Numerical Solution of Linear Quadratic Regulator Problems under PDE Constraints

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Max Planck Institute for Dynamics of Complex Technical Systems
Outline

1. Feedback-Control of Linear Parabolic PDEs
2. Tracking Control
3. Non-linear Systems
Feedback-Control of Linear Parabolic PDEs

1. Feedback-Control of Linear Parabolic PDEs
   - Parabolic PDEs and Abstract Cauchy-Problems
   - LQR Design for Abstract Cauchy Problems
   - Differential Riccati Equations: The case $T_f < \infty$

2. Tracking Control

3. Non-linear Systems
Feedback-Control of Linear Parabolic PDEs

Consider a control problem for a parabolic partial differential equation

\[
\frac{\partial x}{\partial t} + \nabla \cdot (c(x) - k(\nabla x)) + q(x) = v(\xi, t), \quad t \in [0, T_f], \quad (PDE)
\]

on a domain \( \Omega \subset \mathbb{R}^d, d = 1, 2, 3. \)

Here:

- \( q \) uncontrolled sink or source
- \( k \) diffusive part
- \( c \) convection part

For ease of presentation we consider \( T_f = \infty. \)
Consider a control problem for a

parabolic partial differential equation

\[
\frac{\partial \mathbf{x}}{\partial t} + \nabla \cdot (\mathbf{c}(\mathbf{x}) - \mathbf{k}(\nabla \mathbf{x})) + \mathbf{q}(\mathbf{x}) = \mathbf{v}(\xi, t), \quad t \in [0, T_f], \tag{PDE}
\]

on a domain \( \Omega \subset \mathbb{R}^d, d = 1, 2, 3. \)

Here \( \mathbf{v}(\xi, t) = \mathcal{B}(\xi)\mathbf{u}(t) \)

- \( \mathbf{u} \) control
- \( \mathcal{B} \) input operator
Feedback-Control of Linear Parabolic PDEs

Parabolic PDEs and Abstract Cauchy-Problems

Consider a control problem for a

parabolic partial differential equation

\[
\frac{\partial x}{\partial t} + \nabla \cdot (c(x) - k(\nabla x)) + q(x) = v(\xi, t), \quad t \in [0, T_f], \quad (PDE)
\]

on a domain \( \Omega \subset \mathbb{R}^d, d = 1, 2, 3 \).

If (PDE) is linear, then a variational formulation leads to a

Cauchy problem for the

linear evolution equation

\[
\dot{x} = Ax + Bu, \quad x(0) = x_0 \in \mathcal{X}.
\]
Feedback-Control of Linear Parabolic PDEs

LQR Design for Abstract Cauchy Problems (Formulation)

\[ \dot{x} = Ax + Bu, \quad x(0) = x_0 \in \mathcal{X}, \quad \text{(Cauchy)} \]

with linear operators

\[ A : \text{dom}(A) \subset \mathcal{X} \rightarrow \mathcal{X}, \quad B : \mathcal{U} \rightarrow \mathcal{X}, \]

on separable Hilbert spaces \( \mathcal{X} \) (state space), \( \mathcal{U} = \mathbb{R}^k \) (i.e., \( \mathcal{U} \) is finite dim.).
Feedback-Control of Linear Parabolic PDEs

LQR Design for Abstract Cauchy Problems (Formulation)

**Lineare evolution equation**

\[
\dot{x} = Ax + Bu, \quad x(0) = x_0 \in \mathcal{X}, \quad \text{(Cauchy)}
\]

**Output equation**

\[
y = Cx, \quad \text{(output)}
\]

with linear operators

\[
A : \text{dom}(A) \subset \mathcal{X} \to \mathcal{X}, \quad B : \mathcal{U} \to \mathcal{X}, \quad C : \mathcal{X} \to \mathcal{Y},
\]

on separable Hilbert spaces \( \mathcal{X} \) (state space), \( \mathcal{U} = \mathbb{R}^k \) (i.e., \( \mathcal{U} \) is finite dim.) and \( \mathcal{Y} \) (observation space).
Feedback-Control of Linear Parabolic PDEs
LQR Design for Abstract Cauchy Problems (Formulation)

<table>
<thead>
<tr>
<th>lineare evolution equation</th>
<th>output equation</th>
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<td>[ \dot{x} = Ax + Bu, \quad x(0) = x_0 \in \mathcal{X}, \quad (\text{Cauchy}) ]</td>
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Defining \( Q := C^* \hat{Q} C \) with \( \hat{Q} = \hat{Q}^* \geq 0 \), and \( R = R^* > 0 \) we state the

cost function

\[
J(u) = \frac{1}{2} \int_{0}^{\infty} \langle \hat{Q}y, y \rangle + \langle Ru, u \rangle \, dt. \quad (\text{cost})
\]
Feedback-Control of Linear Parabolic PDEs

