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Numerical Solution of Linear Quadratic Regulator Problems under PDE Constraints

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Outline

Feedback-Control of Linear Parabolic PDEs

Feedback-Control of Linear Parabolic PDEs

Parabolic PDEs and Abstract Cauchy-Problems

Consider a control problem for a

parabolic partial differential equation

$$\frac{\partial \mathbf{x}}{\partial t} + \nabla \cdot (\mathbf{c}(\mathbf{x}) - \mathbf{k}(\nabla \mathbf{x})) + \mathbf{q}(\mathbf{x}) = \mathbf{v}(\xi, t), \quad t \in [0, T_f], \qquad \text{(PDE)}$$

on a domain $\Omega \subset \mathbb{R}^d, d = 1, 2, 3$.

Here:

- $\mathbf{q}$ uncontrolled sink or source
- $\mathbf{k}$ diffusive part
- $\mathbf{c}$ convection part

For ease of presentation we consider $T_f = \infty$.

Feedback-Control of Linear Parabolic PDEs

Parabolic PDEs and Abstract Cauchy-Problems

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on a domain $\Omega \subset \mathbb{R}^d, d = 1, 2, 3$.

Here $\mathbf{v}(\xi, t) = \mathcal{B}(\xi)\mathbf{u}(t)$

- $\mathbf{u}$ control
- $\mathcal{B}$ input operator

Feedback-Control of Linear Parabolic PDEs

Parabolic PDEs and Abstract Cauchy-Problems

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on a domain $\Omega \subset \mathbb{R}^d, d = 1, 2, 3$.

If (PDE) is linear, then a variational formulation leads to a Cauchy problem for the

linear evolution equation

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}, \qquad \mathbf{x}(0) = \mathbf{x}_0 \in \mathcal{X}.$$

Feedback-Control of Linear Parabolic PDEs

LQR Design for Abstract Cauchy Problems (Formulation)

lineare evolution equation

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}, \qquad \mathbf{x}(0) = \mathbf{x}_0 \in \mathcal{X}, \qquad \text{(Cauchy)}$$

with linear operators

$$\mathbf{A} : \operatorname{dom}(\mathbf{A}) \subset \mathcal{X} \to \mathcal{X}, \qquad \mathbf{B} : \mathcal{U} \to \mathcal{X},$$

on separable Hilbert spaces $\mathcal{X}$ (state space), $\mathcal{U} = \mathbb{R}^k$ (i.e., $\mathcal{U}$ is finite dim.).

Feedback-Control of Linear Parabolic PDEs

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output equation

$$\mathbf{y} = \mathbf{C}\mathbf{x}, \quad \text{(output)}$$

with linear operators

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on separable Hilbert spaces $\mathcal{X}$ (state space), $\mathcal{U} = \mathbb{R}^k$ (i.e., $\mathcal{U}$ is finite dim.) and $\mathcal{Y}$ (observation space).

Feedback-Control of Linear Parabolic PDEs

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$$\mathbf{y} = \mathbf{C}\mathbf{x}, \qquad \text{(output)}$$

Defining $\mathbf{Q} := \mathbf{C}^*\hat{\mathbf{Q}}\mathbf{C}$ with $\hat{\mathbf{Q}} = \hat{\mathbf{Q}}^* \geq 0$, and $\mathbf{R} = \mathbf{R}^* > 0$ we state the

cost function

$$\mathcal{J}(\mathbf{u}) = \frac{1}{2}\int_0^\infty < \hat{\mathbf{Q}}\mathbf{y}, \mathbf{y} > + < \mathbf{R}\mathbf{u}, \mathbf{u} > dt. \qquad \text{(cost)}$$

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We can now formulate the

LQR–problem.

Minimize (cost) with respect to (Cauchy).

Feedback-Control of Linear Parabolic PDEs

LQR Design for Abstract Cauchy Problems (Solution)

Well understood in the open literature:
Analogously to ODE systems case we find the

optimal state feedback

$$\mathbf{u} = -\mathbf{R}^{-1}\mathbf{B}^*\mathbf{X}_\infty\mathbf{x}.$$

Here $\mathbf{X}_\infty$ is the stabilizing, positive semidefinite, selfadjoint solution to the

Operator–Algebraic–Riccati–Equation

$$0 = \mathcal{R}(\mathbf{X}) := \mathbf{Q} + \mathbf{A}^*\mathbf{X} + \mathbf{X}\mathbf{A} - \mathbf{X}\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^*\mathbf{X}. \qquad \text{(O-ARE)}$$

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e.g. [Lions '71; Lasiecka/Triggiani '00; Bensoussan et al. '92/'06; Pritchard/Salamon '87; Curtain/Zwart '95]

Feedback-Control of Linear Parabolic PDEs

LQR Design for Abstract Cauchy Problems (Solution)

(Cauchy) can now be rewritten as

closed loop system

$$\dot{\mathbf{x}} = (\mathbf{A} - \mathbf{B}\mathbf{R}^{-1}\mathbf{B}^*\mathbf{X}_\infty)\mathbf{x},$$

and the

optimal solution trajectory

is given as

$$\mathbf{x}(t) = \mathbf{S}(t)\mathbf{x}_0,$$

where $\mathbf{S}(t)$ is the operator semigroup generated by $\mathbf{A} - \mathbf{B}\mathbf{R}^{-1}\mathbf{B}^*\mathbf{X}_\infty$.

Feedback-Control of Linear Parabolic PDEs

LQR Design for Abstract Cauchy Problems (Approximation)

Let $(\mathcal{X}_n)_{n\in\mathbb{N}}$ a Galerkin scheme for $\mathcal{X}$. We formulate the

Feedback-Control of Linear Parabolic PDEs

LQR Design for Abstract Cauchy Problems (Approximation)

Let $(\mathcal{X}_n)_{n\in\mathbb{N}}$ a Galerkin scheme for $\mathcal{X}$. We formulate the

n-d evolution equation

$$\dot{x} = A_n x + B_n \mathbf{u}, \qquad \mathcal{X}_n \ni x_n(0) = \mathbf{P}_n \mathbf{x}_0, \qquad \text{(n-d Cauchy)}$$

output equation

$$y_n = C_n x_n, \qquad \text{(n-d output)}$$

with linear operators

$$A_n : \operatorname{dom}(A_n) \subset \mathcal{X}_n \to \mathcal{X}_n, \qquad B_n : \mathcal{U} \to \mathcal{X}_n, \qquad C_n : \mathcal{X}_n \to \mathcal{Y}_n,$$

on n-d Hilbert spaces $\mathcal{X}_n$ (state space) and $\mathcal{Y}_n$ (observation space), but still $\mathcal{U} = \mathbb{R}^k$.

