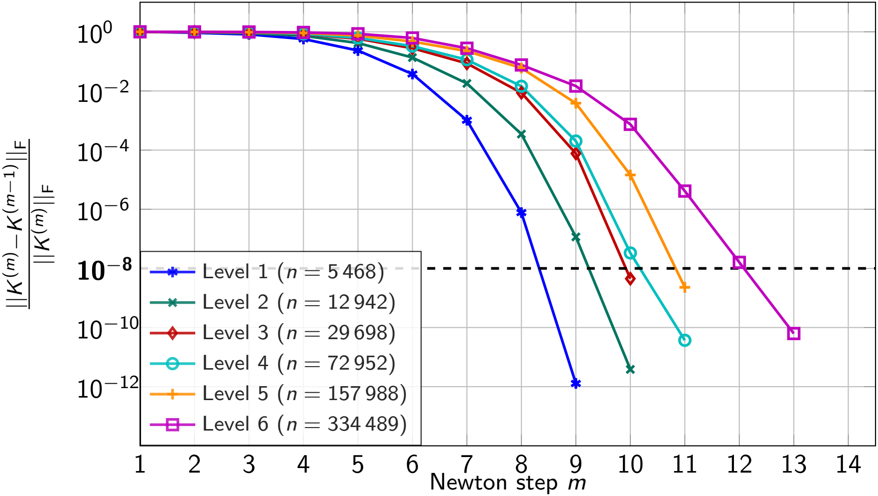


Relative change of feedback matrix K for different refinement levels

($\mathrm{Re} = 500$, $\lambda = 10^{0}$, $tol_{\mathrm{NM}} = 10^{-8}$, $\mathbf{tol_{ADI} = 10^{-7}}$).

Nested Iteration

Newton-ADI vs. Mesh Refinement: Scenario 1 [Bänsch/Benner/S./Weichelt '13]



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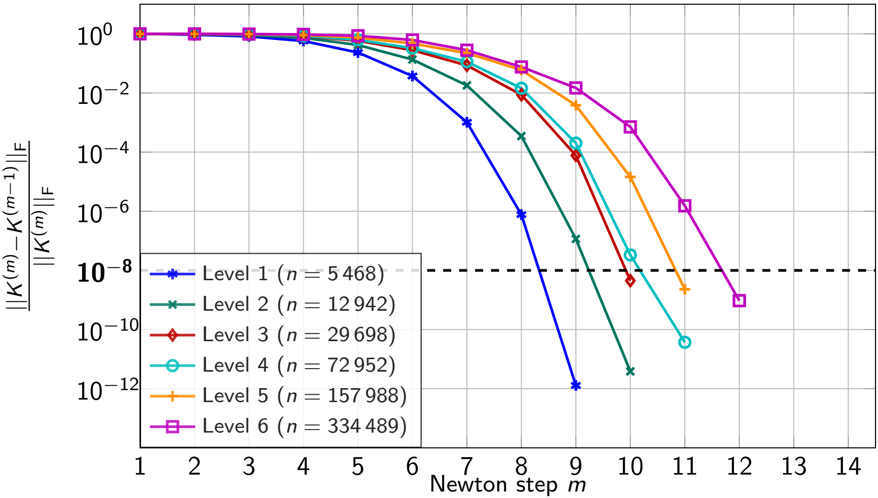
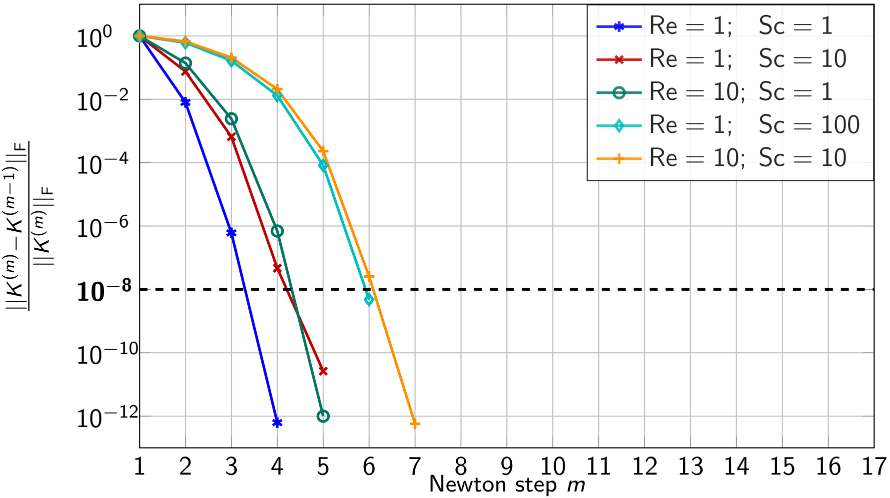