LQR Design for Abstract Cauchy Problems (Formulation)

lineare evolution equation

\[
\dot{x} = A x + B u, \quad x(0) = x_0 \in \mathcal{X}, \quad \text{(Cauchy)}
\]

output equation

\[
y = C x, \quad \text{(output)}
\]

Defining \( Q := C^* \hat{Q} C \) with \( \hat{Q} = \hat{Q}^* \geq 0 \), and \( R = R^* > 0 \) we state the

cost function

\[
\mathcal{J}(u) = \frac{1}{2} \int_{0}^{\infty} \langle Q x, x \rangle + \langle R u, u \rangle \, dt. \quad \text{(cost)}
\]
Feedback-Control of Linear Parabolic PDEs
LQR Design for Abstract Cauchy Problems (Formulation)

**lineare evolution equation**

\[
\dot{x} = Ax + Bu, \quad x(0) = x_0 \in \mathcal{X}, \quad \text{(Cauchy)}
\]

**output equation**

\[
y = Cx, \quad \text{(output)}
\]

**cost function**

\[
J(u) = \frac{1}{2} \int_0^\infty < Qx, x > + < Ru, u > \, dt. \quad \text{(cost)}
\]

We can now formulate the

LQR–problem.

Minimize (cost) with respect to (Cauchy).
Well understood in the open literature:
Analogously to ODE systems case we find the

\[ u = -R^{-1}B^*X_\infty x. \]

Here \( X_\infty \) is the stabilizing, positive semidefinite, selfadjoint solution to the

**Operator–Algebraic–Riccati–Equation**

\[ 0 = \mathcal{R}(X) := Q + A^*X + XA - XBR^{-1}B^*X. \quad (O\text{-ARE}) \]

---

e.g. [Lions ‘71; Lasiecka/Triggiani ‘00; Bensoussan et al. ‘92/‘06; Pritchard/Salamon ‘87; Curtain/Zwart ‘95]
(Cauchy) can now be rewritten as

**closed loop system**

\[ \dot{x} = (A - BR^{-1}B^*X_{\infty})x, \]

and the

**optimal solution trajectory**

is given as

\[ x(t) = S(t)x_0, \]

where \( S(t) \) is the operator semigroup generated by \( A - BR^{-1}B^*X_{\infty} \).
Let \((\mathcal{X}_n)_{n \in \mathbb{N}}\) a Galerkin scheme for \(\mathcal{X}\). We formulate the
Let \((\mathcal{X}_n)_{n \in \mathbb{N}}\) a Galerkin scheme for \(\mathcal{X}\). We formulate the

\[
\dot{x} = A_n x + B_n u, \quad \mathcal{X}_n \ni x_n(0) = P_n x_0, \\
\text{(n-d Cauchy)}
\]

\[
y_n = C_n x_n, \quad \text{(n-d output)}
\]

with linear operators

\[
A_n : \text{dom}(A_n) \subset \mathcal{X}_n \to \mathcal{X}_n, \quad B_n : \mathcal{U} \to \mathcal{X}_n, \quad C_n : \mathcal{X}_n \to \mathcal{Y}_n,
\]

on n-d Hilbert spaces \(\mathcal{X}_n\) (state space) and \(\mathcal{Y}_n\) (observation space), but still \(\mathcal{U} = \mathbb{R}^k\).

\(P_n : \mathcal{X} \to \mathcal{X}_n\) the canonical orthogonal projection.
Feedback-Control of Linear Parabolic PDEs

LQR Design for Abstract Cauchy Problems (Approximation)

Let \((X_n)_{n \in \mathbb{N}}\) a Galerkin scheme for \(X\). We formulate the

\[
\dot{x} = A_n x + B_n u, \quad X_n \ni x_n(0) = P_n x_0, \\
\text{(n-d Cauchy)}
\]

\[
y_n = C_n x_n, \\
\text{(n-d output)}
\]

Defining \(Q_n := C_n^* \hat{Q}_n C_n\) with \(\hat{Q}_n = \hat{Q}_n^* \geq 0\), and \(R = R^* > 0\) we formulate

\[
J_n(u) = \frac{1}{2} \int_{0}^{\infty} <\hat{Q}_n y_n, y_n> + <Ru, u> \, dt. \\
\text{(n-d Cost)}
\]
Feedback-Control of Linear Parabolic PDEs

LQR Design for Abstract Cauchy Problems (Approximation)