$\mathbf{P}_n : \mathcal{X} \to \mathcal{X}_n$ the canonical orthogonal projection.

Feedback-Control of Linear Parabolic PDEs

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$$y_n = C_n x_n, \qquad \text{(n-d output)}$$

Defining $Q_n := C_n^* \hat{Q}_n C_n$ with $\hat{Q}_n = \hat{Q}_n^* \geq 0$, and $\mathbf{R} = \mathbf{R}^* > 0$ we formulate

cost function

$$\mathcal{J}_n(\mathbf{u}) = \frac{1}{2} \int_0^\infty < \hat{Q}_n y_n, y_n > + < \mathbf{R}\mathbf{u}, \mathbf{u} > dt. \qquad \text{(n-d Cost)}$$

Feedback-Control of Linear Parabolic PDEs

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and state the

n-d LQR–problem.

Minimize (n-d Cost) with respect to (n-d Cauchy).

Feedback-Control of Linear Parabolic PDEs

LQR Design for Abstract Cauchy Problems (Approximation)

Analogously to the ∞-dim. case we now find:

optimal state feedback

$$\mathbf{u} = -\mathbf{R}^{-1} B_n^* X_n x_n,$$

where X_n is the stabilizing, positive semidefinite, selfadjoint solution to the

n-d Operator-Algebraic-Riccati-Equation

$$0 = \mathcal{R}_n(X) := Q_n + A_n^* X + X A_n - X B_n \mathbf{R}^{-1} B_n^* X. \qquad \text{(n-d O-ARE)}$$

Feedback-Control of Linear Parabolic PDEs

LQR Design for Abstract Cauchy Problems (Approximation)

As above we can write (n-d Cauchy) as

closed loop system

$$\dot{x}_n = (A_n - B_n \mathbf{R}^{-1} B_n^* X_n) x_n,$$

and the

optimal solution

is given as

$$x_n(t) = S_n(t) \mathbf{P}_n \mathbf{x}_0,$$

also again $S_n(t)$ is the operator semigroup generated by $A_n - B_n \mathbf{R}^{-1} B_n^* X_n$.

Feedback-Control of Linear Parabolic PDEs

LQR Design for Abstract Cauchy Problems (Approximation)

Approximation

The n-d LQR–problems approximate the ∞-dim LQR–problem in the following sense

- $X_n \mathbf{P}_n \mathbf{v} \to \mathbf{X}\mathbf{v}$ for $n \to \infty$ and any $\mathbf{v} \in \mathcal{X}$,
- $S_n(t)\mathbf{P}_n \mathbf{v} \to \mathbf{S}(t)\mathbf{v}$ for $n \to \infty$ and any $\mathbf{v} \in \mathcal{X}$,

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that means in the strong operator-topology.

[Banks/Kunisch'84] distributed control of parabolic PDEs

[Benner/S.'05] boundary control with mixed boundary conditions

[Lasiecka/Triggiani'00] weakens regularity conditions on (Cauchy), also has convergence rates

[Ito'87/'90; Morris'94] general Cauchy problems

[...] many more

Feedback-Control of Linear Parabolic PDEs

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that means in the strong operator-topology.

Remarks:

- For a chosen basis (e.g. from spatial FDM/FEM discretization) all n-d operators have matrix representations and $S(t)$ coincides with the matrix-exponential $e^{(A-BR^{-1}B^T X)t}$.

Feedback-Control of Linear Parabolic PDEs

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Remarks:

- For a chosen basis (e.g. from spatial FDM/FEM discretization) all n-d operators have matrix representations and $S(t)$ coincides with the matrix-exponential $e^{(A-BR^{-1}B^T X)t}$.
- $\mathbf{u}$ and $\mathbf{R}$ are always kept fixed, i.e., $\mathbf{u}$ from computations for an n-d problem can directly be applied in the ∞-d problem.
- Suboptimality can be estimated in terms of the discretization error [Benner/S.'10]

Feedback-Control of Linear Parabolic PDEs

LQR Design for Abstract Cauchy Problems (Numerics)

Main task in numerical solution

Efficient solution of the large sparse matrix–Riccati–equations

$$0 = \mathcal{R}_h(X) := Q_h + A_h^* X + X A_h - X B_h \mathbf{R}^{-1} B_h^* X, \qquad \text{(M-ARE)}$$

with regard to both memory and CPU usage.

Classical methods are not applicable due to their cubic complexity.

Feedback-Control of Linear Parabolic PDEs

LQR Design for Abstract Cauchy Problems (Numerics)

(M-ARE) is non-linear $\Rightarrow$

Newton's method for the ARE

$$\mathcal{R}_h'|_X(N_\ell) = -\mathcal{R}_h(X_\ell), \qquad X_{\ell+1} = X_\ell + N_\ell.$$

The Frechét derivative of $\mathcal{R}_h$ at X is given as the Lyapunov operator

$$\mathcal{R}_h'|_X : \quad Z \mapsto (A_h - B_h R^{-1} B_h^T X)^T Z + Z(A_h - B_h R^{-1} B_h^T X).$$

Thus we find the

Newton update

$$(A_h - B_h R^{-1} B_h^T X_\ell)^T N_{\ell+1} + N_{\ell+1}(A_h - B_h R^{-1} B_h^T X_\ell) = -\mathcal{R}(X_\ell).$$

Feedback-Control of Linear Parabolic PDEs

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Thus we find the

one step Newton-Kleinman iteration

$$(A_h - B_h R^{-1} B_h^T X_\ell)^T X_{\ell+1} + X_{\ell+1}(A_h - B_h R^{-1} B_h^T X_\ell) = -C_h^T Q_h C_h - X_\ell B_h R^{-1} B_h^T X_\ell.$$