Relative change of feedback matrix K for different refinement levels

($\mathrm{Re} = 500$, $\lambda = 10^0$, $tol_{\mathrm{NM}} = 10^{-8}$, $\mathbf{tol_{ADI} = 10^{-9}}$).

Nested Iteration

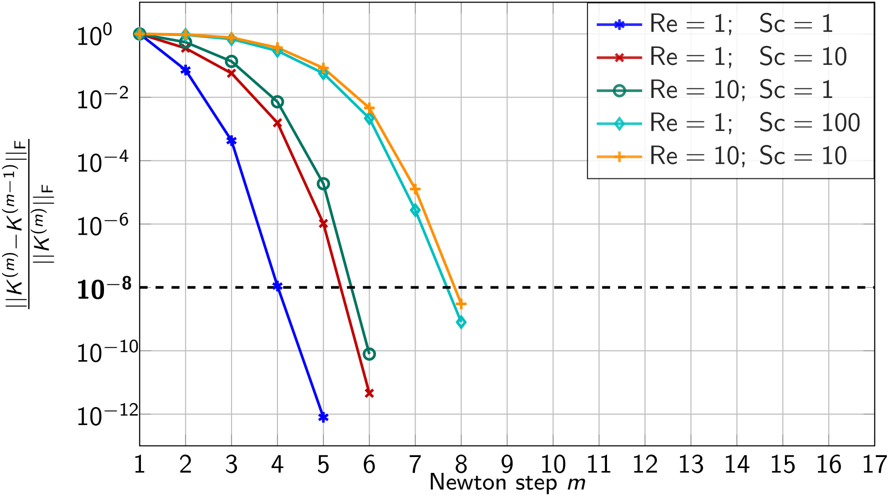
Newton-ADI vs. λ: Scenario 2 [Bänsch/Benner/S./Weichelt '13-2]



Relative change of feedback matrix K: $\lambda = \mathbf{10^{-2}}$
($n = 11\,555$, direct solver, $tol_{\mathrm{NM}} = 10^{-8}$, $tol_{\mathbf{ADI}} = 10^{-7}$).

Nested Iteration

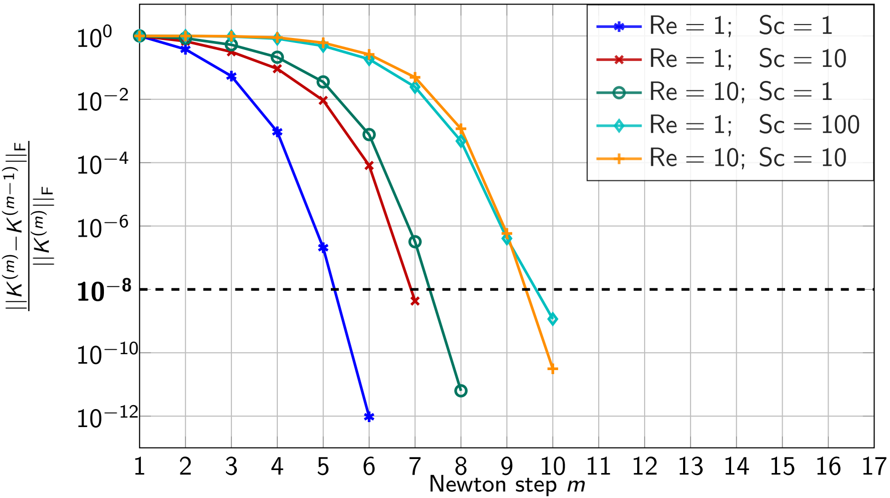
Newton-ADI vs. λ: Scenario 2 [Bänsch/Benner/S./Weichelt '13-2]