Let \( (\mathcal{X}_n)_{n \in \mathbb{N}} \) a Galerkin scheme for \( \mathcal{X} \). We formulate the

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Defining \( Q_n := C_n^* \hat{Q}_n C_n \) with \( \hat{Q}_n = \hat{Q}_n^* \geq 0 \), and \( R = R^* > 0 \) we formulate

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Feedback-Control of Linear Parabolic PDEs

LQR Design for Abstract Cauchy Problems (Approximation)

n-d evolution equation

\[ \dot{x} = A_n x + B_n u, \quad x_n(0) = P_n x_0, \]

(n-d Cauchy)

output equation

\[ y_n = C_n x_n, \]

(n-d output)

cost function

\[ J(u) = \frac{1}{2} \int_0^\infty < Q_n x_n, x_n > + < R u, u > \, dt. \]

(n-d cost)

and state the

n-d LQR–problem.

Minimize (n-d Cost) with respect to (n-d Cauchy).
Feedback-Control of Linear Parabolic PDEs

LQR Design for Abstract Cauchy Problems (Approximation)

Analogously to the $\infty$-dim. case we now find:

\[ u = -R^{-1}B^*_nX_nx_n, \]

where $X_n$ is the stabilizing, positive semidefinite, selfadjoint solution to the

\[ 0 = R_n(X) := Q_n + A^*_nX + XA_n - XB_nR^{-1}B^*_nX. \]  

(n-d O-ARE)
As above we can write (n-d Cauchy) as

**closed loop system**

\[ \dot{x}_n = (A_n - B_nR^{-1}B_n^*X_n)x_n, \]

and the

**optimal solution**

is given as

\[ x_n(t) = S_n(t)P_nx_0, \]

also again \( S_n(t) \) is the operator semigroup generated by \( A_n - B_nR^{-1}B_n^*X_n \).
The n-d LQR–problems approximate the $\infty$-dim LQR–problem in the following sense

- $X_n P_n v \rightarrow X v$ for $n \rightarrow \infty$ and any $v \in \mathcal{X}$,
- $S_n(t) P_n v \rightarrow S(t) v$ for $n \rightarrow \infty$ and any $v \in \mathcal{X}$,
Feedback-Control of Linear Parabolic PDEs

LQR Design for Abstract Cauchy Problems (Approximation)

Approximation

The n-d LQR–problems approximate the $\infty$-dim LQR–problem in the following sense

- $X_n P_n v \to X v$ for $n \to \infty$ and any $v \in \mathcal{X}$,
- $S_n(t) P_n v \to S(t) v$ for $n \to \infty$ and any $v \in \mathcal{X}$,

that means in the strong operator-topology.

[Banks/Kunisch’84] distributed control of parabolic PDEs
[Benner/S.’05] boundary control with mixed boundary conditions
[Lasiecka/Triggiani’00] weakens regularity conditions on (Cauchy), also has convergence rates
[Ito’87/’90; Morris’94] general Cauchy problems

[...] many more
Approximation

The n-d LQR–problems approximate the ∞-dim LQR–problem in the following sense

- $X_n P_n v \to Xv$ for $n \to \infty$ and any $v \in \mathcal{X}$,
- $S_n(t) P_n v \to S(t)v$ for $n \to \infty$ and any $v \in \mathcal{X}$,

that means in the strong operator-topology.

Remarks:

- For a chosen basis (e.g. from spatial FDM/FEM discretization) all n-d operators have matrix representations and $S(t)$ coincides with the matrix-exponential $e^{(A - BR^{-1}B^T)X}t$. 

Feedback-Control of Linear Parabolic PDEs

LQR Design for Abstract Cauchy Problems (Approximation)

Approximation

The n-d LQR–problems approximate the ∞-dim LQR–problem in the following sense

- \( X_n P_n \mathbf{v} \rightarrow X\mathbf{v} \) for \( n \rightarrow \infty \) and any \( \mathbf{v} \in \mathcal{X} \),
- \( S_n(t)P_n \mathbf{v} \rightarrow S(t)\mathbf{v} \) for \( n \rightarrow \infty \) and any \( \mathbf{v} \in \mathcal{X} \),

that means in the strong operator-topology.

Remarks:

- For a chosen basis (e.g. from spatial FDM/FEM discretization) all n-d operators have matrix representations and \( S(t) \) coincides with the matrix-exponential \( e^{(A-BR^{-1}B^TX)t} \).
- \( \mathbf{u} \) and \( \mathbf{R} \) are always kept fixed, i.e., \( \mathbf{u} \) from computations for an n-d problem can directly be applied in the ∞-d problem.
- Suboptimality can be estimated in terms of the discretization error [Benner/S.‘10]
Main task in numerical solution

Efficient solution of the large sparse matrix–Riccati–equations

\[ 0 = \mathcal{R}_h(X) := Q_h + A_h^*X + XA_h - XB_hR^{-1}B_h^*X, \quad (\text{M-ARE}) \]

with regard to both memory and CPU usage.

