Optimal Control-Based Feedback Stabilization in Multi-Field Flow Problems

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Chemnitz University of Technology

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Overview

1. Project Summary
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   - Analysis of the Saddle Point System
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Project Summary

Scientific Goals of the Project

- Derive and investigate numerical algorithms for optimal control-based (normal and tangential) boundary feedback stabilization of multi-field flow problems.
- Explore the potentials and limitations of feedback-based (Riccati) stabilization techniques.
- Extend current methods for flow described by Navier-Stokes equations (NSE) to flow problems coupled with other field equations of increasing complexity.

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Numerical solution of algebraic Riccati equations associated to special LQR problem for linearized Navier-Stokes/Oseen-like equations.

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Optimal Control-Based Stabilization for NSEs

Analytical Solution

[Raymond '05–'07]

Linearized Navier-Stokes Control System

$$\begin{aligned} \partial_t \mathbf{z} - \nu \Delta \mathbf{z} + (\mathbf{z} \cdot \nabla)\mathbf{w} + (\mathbf{w} \cdot \nabla)\mathbf{z} - \omega \mathbf{z} + \nabla p = 0 \ \text{ in } Q_\infty, \\ \operatorname{div} \mathbf{z} = 0 \ \text{ in } Q_\infty, \\ \mathbf{z} = \mathbf{b}\mathbf{u} \text{ in } \Sigma_\infty, \\ \mathbf{z}(0) = \mathbf{z}_0 \ \text{ in } \Omega, \end{aligned}$$

$$J(\mathbf{z}, \mathbf{u}) = \frac{1}{2} \int_0^\infty \langle \mathbf{P}\mathbf{z}, \mathbf{P}\mathbf{z} \rangle_{L_2(\Omega)} + \rho\, \mathbf{u}^T \mathbf{u}\, dt.$$

Proposition [Raymond '05, Bahdra '09]

The solution to the instationary NSEs with perturbed initial data is exponentially controlled to the steady-state solution $\mathbf{w}$ by the feedback law

$$\mathbf{u} = -\rho^{-1} \mathbf{B}^* \mathbf{X} z_H.$$

- $z_H := \mathbf{P}\mathbf{z}$, with $\mathbf{P} : L_2(\Omega) \to V_n^0(\Omega)$ (Helmholtz projector) ($\rightsquigarrow \operatorname{div} \mathbf{z}_H \equiv 0$).
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Previous Work

1. Discretization of Helmholtz-projected linearized NSEs would need divergence-free finite elements.

   Here, we want to use standard discretization.

   (Taylor-Hood elements available in flow solver NAVIER)

   Explicit projection of ansatz functions possible using application of Helmholtz projection, but too expensive in general.

2. Each step of Newton-Kleinman iteration: Solve

$$A_j^T Z_{j+1} Z_{j+1}^T + Z_{j+1} Z_{j+1}^T A_j = -M - K_j^T K_j,$$

   $n_v := \operatorname{rank}(M) =$ dim of ansatz space for velocities.

   $\rightsquigarrow$ need to solve $n_v + m$ linear systems of equations in each step of Newton-ADI iteration!

3. Linearized system (i.e., $A + \omega M$) is unstable in general.

   But to start Newton iteration, a stabilizing initial guess is needed!

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annual meeting Freising 2010

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   [Benner '08-'10] *Partial Stabilization of Descriptor Systems Using Spectral Projectors*; to appear in V. Olshevsky et al (eds.), Numerical Linear Algebra in Signals, Systems and Control, Lecture Notes in Electrical Engineering, Springer-Verlag.

   [Hein '10] *MPC/LQG-Based Optimal Control of Nonlinear Parabolic PDEs*; PhD thesis Chemnitz UT.