Relative change of feedback matrix K: $\lambda = \mathbf{10^{-1}}$
($n = 11\,555$, direct solver, $tol_{\mathrm{NM}} = 10^{-8}$, $tol_{\mathbf{ADI}} = 10^{-7}$).

Nested Iteration

Newton-ADI vs. λ: Scenario 2 [Bänsch/Benner/S./Weichelt '13-2]



Relative change of feedback matrix K: $\lambda = \mathbf{10^0}$
($n = 11\,555$, direct solver, $tol_{\text{NM}} = 10^{-8}$, $tol_{\textbf{ADI}} = 10^{-7}$).

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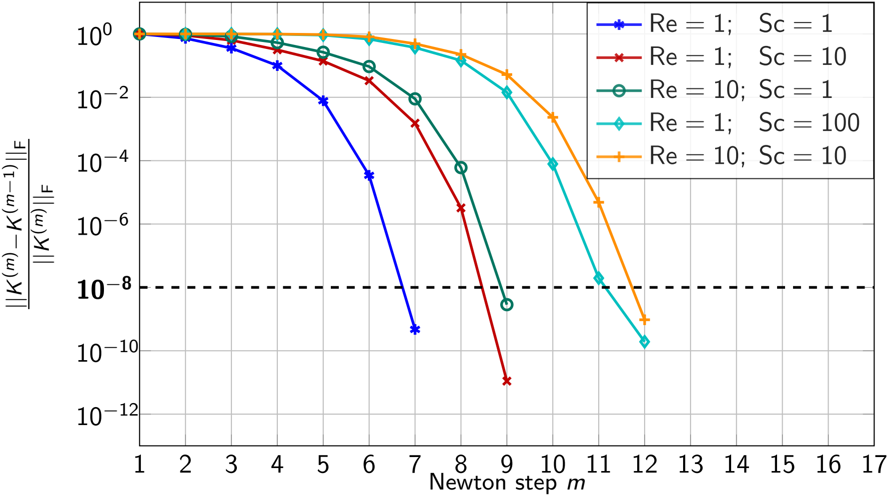

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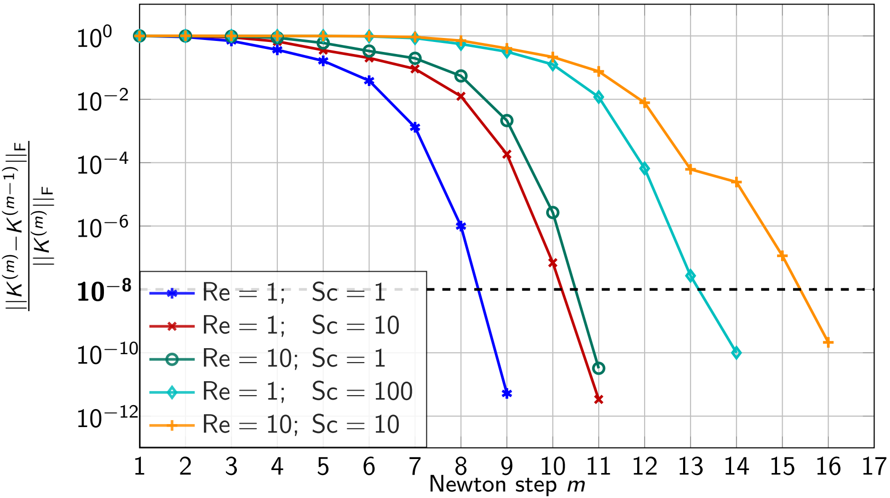

Relative change of feedback matrix K: $\lambda = \mathbf{10^2}$
($n = 11\,555$, direct solver, $tol_{\mathrm{NM}} = 10^{-8}$, $tol_{\mathbf{ADI}} = 10^{-7}$).