Classical methods are not applicable due to their cubic complexity.
(M-ARE) is non-linear ⇒

Newton’s method for the ARE

\[ R'_h|_X(N_\ell) = -R_h(X_\ell), \quad X_{\ell+1} = X_\ell + N_\ell. \]

The Frechét derivative of \( R_h \) at \( X \) is given as the Lyapunov operator

\[ R'_h|_X : \quad Z \mapsto (A_h - B_h R^{-1} B_h^T X)^T Z + Z (A_h - B_h R^{-1} B_h^T X). \]

Thus we find the

Newton update

\[ (A_h - B_h R^{-1} B_h^T X_\ell)^T N_{\ell+1} + N_{\ell+1} (A_h - B_h R^{-1} B_h^T X_\ell) = -R(X_\ell). \]
(M-ARE) is non-linear ⇒

Newton’s method for the ARE

\[ \mathcal{R}'_h|_X(N_\ell) = -\mathcal{R}_h(X_\ell), \quad X_{\ell+1} = X_\ell + N_\ell. \]

The Fréchet derivative of \( \mathcal{R}_h \) at \( X \) is given as the Lyapunov operator

\[ \mathcal{R}'_h|_X : \quad Z \mapsto (A_h - B_hR^{-1}B_h^TX)^T Z + Z(A_h - B_hR^{-1}B_h^TX). \]

Thus we find the one step Newton-Kleinman iteration

\[ (A_h - B_hR^{-1}B_h^TX_\ell)^T X_{\ell+1} + X_{\ell+1}(A_h - B_hR^{-1}B_h^TX_\ell) = -C_h^T Q_h C_h - X_\ell B_h R^{-1} B_h^T X_\ell. \]
In every Newton step we solve a

**Lyapunov equation**

\[ F^T X + XF = -GG^T. \]  
(Lyapunov)
In every Newton step we solve a

\[
F^T X + XF = -GG^T. \tag{Lyapunov}
\]

Available solvers for large sparse Lyapunov equations (Lyapunov):

- **ADI** [Wachpress‘88; Penzl‘99; Benner/Li/Penzl‘08; Li/White‘02];
In every Newton step we solve a Lyapunov equation:

\[ F^T X + X F = -G G^T. \]  

(Lyapunov)

Available solvers for large sparse Lyapunov equations (Lyapunov):

**ADI** [Wachpress‘88; Penzl‘99; Benner/Li/Penzl‘08; Li/White‘02];

**Krylov** [Kasenally/Jaimoukha‘94; Jbilou/Riquet‘06; Simoncini et al.‘06–‘11]

**Smith** [Penzl‘99; Gugercin/Sorensen/Antoulas‘03]

... many more
In every Newton step we solve a

Lyapunov equation

\[ F^T X + XF = -GG^T. \] (Lyapunov)

Available solvers for large sparse Lyapunov equations (Lyapunov)

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... many more

ADI needs shift–paramets; Choice of shifts: [Ellner/Wachspress‘91; Penzl‘00; Benner/Mena/S.‘06; Starke ‘89; Sabino‘06; Wachspress ’08]
In every Newton step we solve a Lyapunov equation:

\[ F^T X + XF = -GG^T. \]

Available solvers for large sparse Lyapunov equations (Lyapunov):

- **ADI** [Wachspress‘88; Penzl‘99; Benner/Li/Penzl‘08; Li/White‘02];
- **Krylov** [Kasenally/Jaimoukha‘94; Jbilou/Riquet‘06; Simoncini et al.‘06–‘11]
- **Smith** [Penzl‘99; Gugercin/Sorensen/Antoulas‘03]
- ... many more

For Systems with very few inputs **Newton-ADI** and **Newton-Smith** can iterate on the feedback \( K_h := R^{-1}B_h^T X \) directly [Penzl‘00; Banks/Ito‘91].
For $T_f < \infty$ one finds ($\ast$-ARE) $\leadsto$ ($\ast$-DRE):
\[
\frac{\partial}{\partial t} \mathcal{R}_h(X) = -\mathcal{R}_h(X)
\]
Feedback-Control of Linear Parabolic PDEs