Feedback-Control of Linear Parabolic PDEs

LQR Design for Abstract Cauchy Problems (Numerics)

In every Newton step we solve a

Lyapunov equation

$$F^T X + XF = -GG^T. \qquad \text{(Lyapunov)}$$

Feedback-Control of Linear Parabolic PDEs

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Available solvers for large sparse Lyapunov equations (Lyapunov)

ADI [Wachspress'88; Penzl'99; Benner/Li/Penzl'08; Li/White'02];

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- **Krylov** [Kasenally/Jaimoukha'94; Jbilou/Riquet'06; Simoncini et al.'06–'11]
- **Smith** [Penzl'99; Gugercin/Sorensen/Antoulas'03]
- ... many more

Feedback-Control of Linear Parabolic PDEs

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ADI needs shift–paramets; Choice of shifts: [Ellner/Wachspress'91; Penzl'00; Benner/Mena/S.'06; Starke '89; Sabino'06; Wachspress '08]

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- Smith [Penzl'99; Gugercin/Sorensen/Antoulas'03]
- ... many more

For Systems with very few inputs Newton-ADI and Newton-Smith can iterate on the feedback $K_h := \mathbf{R}^{-1} B_h^T X$ directly [Penzl'00;Banks/Ito'91].

Feedback-Control of Linear Parabolic PDEs

Differential Riccati Equations: The case $T_f < \infty$

For $T_f < \infty$ one finds ($*$-ARE) $\rightsquigarrow$ ($*$-DRE): $\frac{\partial}{\partial_t}\mathcal{R}_h(X) = -\mathcal{R}_h(X)$

Feedback-Control of Linear Parabolic PDEs

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[Mena 2007]:
ODE solvers of BDF and Rosenbrock type applicable efficiently.

Feedback-Control of Linear Parabolic PDEs

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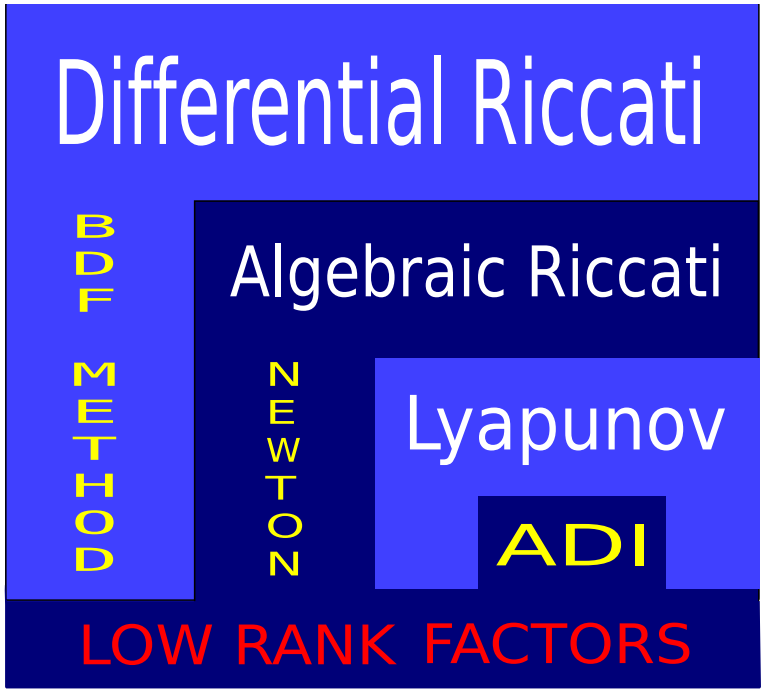

Feedback-Control of Linear Parabolic PDEs

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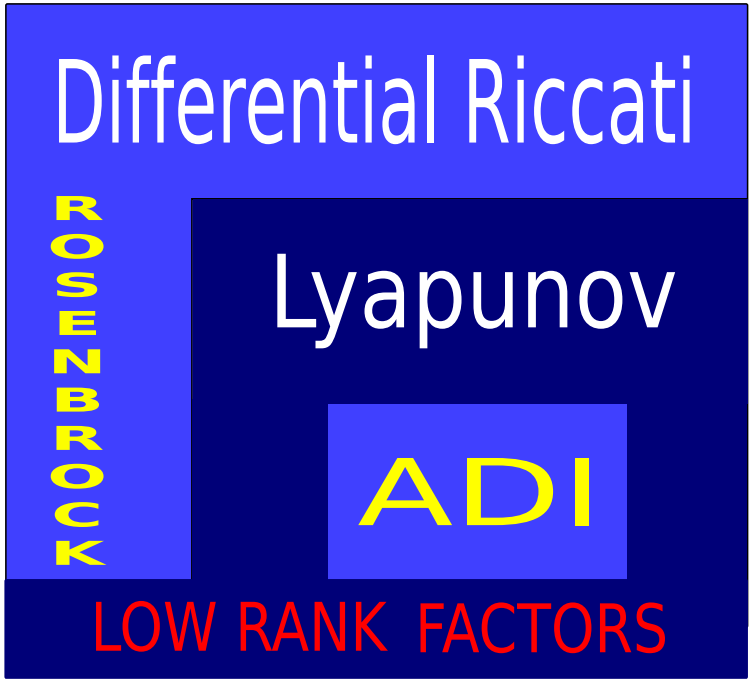