Nested Iteration

ADI vs. SPS solver: Stokes on Scenario 1 [Benner/S./Stoll/Weichelt '13]

tol_{SPS}	$\nu = 10^0$ n_N	n_A	time	$\nu = 10^{-1}$ n_N	n_A	time	$\nu = 10^{-2}$ n_N	n_A	time	$\nu = 10^{-3}$ n_N	n_A	time
10^{-5}	–	–	–	–	–	–	–	–	–	–	–	–
10^{-6}	–	–	–	12	343	536	17	645	1067	23	1266	2036
10^{-7}	**7**	**144**	**279**	**11**	**273**	**504**	17	525	1001	22	1004	1838
10^{-8}	7	139	304	11	247	520	**17**	**457**	**998**	**22**	**686**	**1413**
10^{-9}	7	139	342	11	247	580	17	434	1074	22	616	1437
10^{-10}	7	138	374	11	247	638	17	434	1167	22	612	1568
10^{-11}	7	138	405	11	247	693	17	434	1222	22	612	1707
10^{-12}	7	138	442	11	247	756	17	434	1312	22	606	1856
direct	7	138	/	11	247	/	17	434	/	22	606	/

Table: Number of Newton and ADI steps for varying accuracy of GMRES.

(Inexact) Newton Methods for AREs

Basic Concepts [Kleinman '68, Feitzinger/Hylla/Sachs '09]

Consider
$\mathcal{R}(X) := C^T C + \mathcal{A}^T X \mathcal{M} + \mathcal{M}^T X \mathcal{A} - \mathcal{M}^T X \mathcal{B}\mathcal{B}^T X \mathcal{M} = 0$

Inexact Kleinman's Iteration for the ARE

$$\mathcal{R}'|_{X^{(m)}}(X^{(m+1)}) - \mathcal{R}'|_{X^{(m)}}(X^{(m)}) + \mathcal{R}(X^{(m)}) = R^{(m)}, \qquad m = 0, 1, \ldots$$

i.e., in every Newton step (approximately) solve a

Lyapunov Equation

$$(\mathcal{F}^{(m)})^T X^{(m+1)} \mathcal{M} + \mathcal{M}^T X^{(m+1)} \mathcal{F}^{(m)} = -(\mathcal{W}^{(m)})^T \mathcal{W}^{(m)} + R^{(m)}.$$

(Inexact) Newtons Method for AREs

Convergence Result[Kleinman '68, Lancaster/Rodman '95, Feitzinger/Hylla/Sachs '09]

Theorem

Let Assumption 1 hold,

$$0 \le R^{(m)} \le \mathcal{C}^T\mathcal{C} \quad \text{and} \quad 0 \le R^{(m)} \le \mathcal{M}^T N^{(m)} \mathcal{B}\mathcal{B}^T N^{(m)} \mathcal{M}.$$

Then the iterates defined by

$$(\mathcal{F}^{(m)})^T X^{(m+1)} + X^{(m+1)} \mathcal{F}^{(m)} = -(\mathcal{W}^{(m)})^T \mathcal{W}^{(m)} + R^{(m)},$$

converge to the unique symmetric matrix $X^{(\infty)}$, such that

- $\mathcal{R}(X^{(\infty)}) = 0$
- *and $\mathcal{A} - \mathcal{B}\mathcal{B}^T X^{(\infty)} \mathcal{M}$ is stable.*

Furthermore the convergence is quadratic and monotone with

$$0 \le X^{(\infty)} \le \cdots \le X^{(m+1)} \le X^{(m)} \le \cdots \le X^{(1)}.$$

(Inexact) Newtons Method for AREs

Convergence Result (Remarks)

Weaker Condition [Hylla '10]

Replacing

$$R^{(m)} \leq \mathcal{C}^T\mathcal{C}$$

by

$$R^{(m)} \leq \mathcal{C}^T\mathcal{C} + (\mathcal{K}^{(m)})^T\mathcal{K}^{(m)}$$

keeps the iteration well defined.

