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Second Order to Second Order Balancing for Index-1 Vibrational Systems

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MAX PLANCK INSTITUTE
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TECHNICAL SYSTEMS
MAGDEBURG

Motivation

Adaptive Spindel Support (ASS) with Piezo Actuators

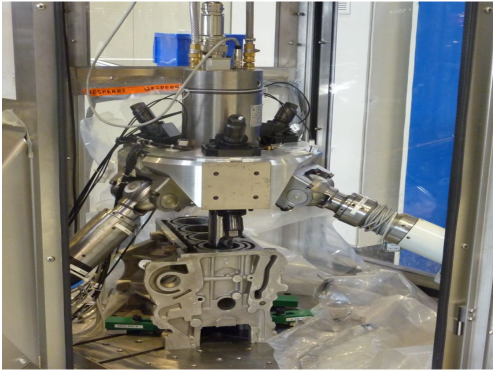


Figure: ASS mounted in Parallel-Kinematic Machine

Source: B. Kranz, Fraunhofer IWU, Dresden, Germany.

Motivation

Adaptive Spindel Support (ASS) with Piezo Actuators

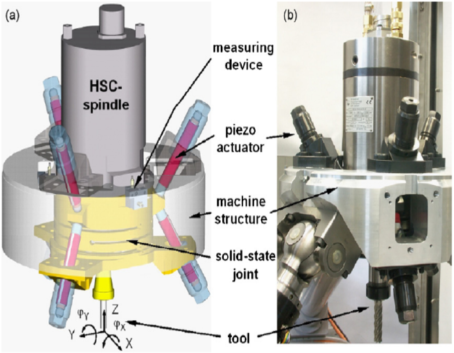


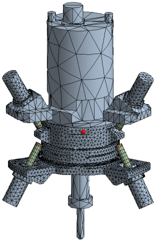
Figure: (a) ASS mounted in (b) Parallel-Kinematic Machine

Source: B. Kranz, Fraunhofer IWU, Dresden, Germany.

Motivation

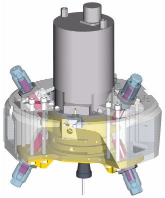
Adaptive Spindel Support (ASS) with Piezo Actuators

Mathematical Model for Controller Design



FEM

Second Order Index-1 System



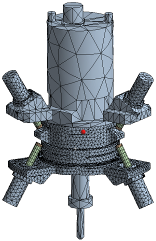
$$\mathcal{M}\ddot{x}(t) + \mathcal{D}\dot{x}(t) + \mathcal{K}x(t) = \mathcal{H}u(t)$$
$$y(t) = \mathcal{H}^T x(t)$$

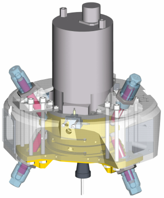
$\mathcal{M}$ mass matrix, $\mathcal{D}$ damping matrix, $\mathcal{K}$ stiffness matrix