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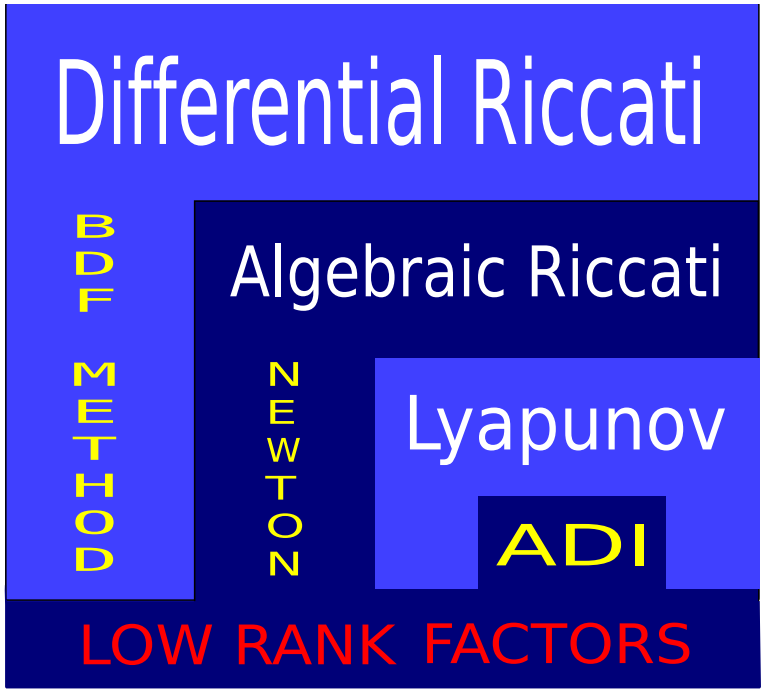

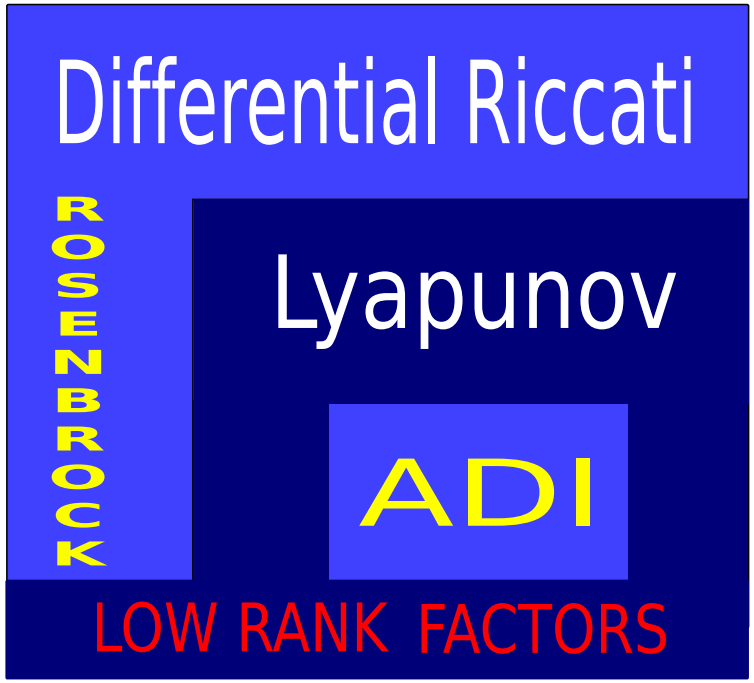

Low Rank Approximation again guarantees efficiency in terms of computational effort and memory usage

Tracking Control

Differential Riccati Equations: The case $T_f < \infty$

Tracking Control

Linear Systems with Inhomogenities

Reminder for systems with

linear inhomogeneous evolution equations

$$\dot{x} = Ax + Bu + f.$$

Let $\hat{x}$ solve the uncontrolled system $\dot{x} = Ax + f$, then

$$f = \dot{\hat{x}} - A\hat{x},$$

and

$$\dot{x} - \dot{\hat{x}} = A(x - \hat{x}) + Bu.$$

We can solve the system

$$\dot{z} = Az + Bu$$

for $z = x - \hat{x}$ to compute the control u.

---

e.g., [Godunov'97]

Tracking Control

Application I: Tracking Control of Parabolic PDEs

Consider $\tilde{\mathbf{x}}$ the state we want to track and the

tracking problem

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{v}, \qquad \mathbf{y} = \mathbf{C}(\mathbf{x} - \tilde{\mathbf{x}}),$$
$$\mathcal{J}(\mathbf{u}) = \frac{1}{2}\int_0^\infty < \mathbf{Q}(\mathbf{x} - \tilde{\mathbf{x}}), \mathbf{x} - \tilde{\mathbf{x}} > + < \mathbf{R}\mathbf{v}, \mathbf{v} > dt. \qquad \text{(tracking)}$$

Define $\mathbf{z} := \mathbf{x} - \tilde{\mathbf{x}}$ and the Cauchy problem

$$\dot{\mathbf{z}} = \mathbf{A}\mathbf{z} + \mathbf{B}\mathbf{v}, \qquad \mathbf{y} = \mathbf{C}\mathbf{z}. \tag{1}$$

The optimal control then is given as $\mathbf{v} = -\mathbf{K}\mathbf{z}$ as above and (1) is equivalent to

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} - \mathbf{B}\mathbf{K}\mathbf{x} + \dot{\tilde{\mathbf{x}}} - \mathbf{A}\tilde{\mathbf{x}} + \mathbf{B}\mathbf{K}\tilde{\mathbf{x}}.$$

Tracking Control

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$$\mathcal{J}(\mathbf{u}) = \frac{1}{2}\int_0^\infty < \mathbf{Q}(\mathbf{x} - \tilde{\mathbf{x}}), \mathbf{x} - \tilde{\mathbf{x}} > + < \mathbf{R}\mathbf{v}, \mathbf{v} > dt. \qquad \text{(tracking)}$$

- $\mathbf{f} := \dot{\tilde{\mathbf{x}}} - \mathbf{A}\tilde{\mathbf{x}} + \mathbf{B}\mathbf{K}\tilde{\mathbf{x}}$ is a known inhomogenity when solving the closed loop system.
- Equations (tracking) and (1) require the same algebraic Riccati equation.