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keeps the iteration well defined.

Large Scale Difficulty

$R^{(m)} = \mathcal{Y}_{n_A}\mathcal{Y}_{n_A}^T$, but column spans are unrelated

In general none of the conditions

- $R^{(m)} \leq \mathcal{C}^T\mathcal{C}$,
- $R^{(m)} \leq \mathcal{C}^T\mathcal{C} + (\mathcal{K}^{(m)})^T\mathcal{K}^{(m)}$,
- $0 \leq R^{(m)} \leq \mathcal{M}^T N^{(m)}\mathcal{B}\mathcal{B}^T N^{(m)}\mathcal{M}$,

can hold in large scale applications.

(Inexact) Newtons Method for AREs

Accuracy control for the (G-)LRCF-ADI

Main Problem:

Can we enforce quadratic convergence without checking

$$0 \leq R^{(m)} \leq \mathcal{C}^T\mathcal{C} \quad \text{and} \quad 0 \leq R^{(m)} \leq \mathcal{M}^T N^{(m)} \mathcal{B}\mathcal{B}^T N^{(m)} \mathcal{M}?$$

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Due to the quadratic nature of $\mathcal{R}(.)$ we have

$$\mathcal{R}(Y) = \mathcal{R}(X) + \mathcal{R}'|_X(Y - X) + \frac{1}{2}\mathcal{R}''|_X(Y - X, Y - X).$$

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Recall the Inexact Kleinman step:

$$R^{(m)} = \mathcal{R}'|_{X^{(m)}}(X^{(m+1)}) - \mathcal{R}'|_{X^{(m)}}(X^{(m)}) + \mathcal{R}(X^{(m)})$$

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and thus

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and thus

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(Inexact) Newtons Method for AREs

Accuracy control for the (G-)LRCF-ADI

Riccati residual

inner Lyapunov residual

New Question

How can we exploit $\mathcal{R}(X^{(m+1)}) = R^{(m)} - \mathcal{M}N^{(m)}\mathcal{B}\mathcal{B}^T N^{(m)}\mathcal{M}$ to control the ADI accuracy?

$$
\begin{aligned}
\mathcal{M}^T N^{(m)}\mathcal{B}\mathcal{B}^T N^{(m)}\mathcal{M} =& \mathcal{M}^T(X^{(m+1)} - X^{(m)})BB^T(X^{(m+1)} - X^{(m)})\mathcal{M} \\
=& (\mathcal{K}^{(m+1)})^T\mathcal{K}^{(m+1)} + (\mathcal{K}^{(m)})^T\mathcal{K}^{(m)} \\
&- (\mathcal{K}^{(m+1)})^T\mathcal{K}^{(m)} - (\mathcal{K}^{(m)})^T\mathcal{K}^{(m+1)} \\
=& (\Delta\mathcal{K}^{(m+1)})^T\Delta\mathcal{K}^{(m+1)}
\end{aligned}
$$

(Inexact) Newtons Method for AREs

Accuracy control for the (G-)LRCF-ADI

Riccati residual — inner Lyapunov residual

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&= (\mathcal{K}^{(m+1)})^T\mathcal{K}^{(m+1)} + (\mathcal{K}^{(m)})^T\mathcal{K}^{(m)} \\
&\quad - (\mathcal{K}^{(m+1)})^T\mathcal{K}^{(m)} - (\mathcal{K}^{(m)})^T\mathcal{K}^{(m+1)} \\
&= (\Delta\mathcal{K}^{(m+1)})^T\Delta\mathcal{K}^{(m+1)}
\end{aligned}
$$