Differential Riccati Equations: The case $T_f < \infty$

For $T_f < \infty$ one finds ($\ast$-ARE) $\leadsto$ ($\ast$-DRE):

$$\frac{\partial}{\partial t} \mathcal{R}_h(X) = -\mathcal{R}_h(X)$$

[**Mena 2007**]:
ODE solvers of BDF and Rosenbrock type applicable efficiently.
Feedback-Control of Linear Parabolic PDEs

Differential Riccati Equations: The case $T_f < \infty$

For $T_f < \infty$ one finds ($\ast$-ARE) $\rightsquigarrow$ ($\ast$-DRE): $\frac{\partial}{\partial t} R_h(X) = -R_h(X)$
Feedback-Control of Linear Parabolic PDEs

Differential Riccati Equations: The case $T_f < \infty$

For $T_f < \infty$ one finds ($\ast$-ARE) $\leadsto$ ($\ast$-DRE):

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Feedback-Control of Linear Parabolic PDEs

Differential Riccati Equations: The case $T_f < \infty$

For $T_f < \infty$ one finds ($\ast$-ARE) $\leadsto$ ($\ast$-DRE):

$$\frac{\partial}{\partial t} \mathcal{R}_h(X) = -\mathcal{R}_h(X)$$

Low Rank Approximation again guarantees efficiency in terms of computational effort and memory usage.
Tracking Control

Differential Riccati Equations: The case $T_f < \infty$

1. Feedback-Control of Linear Parabolic PDEs

2. Tracking Control
   - Linear Systems with Inhomogenities
   - Application I: Tracking Control of Parabolic PDEs
   - Application II: Solution of Inverse Problems with Parabolic PDEs

3. Non-linear Systems
Reminder for systems with linear inhomogeneous evolution equations

\[ \dot{x} = Ax + Bu + f. \]

Let \( \hat{x} \) solve the uncontrolled system \( \dot{x} = Ax + f \), then

\[ f = \dot{\hat{x}} - A\hat{x}, \]

and

\[ \dot{x} - \dot{\hat{x}} = A(x - \hat{x}) + Bu. \]

We can solve the system

\[ \dot{z} = Az + Bu \]

for \( z = x - \hat{x} \) to compute the control \( u \).

e.g., [Godunov’97]
Consider $\tilde{x}$ the state we want to track and the tracking problem

\[
\begin{align*}
\dot{x} &= Ax + Bv, \\
y &= C(x - \tilde{x}), \\
J(u) &= \frac{1}{2} \int_0^\infty < Q(x - \tilde{x}), x - \tilde{x}> + < Rv, v > \, dt.
\end{align*}
\]

Define $z := x - \tilde{x}$ and the Cauchy problem

\[
\begin{align*}
\dot{z} &= Az + Bv, \\
y &= Cz.
\end{align*}
\]  \hspace{1cm} (1)

The optimal control then is given as $v = -Kz$ as above and (1) is equivalent to

\[
\begin{align*}
\dot{x} &= Ax - BKx + \dot{\tilde{x}} - A\tilde{x} + BK\tilde{x}.
\end{align*}
\]
Consider $\tilde{x}$ the state we want to track and the tracking problem

$$\dot{x} = Ax + Bv, \quad y = C(x - \tilde{x}),$$

$$J(u) = \frac{1}{2} \int_0^\infty \langle Q(x - \tilde{x}), x - \tilde{x} \rangle + \langle Rv, v \rangle \, dt.$$  

- $f := \dot{\tilde{x}} - A\tilde{x} + BK\tilde{x}$ is a known inhomogeneity when solving the closed loop system.

- Equations (tracking) and (1) require the same algebraic Riccati equation.
Consider $\tilde{x}$ the state we want to track and the tracking problem:

$$\begin{align*}
\dot{x} &= Ax + Bv, \\
y &= C(x - \tilde{x}), \\
J(u) &= \frac{1}{2} \int_0^\infty < Q(x - \tilde{x}), x - \tilde{x}> + < Rv, v > dt.
\end{align*}$$

- We can compute the Feedback for (tracking) with the above technique for (1) and afterward solve the inhomogeneous closed loop system.
- Method also works for a reference pair $(\tilde{x}, \tilde{u})$ [Benner/Görner/S.‘06]
Consider the instationary heat equation

\[ \frac{\partial}{\partial t} x(\xi, t) = \frac{\lambda}{c \rho} \Delta x(\xi, t) \quad \text{on } \Omega \times (0, T), \]

with boundary coditions:

\[ \lambda \frac{\partial}{\partial \nu} x(\xi, t) = \kappa_i (x(\xi, t) - u_i(t)) \quad \text{on } \Gamma_i \times (0, T), \]

\[ \lambda \frac{\partial}{\partial \nu} x(\xi, t) = \kappa_o (x(\xi, t) - x_{\text{ext}}(t)) \quad \text{on } \Gamma_o \times (0, T), \]