Motivation

Adaptive Spindel Support (ASS) with Piezo Actuators

Mathematical Model for Controller Design



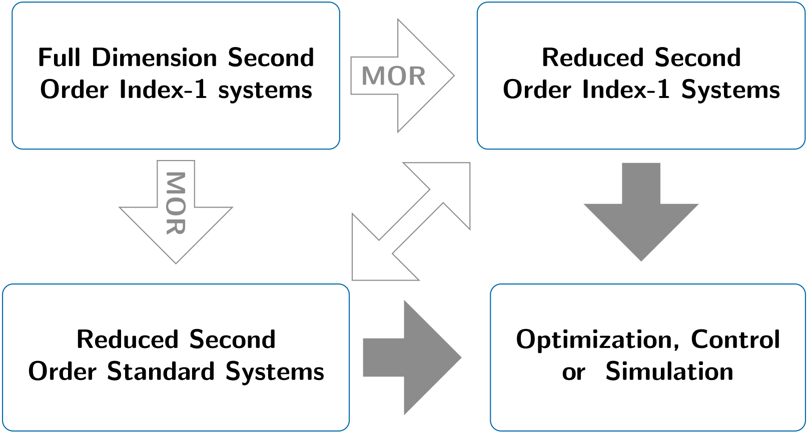


$$\begin{bmatrix} M_1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \ddot{z}(t) \\ \ddot{\varphi}(t) \end{bmatrix} + \begin{bmatrix} D_1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{z}(t) \\ \dot{\varphi}(t) \end{bmatrix} + \begin{bmatrix} K_{11} & K_{12} \\ K_{12}^T & K_{22} \end{bmatrix} \begin{bmatrix} z(t) \\ \varphi(t) \end{bmatrix} = \begin{bmatrix} H_1 \\ H_2 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} H_1^T & H_2^T \end{bmatrix} \begin{bmatrix} z(t) \\ \varphi(t) \end{bmatrix}$$

Motivation

Goal



Balanced Truncation for Large Scale Systems

First Order Systems

- Given LTI continuous-time system

$$\Sigma : \boxed{\dot{x}(t) = Ax(t) + Bu(t), \quad y(t) = Cx(t) + D_a u(t)}$$

where $x(t) \in \mathbb{R}^n, u(t) \in \mathbb{R}^m, y(t) \in \mathbb{R}^p$ and A, B, C and D_a are matrices

- The realization (A, B, C, D_a), of the system Σ, is called balanced, if the solutions P, Q of the Lyapunov equations (LE)

  - $AP + PA^T + BB^T \;=\; 0,$ [controllability]
  - $A^T Q + QA + C^T C \;=\; 0,$ [observability]

satisfy: $P = Q = \mathrm{diag}(\sigma_1, \ldots, \sigma_n)$ where $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_n > 0$

- $\{\sigma_1, \ldots, \sigma_n\}$ are the Hankel singular values (HSVs) of Σ

Balanced Truncation for Large Scale Systems

First Order Systems

- Given LTI continuous-time system

$$\Sigma : \boxed{\dot{x}(t) = Ax(t) + Bu(t), \quad y(t) = Cx(t) + D_a u(t)}$$

- A balanced realization is computed via state space transformation

$$\mathcal{T} : (A, B, C, D_a) \mapsto (TAT^{-1}, TB, CT^{-1}, D_a)$$
$$= \left(\begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}, \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \begin{bmatrix} C_1 & C_2 \end{bmatrix}, D_a \right)$$

- Form $k \ll n$ dimensional reduced order model:

$$\hat{\Sigma} : \boxed{\dot{\hat{x}}(t) = \hat{A}\hat{x}(t) + \hat{B}u(t), \quad \hat{y}(t) = \hat{C}\hat{x}(t) + \hat{D}_a u(t)}$$

where $(\hat{A}, \hat{B}, \hat{C}, \hat{D}_a) = (A_1, B_1, C_1, D_a)$

- Such that $\|y - \hat{y}\|_\infty$ is small enough

Balanced Truncation for Large Scale Systems

First Order Systems

The SR Method (Implementation of BT)

1. Compute (Cholesky) factors of the solutions to the Lyapunov equation,
$$P = S^T S, \quad Q = R^T R$$
2. Compute singular value decomposition
$$SR^T = [\, U_1,\ U_2 \,] \begin{bmatrix} \Sigma_1 & \\ & \Sigma_2 \end{bmatrix} \begin{bmatrix} V_1^T \\ V_2^T \end{bmatrix}$$
3. Define
$$W := R^T V_1 \Sigma_1^{-1/2} \in \mathbb{R}^{n \times k} \qquad V := S^T U_1 \Sigma_1^{-1/2} \in \mathbb{R}^{n \times k}$$
4. Then
$$\hat{\Sigma} = (\hat{A}, \hat{B}, \hat{C}, \hat{D}_a) = (W^T A V, W^T B, C V, D_a)$$

