Riccati-Based Boundary Feedback Stabilization of Incompressible Flow Problems

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Overview

Introduction

Motivation

- Asymptotic stabilization of partial differential equations.
- Feedback control approach applied to multi-field flow problems.
- Coupling of flow with other field equations.
- Suggest different scenarios with increasing difficulty.
- First proof of concepts:
  - "von Kármán vortex street",
  - Navier-Stokes equations to describe flow.

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Navier-Stokes Equations

$$\left.\begin{aligned} \frac{\partial}{\partial t}\mathbf{v} - \frac{1}{\mathrm{Re}}\Delta\mathbf{v} + \mathbf{v}\cdot\nabla\mathbf{v} + \nabla p &= \mathbf{0} \\ \nabla\cdot\mathbf{v} &= \mathbf{0} \end{aligned}\right\} \quad \text{on } (0,\infty)\times\Omega,$$

with $\Omega \subset \mathbb{R}^2$ and bounded with $\Gamma = \partial\Omega$.

Introduction

Basic Ideas

- Riccati-based feedback stabilization with boundary control input.
- Analytical approach by [Raymond '05–'07].
  ↪ Uses *Leray* projector to project onto the correct subspace.
- Ideas for numerical treatment based on [Bänsch/Benner '10].
  ↪ Consider linearized Navier-Stokes equations for 2D.
  ↪ Discrete projection idea by [Heinkenschloss/Sorensen/Sun '08].

Discretized Control System

Finite Element Discretization

- Standard finite element discretization of linearized NSE yields

$$M\frac{d}{dt}\mathbf{z}(t) = A\mathbf{z}(t) + G\mathbf{p}(t) + \mathbf{f}(t), \tag{1a}$$

$$\mathbf{0} = G^T\mathbf{z}(t). \tag{1b}$$

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\end{aligned}
$$

Properties

- Differential algebraic system (DAE) of D-index 2.
- Matrix pencil:

$$
\left(\begin{bmatrix} A & G \\ G^T & 0 \end{bmatrix}, \begin{bmatrix} M & 0 \\ 0 & 0 \end{bmatrix} \right).
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- Descriptor system with multiple inputs and outputs (MIMO).
- Index reduction to apply linear quadratic control approach (LQR).

Discretized Control System

Projection Method [Heinkenschloss/Sorensen/Sun '08]

- Index reduction for Lyapunov-solver.
- Projector:

$$\Pi := I - G(G^T M^{-1} G)^{-1} G^T M^{-1}.$$

- For $G^T \mathbf{z}(t) = \mathbf{0} \Leftrightarrow \Pi^T \mathbf{z}(t) = \mathbf{z}(t)$.
- Correct solution manifold (*hidden manifold*)

$$\mathbf{0} = G^T M^{-1} A\mathbf{z}(t) + G^T M^{-1} G\mathbf{p}(t) + G^T M^{-1} B\mathbf{u}(t),$$

is invariant under Π^T.

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Discrete *Leray* Projector [Benner/Saak/W. '11]

- Lyapunov-solver used during Newton iteration.
- Π^T is symmetric w.r.t. $(.\,,.)_M$ (i.e., L_2-orthogonal).
- Π^T is discrete version of *Leray* projector.

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- Projected version of System (1):

$$\mathcal{M}\frac{d}{dt}\tilde{\mathbf{z}}(t) = \mathcal{A}\tilde{\mathbf{z}}(t) + \mathcal{B}\mathbf{u}(t),$$
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with $\tilde{\mathbf{z}} \in \mathbb{R}^{n_v - n_p}$ and $\mathcal{M} = \mathcal{M}^T \succ 0$.

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- For balanced truncation the generalized Lyapunov equations

$$\mathcal{A}\tilde{P}\mathcal{M}^T + \mathcal{M}\tilde{P}\mathcal{A}^T = -\mathcal{B}\mathcal{B}^T,$$
$$\mathcal{A}^T\tilde{Q}\mathcal{M} + \mathcal{M}^T\tilde{Q}\mathcal{A} = -\mathcal{C}^T\mathcal{C},$$

have to be solved.

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Feedback Control Approach

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Generalized algebraic Riccati equation:

$$\mathcal{R}(X) = \mathcal{C}^T\mathcal{C} + \mathcal{A}^T X\mathcal{M} + \mathcal{M}^T X\mathcal{A} - \mathcal{M}^T X\mathcal{B}\mathcal{B}^T X\mathcal{M} = \mathbf{0}.$$

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Lyapunov equation with low-rank rhs:

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Compare: [Heinkenschloss/Sorensen/Sun '08]

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Saddle Point System: [Heinkenschloss/Sorensen/Sun '08]

$$\begin{bmatrix} A^T - M^T X^{(m)} B B^T + p_i M^T & G \\ G^T & 0 \end{bmatrix} \begin{bmatrix} \Lambda \\ * \end{bmatrix} = \begin{bmatrix} Y \\ 0 \end{bmatrix}.$$

Solving Large-Scale Saddle Point Systems

Properties of Saddle Point System

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Current Research

Preconditioned Iterative Method

Block Preconditioner [Elman/Silvester/Wathen '05]

$$\mathbf{P} = \begin{bmatrix} P_F & 0 \\ G^T & -P_{SC} \end{bmatrix} \qquad (2)$$

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- *Multigrid*-methods: $P_F \approx F = A^T + p_i M^T$ (not yet implemented).
- Least-squares commutator approach:

$$P_{SC} \approx S_p F_p^{-1} M_p \Rightarrow P_{SC}^{-1} \approx M_p^{-1} F_p S_p^{-1}.$$

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- Iterative Solver: GMRES with preconditioner (2).
- Method is robust with respect to the mesh parameter.
- Reynolds number Re and ADI shifts p_i influence convergence rate.