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Key to Numerical Solution: Saddle Point System

Linearized NSE:

$$\partial_t \mathbf{z} - \nu \Delta \mathbf{z} + (\mathbf{z} \cdot \nabla)\mathbf{w} + (\mathbf{w} \cdot \nabla)\mathbf{z} + \nabla p = 0$$
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Newton iteration: $X^{k+1} = X^k + \tilde{N}^k$, where $\tilde{N}^k$ is solution of
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Lyapunov equation $\Rightarrow$ ADI-Method:
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Saddle Point System: [Heinkenschloss, Sorensen, Sun '08]

$$\begin{bmatrix} A^T + p_j M - K^k B^T & G \\ G^T & 0 \end{bmatrix} \begin{bmatrix} \Lambda \\ * \end{bmatrix} = \begin{bmatrix} W \\ 0 \end{bmatrix}$$

p_j ... ADI shift parameter

$K^k := MX^k B$ ... Feedback operator

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Analysis of the Saddle Point System

Properties

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- $A^T \neq A \in \mathbb{R}^{n_v \times n_v}$, sparse, discretized lin. Nav.-St. operator, constant
- $p_j \in \mathbb{R}^-$, changes in every ADI step
- $M \in \mathbb{R}^{n_v \times n_v}$, spd, sparse, mass matrix, constant
- $K^k \in \mathbb{R}^{n_v \times n_r}$, dense, feedback operator, changes in every Newton step
- $B \in \mathbb{R}^{n_v \times n_r}$, highly sparse, boundary operator, constant
- $G \in \mathbb{R}^{n_v \times n_p}$, sparse, full column rank, discretized gradient, constant

Solver

- Try to avoid direct solvers because of high dimensions.
- Use Krylov subspace method for non symmetric problems. $\Rightarrow$ GMRES

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Left Preconditioner

[Elman, Silvester, Wathen '06]

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$$P_l = \begin{bmatrix} \mathbf{P_F} & 0 \\ G^T & -P_{SC} \end{bmatrix} \Rightarrow P_l^{-1} = \begin{bmatrix} \mathbf{P_F}^{-1} & 0 \\ P_{SC}^{-1} G^T \mathbf{P_F}^{-1} & -P_{SC}^{-1} \end{bmatrix}$$

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$$P_l = \begin{bmatrix} \mathbf{P_F} & 0 \\ G^T & -P_{SC} \end{bmatrix} \Rightarrow P_l^{-1} = \begin{bmatrix} \mathbf{P_F}^{-1} & 0 \\ P_{SC}^{-1} G^T \mathbf{P_F}^{-1} & -P_{SC}^{-1} \end{bmatrix}$$

- Choice of $\mathbf{P_F}, P_{SC}$:

Preconditioned GMRES

Left Preconditioner

[Elman, Silvester, Wathen '06]

Derivation

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Schur Complement

- Choose *Schur complement* for lower right block: $P_{SC} = G^T \mathbf{F}^{-1} G$
- Can not build this matrix (dense, high dimension).
- Have to find a good approximation for P_{SC}.

Preconditioned GMRES

Schur Complement

Section 8.2 [Elman, Silvester, Wathen '06]

Approximation to the Schur Complement Operator

Consider the discrete version of a shifted Navier-Stokes operator:

$$\mathcal{L} = -\nu\nabla^2 + \vec{w}_h \cdot \nabla + \nabla \cdot \vec{w}_h + p_j I$$

Preconditioned GMRES

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Suppose analogous operator on pressure space:

$$\mathcal{L}_p = (-\nu\nabla^2 + \vec{w}_h \cdot \nabla + \nabla \cdot \vec{w}_h + p_j I)_p$$

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Consider a small commutator of $\mathcal{L}$ with the gradient:

$$\mathcal{E} = (-\nu\nabla^2 + \vec{w}_h \cdot \nabla + \nabla \cdot \vec{w}_h + p_j I)\nabla - \nabla(-\nu\nabla^2 + \vec{w}_h \cdot \nabla + \nabla \cdot \vec{w}_h + p_j I)_p$$

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Discrete version of commutator in terms of finite element matrices:

$$\mathcal{E}_h = (M^{-1}\mathbf{F})(M^{-1}G) - (M^{-1}G)(M_p^{-1}F_p)$$

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$$G^T\mathbf{F}^{-1}G \approx G^T M^{-1} G F_p^{-1} M_p$$

Preconditioned GMRES

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Schur complement approximation:

$$P_{SC} := S_p F_p^{-1} M_p$$

$$\Rightarrow P_{SC}^{-1} = M_p^{-1} F_p S_p^{-1}$$

Preconditioned GMRES

Preconditioner

[Elman, Silvester, Wathen '06]

Approximation of $\mathbf{P_F}$

- Need approximation for $\mathbf{P_F}$.
- For high Reynolds numbers the approximation has to get more accurate.
- In the future we want to use some V-cycle `Multigrid` steps.