Tracking Control

Application I: Tracking Control of Parabolic PDEs

Consider $\tilde{\mathbf{x}}$ the state we want to track and the

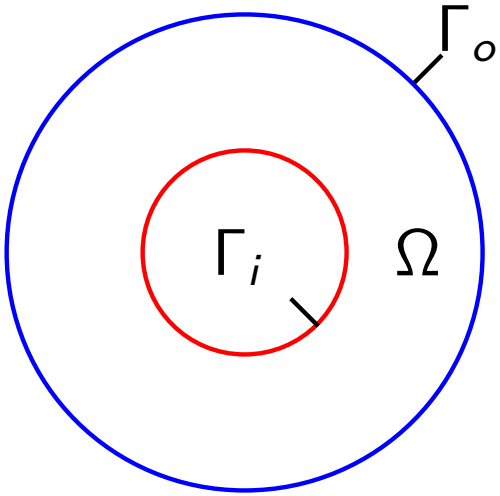
tracking problem

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{v}, \qquad \mathbf{y} = \mathbf{C}(\mathbf{x} - \tilde{\mathbf{x}}),$$
$$\mathcal{J}(\mathbf{u}) = \frac{1}{2}\int_0^\infty < \mathbf{Q}(\mathbf{x} - \tilde{\mathbf{x}}), \mathbf{x} - \tilde{\mathbf{x}} > + < \mathbf{R}\mathbf{v}, \mathbf{v} > dt. \qquad \text{(tracking)}$$

- We can compute the Feedback for (tracking) with the above technique for (1) and afterward solve the inhomogeneous closed loop system.
- Method also works for a reference pair $(\tilde{\mathbf{x}}, \tilde{\mathbf{u}})$ [Benner/Görner/S.'06]

Tracking Control

Application II: Solution of Inverse Problems with Parabolic PDEs



Consider the instationary heat equation

$$\frac{\partial}{\partial t}x(\xi,t) = \frac{\lambda}{c\rho}\Delta x(\xi,t) \qquad \text{on } \Omega \times (0,T),$$

with boundary coditions:

$$\lambda\frac{\partial}{\partial \nu}x(\xi,t) = \kappa_i(x(\xi,t) - u_i(t)) \qquad \text{on } \Gamma_i \times (0,T),$$

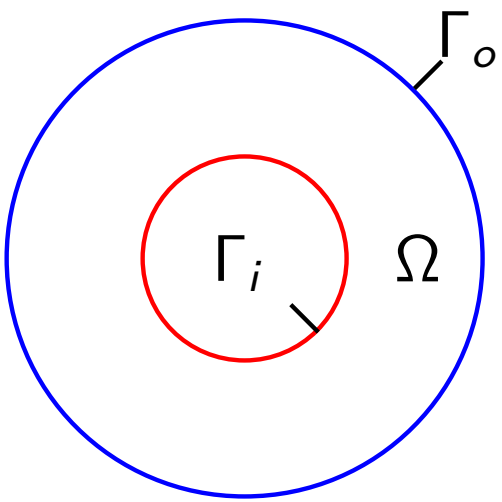
$$\lambda\frac{\partial}{\partial \nu}x(\xi,t) = \kappa_o(x(\xi,t) - x_{ext}(t)) \qquad \text{on } \Gamma_o \times (0,T),$$

Identification Problem [Benner/Lang/S.'11]

Knowing measurements $\hat{y}$ use the output $y(t) = Cx(\xi,t) \in \mathbb{R}^k$ to find the heat u_i induced via the inner boundary Γ_i.

Tracking Control

Application II: Solution of Inverse Problems with Parabolic PDEs



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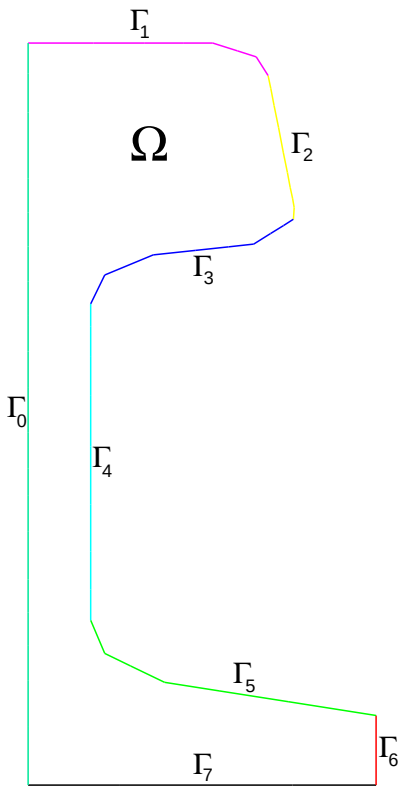
- Allowing for measurement uncertainties $\rightsquigarrow$ LQG-design, e.g., [Hein '10] (allows nonlinear treatment of material parameters).
- Combine LQG-design to identify $x(\xi,t)$ with tracking $y \to \hat{y}$.

Ongoing work in the graduation thesis of N. Lang.

Non-linear Systems

Heat Distribution in Steel Profiles, a Model Problem



Non-linear Systems

Heat Distribution in Steel Profiles, a Model Problem

The active cooling of steel profiles in a rolling facility serves as a model problem. We consider the in stationary heat equation

$$\begin{aligned} c(x)\rho(x)\tfrac{\partial}{\partial t}x(\xi,t) &= \nabla.(\lambda(x)\nabla x(\xi,t)) && \text{on } \Omega\times(0,T),\\ -\lambda(x)\tfrac{\partial}{\partial \nu}x(\xi,t) &= \kappa_i(x(\xi,t)-u_i(t)) && \text{on } \Gamma_i\times(0,T), \qquad \text{(heat)}\\ x(\xi,0) &= x_0(\xi) && \text{on } \Omega, \end{aligned}$$

x	state, temperature	$c(x)$	specific heat capacity
u	control	$\varrho(x)$	density
$T\in\mathbb{R}\cup\{\infty\}$	final time	$\lambda(x)$	heat conductivity

Non-linear Systems

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(heat) obviously is non-linear due to c, ϱ and λ depending on the temperature x.

Idea

Freeze the material parameters for one or more time steps. $\Rightarrow$ Linearization $\Rightarrow$ method from the introduction can be applied.