**Identification Problem**

Knowing measurements \( \hat{y} \) use the output \( y(t) = Cx(\xi, t) \in \mathbb{R}^k \) to find

the heat \( u_i \) induced via the inner boundary \( \Gamma_i \).
Consider the instationary heat equation

\[
\frac{\partial}{\partial t} x(\xi, t) = \frac{\lambda}{c \rho} \Delta x(\xi, t)
\]

on \( \Omega \times (0, T) \),

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\]

\[
\lambda \frac{\partial}{\partial \nu} x(\xi, t) = \kappa_o (x(\xi, t) - x_{\text{ext}}(t)) \quad \text{on} \ \Gamma_o \times (0, T),
\]

- Allowing for measurement uncertainties \( \rightsquigarrow \) LQG-design, e.g., [Hein ‘10] (allows nonlinear treatment of material parameters).
- Combine LQG-design to identify \( x(\xi, t) \) with tracking \( y \rightarrow \hat{y} \).

Ongoing work in the graduation thesis of N. Lang.
Non-linear Systems

Heat Distribution in Steel Profiles, a Model Problem

1. Feedback-Control of Linear Parabolic PDEs

2. Tracking Control

3. Non-linear Systems
   - Heat Distribution in Steel Profiles, a Model Problem
   - Linearization and Results
   - Optimal Control-Based Stabilization for NSEs
   - Solution to 1. Problem/no need for divergence free FE
   - Solving the Projected Matrix Equations
   - Navier-Stokes Coupled with (Passive) Transport of (Reactive) Species
   - Results
Non-linear Systems

Heat Distribution in Steel Profiles, a Model Problem

The active cooling of steel profiles in a rolling facility serves as a model problem. We consider the in stationary heat equation

\[ c(x) \rho(x) \frac{\partial}{\partial t} x(\xi, t) = \nabla \cdot (\lambda(x) \nabla x(\xi, t)) \quad \text{on } \Omega \times (0, T), \]
\[ -\lambda(x) \frac{\partial}{\partial \nu} x(\xi, t) = \kappa_i(x(\xi, t) - u_i(t)) \quad \text{on } \Gamma_i \times (0, T), \quad \text{(heat)} \]
\[ x(\xi, 0) = x_0(\xi) \quad \text{on } \Omega, \]

\[ x \quad \text{state, temperature} \]
\[ u \quad \text{control} \]
\[ T \in \mathbb{R} \cup \{\infty\} \quad \text{final time} \]

\[ c(x) \quad \text{specific heat capacity} \]
\[ \rho(x) \quad \text{density} \]
\[ \lambda(x) \quad \text{heat conductivity} \]
Non-linear Systems
Heat Distribution in Steel Profiles, a Model Problem

The active cooling of steel profiles in a rolling facility serves as a model problem. We consider the in stationary heat equation

\[
\begin{align*}
  c(x)\rho(x) \frac{\partial}{\partial t} x(\xi, t) &= \nabla \cdot (\lambda(x) \nabla x(\xi, t)) & \text{on } \Omega \times (0, T), \\
  -\lambda(x) \frac{\partial}{\partial \nu} x(\xi, t) &= \kappa_i(x(\xi, t) - u_i(t)) & \text{on } \Gamma_i \times (0, T), \quad \text{(heat)} \\
  x(\xi, 0) &= x_0(\xi) & \text{on } \Omega,
\end{align*}
\]

(heat) obviously is non-linear due to \( c, \rho \) and \( \lambda \) depending on the temperature \( x \).

Idea
Freeze the material parameters for one or more time steps. \( \Rightarrow \) 
Linearization \( \Rightarrow \) method from the introduction can be applied.
**Non-linear Systems**

**Linearization and Results**

**Idea**

Freeze the material parameters for one or more time steps. ⇒ Linearization ⇒ method from the introduction can be applied.