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Simplification

We considered $\mathbf{F}$ instead of $A^T + p_j M - KB^T$.
Why can we do this?
$\Rightarrow$ *Feedback-invariance of Krylov Subspaces*

Preconditioned GMRES

Feedback-invariance of the Krylov Subspaces

Saddle Point System

$$\begin{bmatrix} A^T + p_j M - K^k B^T & G \\ G^T & 0 \end{bmatrix} \begin{bmatrix} \Lambda \\ * \end{bmatrix} = \begin{bmatrix} W \\ 0 \end{bmatrix}$$

Preconditioned GMRES

Feedback-invariance of the Krylov Subspaces

Saddle Point System

$$\begin{bmatrix} A^T + p_j M - K^k B^T & G \\ G^T & 0 \end{bmatrix} \begin{bmatrix} \Lambda \\ * \end{bmatrix} = \begin{bmatrix} [C^T K^k] \\ 0 \end{bmatrix}$$

Preconditioned GMRES

Feedback-invariance of the Krylov Subspaces

Saddle Point System

$$\left(\begin{bmatrix} A^T + p_j M & G \\ G^T & 0 \end{bmatrix} - \begin{bmatrix} K^k \\ 0 \end{bmatrix} \cdot \begin{bmatrix} B \\ 0 \end{bmatrix}^T \right) \begin{bmatrix} \Lambda \\ * \end{bmatrix} = \left[\begin{bmatrix} C^T \\ 0 \end{bmatrix} \begin{bmatrix} K^k \\ 0 \end{bmatrix} \right]$$

Preconditioned GMRES

Feedback-invariance of the Krylov Subspaces

Saddle Point System

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Feedback Invariance of Subspaces [Benner/Beckermann '11]

$$\mathcal{K}_m(\mathbf{A}^k, \mathbf{W}^k) = \mathcal{K}_m(\mathbf{F_S}, \mathbf{W}^k)$$

Preconditioned GMRES

Feedback-invariance of the Krylov Subspaces

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\mathcal{K}_m(\mathbf{A}^k, \mathbf{W}^k) &= \mathcal{K}_m(\mathbf{F_S} - \mathbf{K}^k\mathbf{B}^T), [\mathbf{C}^T\,\mathbf{K}^k])\\
&= \mathrm{sp}[\mathbf{C}^T, \mathbf{K}^k, \mathbf{F_S}\mathbf{C}^T - \mathbf{K}^k\mathbf{B}^T\mathbf{C}^T, \mathbf{F_S}\mathbf{K}^k - \mathbf{K}^k\mathbf{B}^T\mathbf{K}^k, \ldots]\\
&= \mathrm{sp}[\mathbf{C}^T, \mathbf{K}^k, \mathbf{F_S}\mathbf{C}^T, \mathbf{F_S}\mathbf{K}^k, \ldots]\\
&= \mathcal{K}_m(\mathbf{F_S}, [\mathbf{C}^T\,\mathbf{K}^k]) = \mathcal{K}_m(\mathbf{F_S}, \mathbf{W}^k)
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Preconditioned GMRES

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Preconditioned GMRES

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$$\mathrm{sp}[-\mathbf{K}^k\mathbf{B}^T\mathbf{C}^T] \subseteq \mathrm{sp}[\mathbf{K}^k]$$

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\end{aligned}$$

Preconditioned GMRES

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&= \mathcal{K}_m(\mathbf{F_S}, [\mathbf{C}^T\,\mathbf{K}^k]) = \mathcal{K}_m(\mathbf{F_S}, \mathbf{W}^k)
\end{aligned}$$