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Lyapunov equation with low-rank rhs: $\Rightarrow$ ADI-Method

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Outermost loop:

Stopping criterion: tol_{newt}

$\hookrightarrow$ Number of Newton-steps

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Central loop:

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ADI shifts: p_i

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Innermost loop:

Stopping criterion: tol_{GMRES}

Preconditioner

↪ Number of GMRES-steps

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Lyapunov equation with low-rank rhs: ⇒ ADI-Method

$$(\mathcal{A}^{(m)})^T X^{(m+1)} \mathcal{M} + \mathcal{M}^T X^{(m+1)} \mathcal{A}^{(m)} = -(\mathcal{W}^{(m)})^T \mathcal{W}^{(m)}.$$

Use GMRES to solve Saddle Point System:

$$\begin{bmatrix} A^T + p_i M^T & G \\ G^T & 0 \end{bmatrix} \begin{bmatrix} \Lambda \\ * \end{bmatrix} = \begin{bmatrix} \tilde{Y} \\ 0 \end{bmatrix}.$$

Conclusions

Nested Iteration

State space system:

Outermost loop:

Stopping criterion: tol_{newt}

$\hookrightarrow$ Number of Newton-steps

Central loop:

Stopping criterion: tol_{ADI}

ADI shifts: p_i

$\hookrightarrow$ Number of ADI-steps

Innermost loop:

Stopping criterion: tol_{GMRES}

Preconditioner

$\hookrightarrow$ Number of GMRES-steps

...lized algebraic Riccati ...lation:

$$\ldots = \mathcal{C}^T\mathcal{C} + \mathcal{A}^T X \mathcal{M} + \mathcal{M}^T X \mathcal{A} - \mathcal{M}^T X \mathcal{B}\mathcal{B}^T X \mathcal{M}\ldots$$

Newton iteration: $X^{(m+1)} = X^{(m)} + N^{(m)}$, where $N^{(m)}$ is solution of

$$(\mathcal{A} - \mathcal{B}\mathcal{B}^T X^{(m)} \mathcal{M})^T N^{(m)} \mathcal{M} + \mathcal{M}^T N^{(m)} (\mathcal{A} - \mathcal{B}\mathcal{B}^T X^{(m)} \mathcal{M}) = -\mathcal{R}(X^{(m)}).$$

$\downarrow$

Lyapunov equation with low-rank rhs: $\Rightarrow$ ADI-Method

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Conclusions – Numerical Example

Accuracy of GMRES vs. time for a Newton step (Re = 10, DoF= 3 905)

tol_{GMRES}	∅ GMRES-steps	# ADI-steps	time
10^{-6}	13	> 500	> 2800 sec.
10^{-7}	15	82	375 sec.
10^{-8}	16	71	348 sec.
10^{-9}	18	65	330 sec.
10^{-10}	19	55	308 sec.
10^{-11}	20	55	335 sec.
10^{-12}	21	55	341 sec.
10^{-13}	23	55	386 sec.
10^{-14}	23	55	458 sec.
10^{-15}	24	55	494 sec.
10^{-16}	25	55	491 sec.
"direct solver"	- -	55	7 sec.

Conclusions

Outlook

- Improve choice of ADI shifts.

  $\hookrightarrow$ Main problem: computation of largest finite eigenvalues of the DAE-pencil.
- Recycling techniques for iterative solves with multiple rhs.
- Non-conforming finite elements that guarantee $G^T\mathbf{v} = 0$ for the solution (not divergence free FEM-basis).

  $\hookrightarrow$ Start cooperation with F. Schieweck, OvGU Magedburg.

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Many thanks for your attention!

Literature

- E. Bänsch and P. Benner, *Stabilization of Incompressible Flow Problems by Riccati-Based Feedback*, in Constrained Optimization and Optimal Control for Partial Differential Equations, G. Leugering, S. Engell, A. Griewank, M. Hinze, R. Rannacher, V. Schulz, M. Ulbrich, and S. Ulbrich, eds., vol. 160 of International Series of Numerical Mathematics, Birkhäuser, 2012, pp. 5–20.
- H. Elman, D. Silvester, and A. Wathen, *Finite Elements and Fast Iterative Solvers: with applications in incompressible fluid dynamics*, Oxford University Press, Oxford, 2005.
- M. Heinkenschloss, D. C. Sorensen, and K. Sun, *Balanced truncation model reduction for a class of descriptor systems with application to the Oseen equations*, SIAM Journal on Scientific Computing, 30 (2008), pp. 1038–1063.
- J. Raymond, *Feedback boundary stabilization of the two-dimensional Navier-Stokes equations*, SIAM Journal on Control and Optimization, 45 (2006), pp. 790–828.