**Numerics**  semi-implicit discretization

**Theory**  embeds to model predictive control. [Benner/S.‘07]

---

[Images of temperature distribution over time]
Optimal Control-Based Stabilization for NSEs

Analytical Solution [Raymond ’05–’07]

Linearized Navier-Stokes control system:

\[
\begin{align*}
\partial_t z + (z \cdot \nabla)w + (w \cdot \nabla)z - \frac{1}{Re} \Delta z - \omega z + \nabla p &= 0 \text{ in } Q_\infty \\
\text{div } z &= 0 \text{ in } Q_\infty \\
z &= bu \text{ in } \Sigma_\infty \\
z(0) &= z_0 \text{ in } \Omega,
\end{align*}
\]

ωz with ω > 0 de-stabilizes the system further, needed to guarantee exponential stabilization, ω controls decay rate!
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\text{div } \mathbf{z} &= 0 \quad \text{in } Q_{\infty} \quad (1b) \\
\mathbf{z} &= \mathbf{b}u \quad \text{in } \Sigma_{\infty} \quad (1c) \\
\mathbf{z}(0) &= \mathbf{z}_0 \quad \text{in } \Omega, \quad (1d)
\end{align*}\]

\(\omega \mathbf{z}\) with \(\omega > 0\) de-stabilizes the system further, needed to guarantee exponential stabilization, \(\omega\) controls decay rate!

Cost functional (with \(\mathbf{P} = \text{Helmholtz projector}\))

\[J(\mathbf{z}, u) = \frac{1}{2} \int_0^\infty \langle \mathbf{Pz}, \mathbf{Pz} \rangle_{L^2(\Omega)} + \rho u(t)^2 \, dt, \quad (2)\]

the linear-quadratic optimal control problem associated to (1) becomes

\[\inf \{ J(\mathbf{z}, u) \mid (\mathbf{z}, u) \text{ satisfies (1)}, \ u \in L^2(0, \infty) \}. \quad (3)\]
Proposition [Raymond ’05, Bahdra ’09]

The solution to the instationary Navier-Stokes equations with perturbed initial data is exponentially controlled to the steady-state solution \( w \) by the feedback law

\[
u = -\rho^{-1}B^*Xz_H,
\]

where

\( Pz \):

\( P \) : \( L^2(\Omega) \rightarrow V_0(\Omega) \) being the Helmholtz projector

\( \capz : \div z_H \equiv 0 \); 

\( X = X^* \in L(V_0(\Omega)) \) is the unique nonnegative semidefinite weak solution of the operator Riccati equation

\[
0 = I + (A + \omega I)^*X + X(A + \omega I) - X(B_{\tau}B_{\tau}^* + \rho^{-1}B_{n}B_{n}^*)X,
\]

\( A \) is the linearized Navier-Stokes operator restricted to \( V_0 \); 

\( B_{\tau} \) and \( B_{n} \) correspond to the projection of the control action in the tangential and normal directions.
Optimal Control-Based Stabilization for NSEs

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$A$ is the linearized Navier-Stokes operator restricted to $V_0^0$; $B^T$ and $B_n$ correspond to the projection of the control action in the tangential and normal directions.
Discretization of Helmholtz-projected linearized Navier-Stokes equations would need divergence-free finite elements. Here, we want to use standard discretization. Explicit projection of ansatz functions possible using application of Helmholtz projection, but too expensive in general.
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Each step of Newton-Kleinman iteration: solve

\[ A_j^T Z_{j+1} Z_{j+1}^T + Z_{j+1} Z_{j+1}^T A_j = - \dot{M} - K_j^T K_j \]

\[ n_v := \text{rank}(\dot{M}) = \text{dim of ansatz space for velocities.} \]

\[ \sim \text{need to solve } n_v + m \text{ linear systems of equations in each step of Newton-ADI iteration!} \]
Problems with Newton-Kleinman

1. Discretization of Helmholtz-projected linearized Navier-Stokes equations would need divergence-free finite elements. Here, we want to use standard discretization. Explicit projection of ansatz functions possible using application of Helmholtz projection, but too expensive in general.

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3. Linearized system (i.e., \( A + \omega M \)) is unstable in general. To start Newton iteration, a stabilizing initial guess is needed!
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1. Discretization of Helmholtz-projected linearized Navier-Stokes equations would need divergence-free finite elements.
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   Ex. [Morris/Navasca '08] Helmholtz functions possible using application of Helmholtz projection, but too expensive in general.

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\[ \square \]
Optimal Control-Based Stabilization for NSEs
Solving the Helmholtz-projected Navier-Stokes ARE

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\[ \Rightarrow [\text{HEIN '10, BENNER '11}] \text{ linear systems of equations in each step of Newton-ADI iteration!} \]

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Non-linear Systems
Solution to 1. Problem/no need for divergence free FE

- incompressible Navier-Stokes-Equations

\[
\frac{\partial \mathbf{v}}{\partial t} - \frac{1}{\text{Re}} \Delta \mathbf{v} + \mathbf{v} \cdot \nabla \mathbf{v} + \nabla p = 0 \quad + \text{B.C.}
\]
\[
\nabla \cdot \mathbf{v} = 0
\]
Non-linear Systems