Key Idea:

$$\|\mathcal{R}(X^{(m+1)})\| = \|R^{(m)} - (\Delta\mathcal{K}^{(m+1)})^T\Delta\mathcal{K}^{(m+1)}\| \leq \alpha\|\mathcal{R}(X^{(m)})\|^2$$

(Inexact) Newtons Method for AREs

Accuracy control for the (G-)LRCF-ADI

Overestimation approach (good steps are bad)

$$\|\mathcal{R}(X^{(m+1)})\| \leq \|R^{(m)}\| + \|(\Delta\mathcal{K}^{(m+1)})^T\Delta\mathcal{K}^{(m+1)}\| \leq \alpha\|\mathcal{R}(X^{(m)})\|^2$$

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Low-rank residual approach (several version with disadvantages)

$$\begin{aligned}\|R^{(m)} - (\Delta\mathcal{K}^{(m+1)})^T\Delta\mathcal{K}^{(m+1)}\| &= \|\mathcal{Y}^{(m)}(\mathcal{Y}^{(m)})^T - (\Delta\mathcal{K}^{(m+1)})^T\Delta\mathcal{K}^{(m+1)}\| \\ &= \|\mathcal{U}^{(m)}(\mathcal{U}^{(m)})^T\| \\ \mathcal{U}^{(m)} &= [\mathcal{Y}^{(m)}, \imath(\Delta\mathcal{K}^{(m+1)})^T]\end{aligned}$$

(Inexact) Newtons Method for AREs

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\|R^{(m)} - (\Delta\mathcal{K}^{(m+1)})^T\Delta\mathcal{K}^{(m+1)}\| &= \|\mathcal{Y}^{(m)}(\mathcal{Y}^{(m)})^T - (\Delta\mathcal{K}^{(m+1)})^T\Delta\mathcal{K}^{(m+1)}\| \\
&= \|\mathcal{U}^{(m)}(\mathcal{U}^{(m)})^T\| \\
\mathcal{U}^{(m)} &= [\mathcal{Y}^{(m)}, \imath(\Delta\mathcal{K}^{(m+1)})^T]
\end{aligned}$$

- $\|.\| = \|.\|_2 \rightsquigarrow$ eigensolver convergence?
- $\mathcal{U}$ is complex $\rightsquigarrow$ *wrong* outer product
- $\|.\| = \|.\|_F \rightsquigarrow$ inner product version reformulation may suffer from numerical cancelation?

(Inexact) Newtons Method for AREs

Accuracy Control of the SPS Solver

Exact Solution of the SPS [Benner/Kürschner/S. '13]

- $\mathcal{Z}_i^{(m)} = [\mathcal{V}_1, \mathcal{V}_2, \ldots, \mathcal{V}_i]$.
- $\operatorname{rank}\left(R^{(m)}\right) = \operatorname{rank}\left(\mathcal{Y}_i^{(m)}\right) = p + q \qquad (B \in \mathbb{R}^{n \times p},\ C \in \mathbb{R}^{q \times n}),$

(Inexact) Newtons Method for AREs

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Inexact/Iterative Solution of the SPS

- $\mathcal{Z}_i^{(m)} = [\mathcal{V}_1 + \mathcal{E}_1, \mathcal{V}_2 + \mathcal{E}_2, \ldots, \mathcal{V}_i + \mathcal{E}_i]$,
- $\operatorname{rank}\left(R^{(m)}\right) = (2i + 1) \cdot (p + q)$,
- $\mathcal{Y}_i^{(m)}$ can serve as a convergence indicator, but is not the actual residual factor.

Summary

Conclusion

- Explained idea of feedback stabilization for mulit-field flow problems.
- Recalled and adapted the concept of inexact Newtons method for the arising projected AREs.
- Discovered a gap in the theory.
- Showed possible computationally efficient criteria to use once this gap has been closed.

Outlook

- Investigate inexact Newton theory to close the gap.
- Extend ideas to the whole nested iteration, i.e. accuracy control for the SPS solvers.
- Implement and test new ideas.

Literatur

Thank you for your time!

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