Problem Setting

Example: KÃ¡rmÃ¡n vortex street

- Create matrices with FEM software NAVIER.
- Discretization of the domain with conformal *Taylor-Hood elements*.
- P2-P1 elements fulfill the LBB condition.
  $\Rightarrow$ initial stable discretization

Level	n_v	n_p
1	3 452	453
2	8 726	1 123
3	20 512	2 615
4	45 718	5 783
5	99 652	12 566
6	211 452	26 572

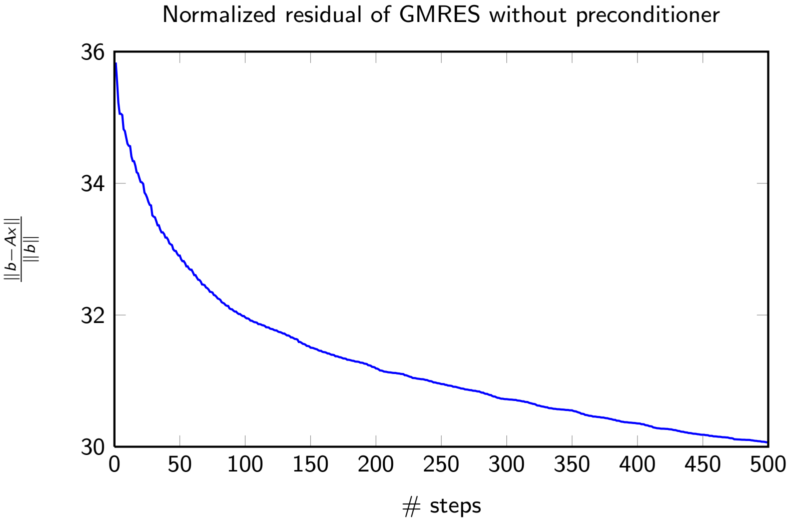
Tabelle: Levels of refinement

GMRES

Computation Comparison

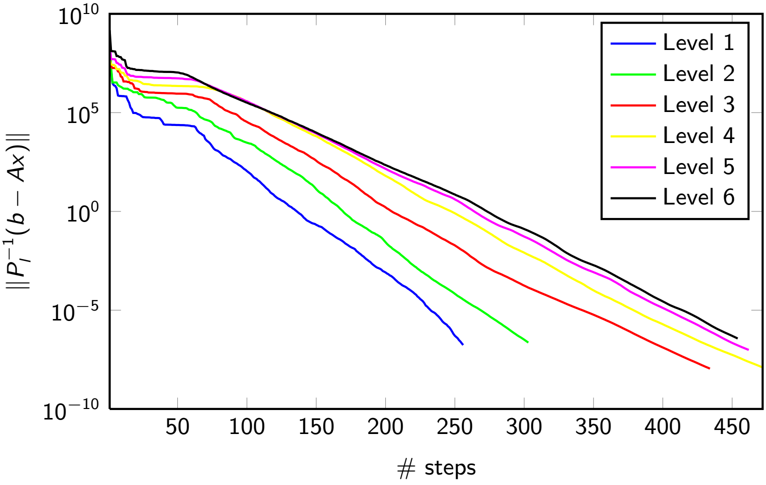
- GMRES with P_l as left preconditioner.
- Initial data: $b = rand(n_v + n_p, 1), x^{(0)} = zeros(n_v + n_p, 1)$
- Computations using MATLAB 2010b on `editha`:
  CPU type: Intel®Xeon®@2.67GHz,
  #CPUs: 4, #Cores: 32 (8 per CPU), RAM: 1 TB

GMRES


Normalized residual of GMRES without preconditioner
36
34
32
30
0
50
100
150
200
250
300
350
400
450
500
$\frac{\|b-Ax\|}{\|b\|}$
steps

GMRES

Preconditioned residuals in GMRES



GMRES

Computation Comparison

- GMRES with P_l as left preconditioner.
- Initial data: $b = rand(n_v + n_p, 1), x^{(0)} = zeros(n_v + n_p, 1)$
- Computations using MATLAB 2010b on `editha`:
  CPU type: Intel®Xeon®@2.67GHz,
  #CPUs: 4, #Cores: 32 (8 per CPU), RAM: 1 TB