Non-linear Systems

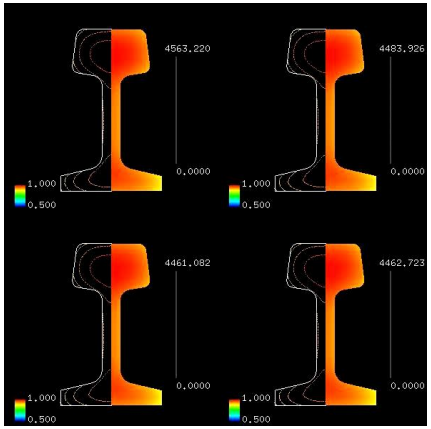
Linearization and Results

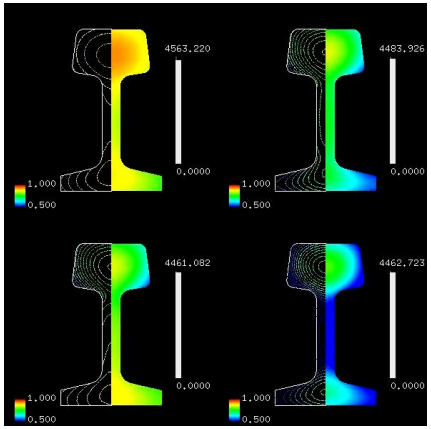
Idea

Freeze the material parameters for one or more time steps. ⇒ Linearization ⇒ method from the introduction can be applied.

Numerics semi-implicit discretization

Theory embeds to model predictive control. [Benner/S.'07]





Optimal Control-Based Stabilization for NSEs

Analytical Solution [Raymond '05–'07]

Linearized Navier-Stokes control system:

$$\partial_t \mathbf{z} + (\mathbf{z} \cdot \nabla)\mathbf{w} + (\mathbf{w} \cdot \nabla)\mathbf{z} - \frac{1}{Re}\Delta \mathbf{z} - \omega \mathbf{z} + \nabla \mathbf{p} = 0 \text{ in } Q_\infty \tag{1a}$$

$$\operatorname{div} \mathbf{z} = 0 \text{ in } Q_\infty \tag{1b}$$

$$\mathbf{z} = \mathbf{b}\mathbf{u} \text{ in } \Sigma_\infty \tag{1c}$$

$$\mathbf{z}(0) = \mathbf{z}_0 \text{ in } \Omega, \tag{1d}$$

$\omega \mathbf{z}$ with $\omega > 0$ de-stabilizes the system further, needed to guarantee exponential stabilization, ω controls decay rate!

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Cost functional (with $\mathbf{P}$ = Helmholtz projector)

$$J(z,u) = \frac{1}{2}\int_0^\infty \langle \mathbf{Pz}, \mathbf{Pz}\rangle_{L_2(\Omega)} + \rho\mathbf{u}(t)^2\,dt, \tag{2}$$

the linear-quadratic optimal control problem associated to (1) becomes

$$\inf\left\{J(\mathbf{z},\mathbf{u}) \mid (\mathbf{z},\mathbf{u}) \text{ satisfies (1)},\ \mathbf{u}\in L_2(0,\infty)\right\}. \tag{3}$$

Optimal Control-Based Stabilization for NSEs

Analytical Solution [Raymond '05–'07]

Proposition [Raymond '05, Bahdra '09]

The solution to the instationary Navier-Stokes equations with perturbed initial data is exponentially controlled to the steady-state solution w by the feedback law

$$\mathbf{u} = -\rho^{-1}\mathbf{B}^*\mathbf{X}z_H,$$

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- $\mathbf{z}_H := \mathbf{P}\mathbf{z}$, with $\mathbf{P} : L_2(\Omega) \mapsto V_n^0(\Omega)$ being the Helmholtz projector ($\rightsquigarrow \operatorname{div} \mathbf{z}_H \equiv 0$);

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- $\mathbf{X} = \mathbf{X}^* \in \mathcal{L}(V_n^0(\Omega))$ is the unique nonnegative semidefinite weak solution of the operator Riccati equation

$$0 = \mathbf{I} + (\mathbf{A} + \omega\mathbf{I})^*\mathbf{X} + \mathbf{X}(\mathbf{A} + \omega\mathbf{I}) - \mathbf{X}(\mathbf{B}_\tau\mathbf{B}_\tau^* + \rho^{-1}\mathbf{B}_n\mathbf{B}_n^*)\mathbf{X},$$

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  $\mathbf{A}$ is the linearized Navier-Stokes operator restricted to V_n^0; $\mathbf{B}_\tau$ and $\mathbf{B}_n$ correspond to the projection of the control action in the tangential and normal directions.

Optimal Control-Based Stabilization for NSEs

Solving the Helmholtz-projected Navier-Stokes ARE

Problems with Newton-Kleinman

1. Discretization of Helmholtz-projected linearized Navier-Stokes equations would need divergence-free finite elements.

   Here, we want to use standard discretization.

   Explicit projection of ansatz functions possible using application of Helmholtz projection, but too expensive in general.

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2. Each step of Newton-Kleinman iteration: solve

   $$A_j^T Z_{j+1} Z_{j+1}^T + Z_{j+1} Z_{j+1}^T A_j = -M - K_j^T K_j$$

   $n_v := \operatorname{rank}(M) =$ dim of ansatz space for velocities.

   $\rightsquigarrow$ need to solve $n_v + m$ linear systems of equations in each step of Newton-ADI iteration!

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   $\rightsquigarrow$ [Hein '10, Benner '11] near systems of equations in each step of New[illegible]-ADI iteration! ☑

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Non-linear Systems

Solution to 1. Problem/no need for divergence free FE

- incompressible Navier-Stokes-Equations

$$\frac{\partial \mathbf{v}}{\partial t} - \frac{1}{\mathrm{Re}}\Delta \mathbf{v} + \mathbf{v}.\nabla \mathbf{v} + \nabla \mathbf{p} = 0 \quad + \text{B.C.}$$
$$\nabla . \mathbf{v} = 0 \qquad \text{(NSE)}$$

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- Spatial FE discretization

$$M\dot{v}(t) = K(v)v(t) - Gp(t) + B_1\mathbf{u}(t)$$
$$0 = G^T v(t) \qquad \text{(SNSE)}$$

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- Linearization and change of notation

$$E_{11}\dot{v}(t) = A_{11}v(t) + A_{12}p(t) + B_1\mathbf{u}(t)$$
$$0 = A_{12}^T v(t) \qquad \text{(DANSE)}$$