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- Spatial FE discretization

\[
M \dot{\mathbf{v}}(t) = K(\mathbf{v})\mathbf{v}(t) - Gp(t) + B_1\mathbf{u}(t)
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  (NSE)

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  \]
  \[
  0 = G^T \nu(t)
  \]
  (SNSE)

- Linearization and change of notation
  \[
  E_{11} \dot{\nu}(t) = A_{11} \nu(t) + A_{12} \rho(t) + B_1 \mathbf{u}(t)
  \]
  \[
  0 = A_{12}^T \nu(t)
  \]
  (DANSE)
Non-linear Systems

Solution to 1. Problem/no need for divergence free FE

\[ E_{11} \dot{v}(t) = A_{11} v(t) + A_{12} p(t) + B_1 u(t) \]
\[ 0 = A_{12}^T v(t) \]

Multiplication of line one from the left by \( A_{12}^T E_{11}^{-1} \) together with
\[ 0 = A_{12}^T v(t) \Rightarrow 0 = A_{12}^T \dot{v}(t) \]
reveals the hidden manifold

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Inserting \( p(t) \) above leads to:
Non-linear Systems

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**Definition**

\[ \Pi := I - A_{12} \left( A_{12}^T E_{11}^{-1} A_{12} \right)^{-1} A_{12}^T E_{11}^{-1} \]
Non-linear Systems

Solution to 1. Problem/no need for divergence free FE

Definition

\[ \Pi := I - A_{12} \left( A_{12}^T E_{11}^{-1} A_{12} \right)^{-1} A_{12}^T E_{11}^{-1} \]

Leads to

projected Riccati equation

\[ \Pi C^T C \Pi^T + \Pi A_{11}^T \Pi^T X \Pi E_{11} \Pi^T + \Pi E_{11}^T \Pi^T X \Pi A_{11} \Pi^T - \Pi E_{11}^T \Pi^T X \Pi B_1 B_1^T \Pi^T X \Pi E_{11} \Pi^T = 0 \]

\[ \Pi^T X \Pi = X. \]
Non-linear Systems
Solving the Projected Matrix Equations

Apply factored-Newton-ADI

Central question
How do we solve systems of equations

\[
A_i := A_{11} + BK_i
\]

\[
Z = \Pi^T \Pi Z, \quad \Pi (E_{11} + p_i A_i) \Pi^T Z = \Pi \tilde{G}
\]
in the (inner) ADI steps avoiding the computation of \( \Pi \)?
Non-linear Systems
Solving the Projected Matrix Equations

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For \( A_i = A_{11} \)

Lemma

\[ \Pi (E_{11} + p_i A_{11}) \Pi^T Z = \Pi \tilde{G} \iff \begin{bmatrix} E_{11} + p_i A_{11} & A_{12} \\ A_{12}^T & 0 \end{bmatrix} \begin{bmatrix} Z \\ \Lambda \end{bmatrix} = \begin{bmatrix} \tilde{G} \\ 0 \end{bmatrix} \]

[Heinkenschloss/Sorensen/Sun ’08]
Non-linear Systems

Navier-Stokes Coupled with (Passive) Transport of (Reactive) Species

Goal: stabilize concentration at certain level

Model equations:

\[ \partial_t \mathbf{v} - \frac{1}{Re} \Delta \mathbf{v} + \mathbf{v} \cdot \nabla \mathbf{v} + \nabla p = f \]

\[ \text{div} \mathbf{v} = 0 \]

\[ \partial_t c + \mathbf{v} \cdot \nabla c - \frac{1}{Re \cdot Sc} \Delta c = 0 \]

with boundary conditions:

\[ \mathbf{v} = \mathbf{v}_0 \quad c = c_0 = \text{const} \quad \text{on } \Gamma_{\text{in}} \]
\[ \mathbf{v} = 0 \quad \partial_n c = 0 \quad \text{on } \Gamma_{\text{wall}} \]
\[ \mathbf{v} = 0 \quad c = 0 \quad \text{on } \Gamma_r, \]
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\text{div} \mathbf{v} &= 0 \\
\partial_t c + \mathbf{v} \cdot \nabla c - \frac{1}{Re \cdot Sc} \Delta c &= 0
\end{align*}
\]

Domain:
Non-linear Systems

Results for $Re = 10$, $Sc = 10$

![Graph showing the amount of reactive substance over time with different feedback scenarios.](image-url)
Non-linear Systems

Results for $Re = 10, Sc = 10$ shown at $3 \times$ speed

no control

piecewise constant feedback

Computations by Heiko Weichelt
Thank you for your attention!