Level	# it	$\frac{\|\|P_l^{-1}(b-Ax^{end})\|\|}{1}$	$\frac{\|\|b-Ax^{end}\|\|}{\|\|b\|\|}$	$\frac{res_{end}}{res_0}$
1	256	$8.6 \cdot 10^{-9}$	$7.5 \cdot 10^{-6}$	$2.8 \cdot 10^{-16}$
2	303	$2.4 \cdot 10^{-11}$	$9.5 \cdot 10^{-6}$	$2.7 \cdot 10^{-16}$
3	434	$5.3 \cdot 10^{-10}$	$4.3 \cdot 10^{-6}$	$2.4 \cdot 10^{-16}$
4	472	$9.9 \cdot 10^{-10}$	$8.8 \cdot 10^{-6}$	$2.5 \cdot 10^{-16}$
5	462	$4.5 \cdot 10^{-10}$	$1.5 \cdot 10^{-5}$	$2.4 \cdot 10^{-16}$
6	454	$3.5 \cdot 10^{-10}$	$1.6 \cdot 10^{-5}$	$2.5 \cdot 10^{-16}$
no P_l	500	$8.1 \cdot 10^{-1}$	$8.1 \cdot 10^{-1}$	$8.4 \cdot 10^{-1}$

GMRES

Computation Comparison

- GMRES with P_l as left preconditioner.
- Initial data: $b = rand(n_v + n_p, 1), x^{(0)} = zeros(n_v + n_p, 1)$
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Level	# it	$\frac{\|\|P_l^{-1}(b-Ax^{end})\|\|}{1}$	$\frac{\|\|b-Ax^{end}\|\|}{\|\|b\|\|}$	$\frac{res_{end}}{res_0}$
1	256	$8.6 \cdot 10^{-9}$	$7.5 \cdot 10^{-6}$	$2.8 \cdot 10^{-16}$
2	303	$2.4 \cdot 10^{-11}$	$9.5 \cdot 10^{-6}$	$2.7 \cdot 10^{-16}$
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6	454	$3.5 \cdot 10^{-10}$	$1.6 \cdot 10^{-5}$	$2.5 \cdot 10^{-16}$
no P_l	500	$8.1 \cdot 10^{-1}$	$8.1 \cdot 10^{-1}$	eps $= 2.22 \cdot 10^{-16}$

GMRES

Computation Comparison

- GMRES with P_l as left preconditioner.
- Initial data: $b = rand(n_v + n_p, 1), x^{(0)} = zeros(n_v + n_p, 1)$
- Computations using MATLAB 2010b on `editha`:
  CPU type: Intel®Xeon®@2.67GHz,
  #CPUs: 4, #Cores: 32 (8 per CPU), RAM: 1 TB

Many thanks for your attention.

Level	# it	$\lVert P_l^{-1}(b - Ax^{end})\rVert$	$\frac{\lVert b - Ax^{end}\rVert}{\lVert b\rVert}$	$\frac{res_{end}}{res_0}$
1	256	$8.6 \cdot 10^{[illegible]}$	$7.5 \cdot 10^{-6}$	$2.8 \cdot 10^{-16}$
2	303	$2.4 \cdot 10^{-11}$	$9.5 \cdot 10^{-6}$	$2.7 \cdot 10^{-16}$
3	434	$5.3 \cdot 10^{-10}$	$4.3 \cdot 10^{-6}$	$2.4 \cdot 10^{-16}$
4	472	$9.9 \cdot 10^{-10}$	$8.8 \cdot 10^{-6}$	$2.5 \cdot 10^{-16}$
5	462	$4.5 \cdot 10^{-10}$	$1.5 \cdot 10^{-5}$	$2.4 \cdot 10^{-16}$
6	454	$3.5 \cdot 10^{-10}$	$1.6 \cdot 10^{-5}$	$2.5 \cdot 10^{-16}$
no P_l	500	$8.1 \cdot 10^{-1}$	$8.1 \cdot 10^{-1}$	$8.4 \cdot 10^{-1}$

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