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Multiplication of line one from the left by $A_{12}^T E_{11}^{-1}$ together with $0 = A_{12}^T v(t) \Rightarrow 0 = A_{12}^T \dot{v}(t)$ reveals the

hidden manifold

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Inserting $p(t)$ above leads to:

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Definition [Heinkenschloss/Sorensen/Sun '08]

$$\Pi := I - A_{12}\left(A_{12}^T E_{11}^{-1} A_{12}\right)^{-1} A_{12}^T E_{11}^{-1}$$

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$$\Pi := I - A_{12}\left(A_{12}^T E_{11}^{-1} A_{12}\right)^{-1} A_{12}^T E_{11}^{-1}$$

Leads to

projected Riccati equation

$$\begin{aligned} \Pi C^T C \Pi^T + \Pi A_{11}^T \Pi^T X \Pi E_{11} \Pi^T + \Pi E_{11}^T \Pi^T X \Pi A_{11} \Pi^T \\ - \Pi E_{11}^T \Pi^T X \Pi B_1 B_1^T \Pi^T X \Pi E_{11} \Pi^T = 0 \\ \Pi^T X \Pi = X. \end{aligned}$$

Non-linear Systems

Solving the Projected Matrix Equations

Apply factored-Newton-ADI

Central question

How do we solve systems of equations $(A_i := A_{11} + BK_i)$

$$Z = \Pi^T Z, \qquad \Pi\,(E_{11} + p_i A_i)\,\Pi^T Z = \Pi \tilde{G}$$

in the (inner) ADI steps avoiding the computation of Π?

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For $A_i = A_{11}$

Lemma

[Heinkenschloss/Sorensen/Sun '08]

$$\begin{aligned} Z &= \Pi^T Z \\ \Pi\,(E_{11} + p_i A_{11})\,\Pi^T Z &= \Pi \tilde{G} \end{aligned} \quad \Leftrightarrow \quad \begin{bmatrix} E_{11} + p_i A_{11} & A_{12} \\ A_{12}^T & 0 \end{bmatrix} \begin{bmatrix} Z \\ \Lambda \end{bmatrix} = \begin{bmatrix} \tilde{G} \\ 0 \end{bmatrix}$$

Non-linear Systems

Navier-Stokes Coupled with (Passive) Transport of (Reactive) Species

Goal: stabilize concentration at certain level

Model equations:

$$\partial_t \mathbf{v} - \frac{1}{Re}\Delta \mathbf{v} + \mathbf{v}.\nabla \mathbf{v} + \nabla \mathbf{p} = \mathbf{f}$$

$$\operatorname{div} \mathbf{v} = 0$$

$$\partial_t \mathbf{c} + \mathbf{v}.\nabla \mathbf{c} - \frac{1}{Re \cdot Sc}\Delta \mathbf{c} = 0$$

with boundary conditions:

$$\begin{aligned} \mathbf{v} &= \mathbf{v}_0 & \mathbf{c} &= \mathbf{c}_0 = const & &\text{on } \Gamma_{in} \\ \mathbf{v} &= 0 & \partial_\nu \mathbf{c} &= 0 & &\text{on } \Gamma_{wall} \\ \mathbf{v} &= 0 & \mathbf{c} &= 0 & &\text{on } \Gamma_r, \end{aligned}$$

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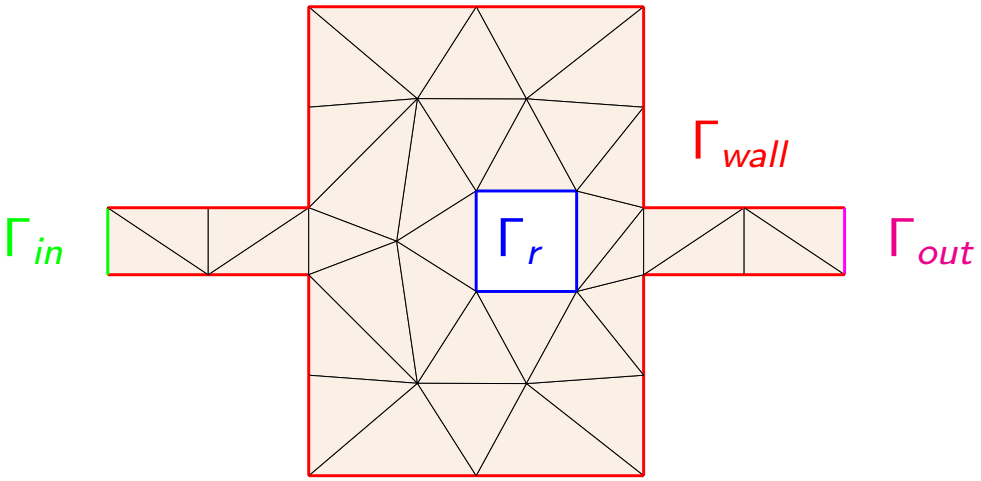
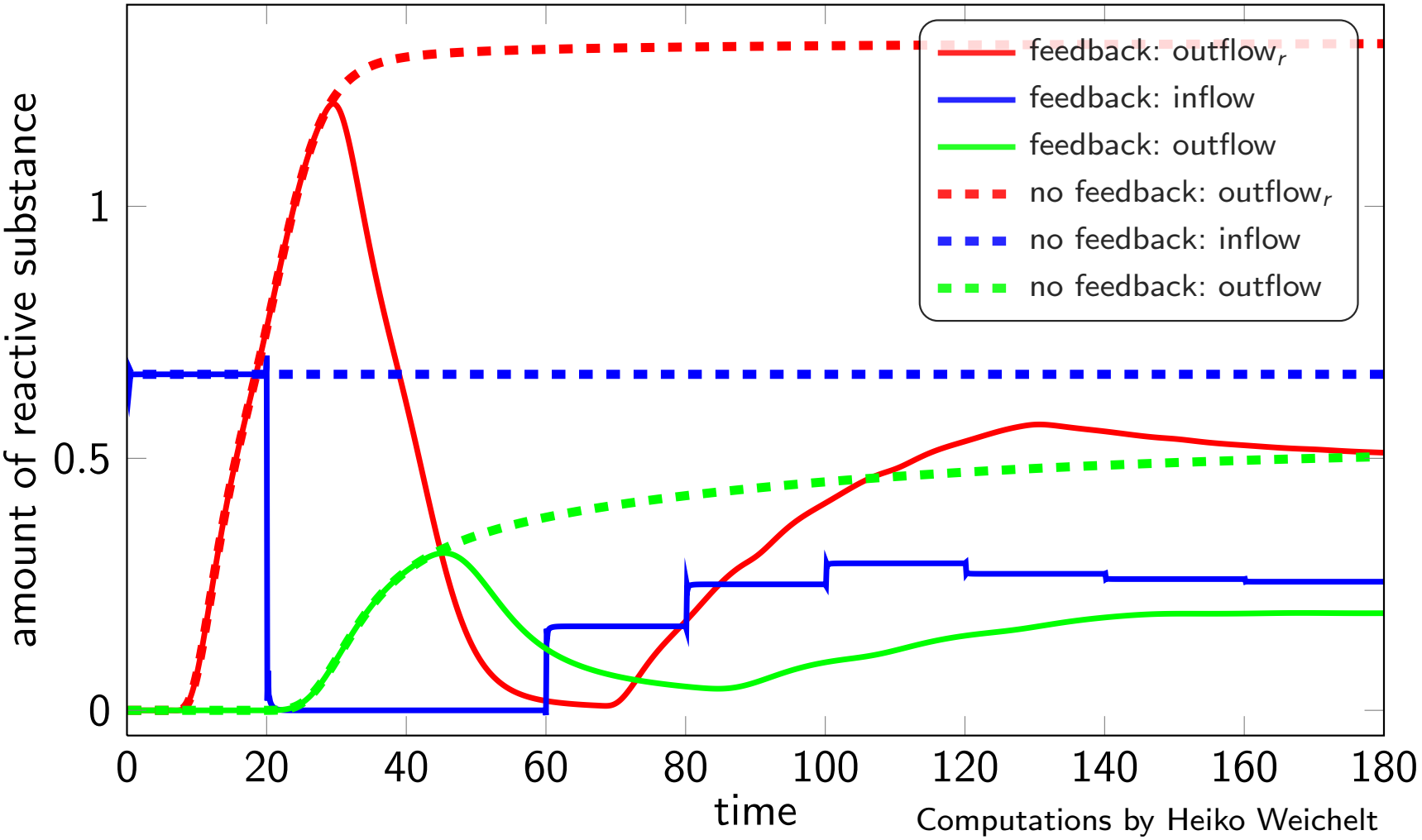
Domain:



Non-linear Systems

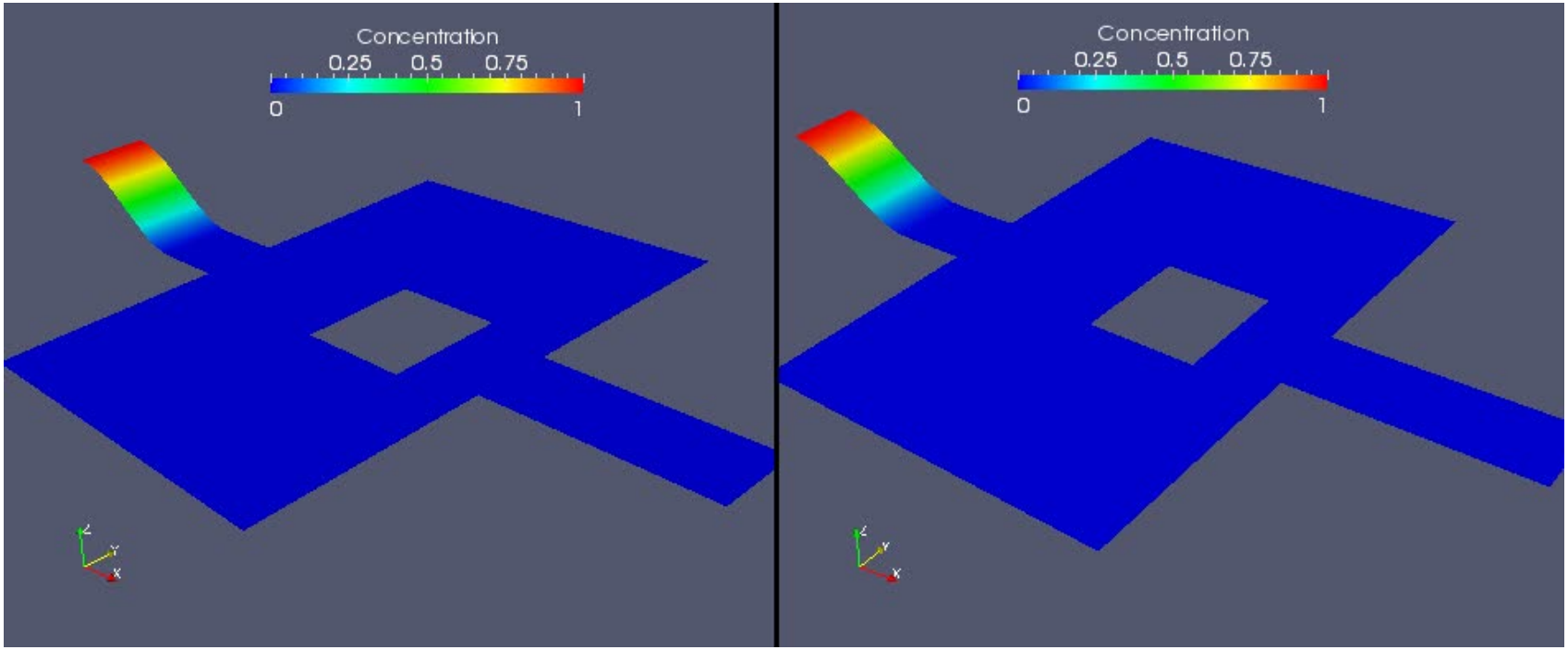
Results for $Re = 10$, $Sc = 10$



Computations by Heiko Weichelt

Non-linear Systems

Results for $Re = 10$, $Sc = 10$ shown at $3\times$speed



no control piecewise constant feedback

Computations by Heiko Weichelt

The End

Thank you for your attention!