Balanced Truncation for Large Scale Systems

Second Order Systems

Second Order Form

$$
\begin{aligned}
M\ddot{x} + D\dot{x} + Kx &= Hu \\
y &= Lx
\end{aligned}
$$

First Order Form

$$
\begin{aligned}
\mathcal{E}\dot{z} &= \mathcal{A}z + \mathcal{B}u \\
y &= \mathcal{C}z
\end{aligned}
$$

- x displacements
- $z = \left[\dot{x}^T, x^T\right]^T$
- $M, D, K \in \mathbb{R}^{n_1 \times n_1}$ invertible
- J arbitrary but invertible

$$
\mathcal{E} = \begin{bmatrix} 0 & J \\ M & D \end{bmatrix}, \quad \mathcal{A} = \begin{bmatrix} J & 0 \\ 0 & -K \end{bmatrix},
$$

$$
\mathcal{B} = \begin{bmatrix} 0 \\ H \end{bmatrix}, \quad \mathcal{C} = \begin{bmatrix} 0 & L \end{bmatrix}
$$

Balanced Truncation for Large Scale Systems

Second Order Systems

Second Order Form

$$M\ddot{x} + D\dot{x} + Kx = Hu$$
$$y = Lx$$

First Order Form

$$\mathcal{E}\dot{z} = \mathcal{A}z + \mathcal{B}u$$
$$y = \mathcal{C}z$$

- x displacements
- $z = \left[\dot{x}^T, x^T\right]^T$
- $M, D, K \in \mathbb{R}^{n_1 \times n_1}$ invertible
- $J = I$ identity matrix

$$\mathcal{E} = \begin{bmatrix} 0 & I \\ M & D \end{bmatrix}, \quad \mathcal{A} = \begin{bmatrix} I & 0 \\ 0 & -K \end{bmatrix},$$
$$\mathcal{B} = \begin{bmatrix} 0 \\ H \end{bmatrix}, \quad \mathcal{C} = \begin{bmatrix} 0 & L \end{bmatrix}$$

Balanced Truncation for Large Scale Systems

Second Order Systems

Second Order Form

$$M\ddot{x} + D\dot{x} + Kx = Hu$$
$$y = Lx$$

First Order Form

$$\mathcal{E}\dot{z} = \mathcal{A}z + \mathcal{B}u$$
$$y = \mathcal{C}z$$

- x displacements
- $z = \left[\dot{x}^T, x^T\right]^T$
- $M, D, K \in \mathbb{R}^{n_1 \times n_1}$ invertible
- $J = M$ for symmetric system

$$\mathcal{E} = \begin{bmatrix} 0 & M \\ M & D \end{bmatrix}, \quad \mathcal{A} = \begin{bmatrix} M & 0 \\ 0 & -K \end{bmatrix},$$
$$\mathcal{B} = \begin{bmatrix} 0 \\ H \end{bmatrix}, \quad \mathcal{C} = \begin{bmatrix} 0 & L \end{bmatrix}$$

- First order form is symmetric since $\mathcal{A}$ and $\mathcal{E}$ symmetric
- Apply BT methods on first order systems

Balanced Truncation for Large Scale Systems

Second Order Systems

Second Order Systems

$$M\ddot{x} + D\dot{x} + Kx = Hu, \quad y = Lx$$

Projectors

$$W, V \in \mathbb{R}^{n_1 \times k}?$$

Second Order Projected Systems

$$W^T M V \ddot{\hat{x}} + W^T D V \dot{\hat{x}} + W^T K V \hat{x} = W^T H u$$
$$\hat{y} = L V \hat{x}$$

Second Order Reduced Systems

$$\hat{M}\ddot{\hat{x}} + \hat{D}\dot{\hat{x}} + \hat{K}\hat{x} = \hat{H}u, \quad \hat{y} = \hat{L}\hat{x}$$

Balanced Truncation for Large Scale Systems

Second Order Systems

Second Order Systems

$$M\ddot{x} + D\dot{x} + Kx = Hu, \quad y = Lx$$

First Order Form

$$\mathcal{E}\dot{z} = \mathcal{A}z + \mathcal{B}u, \; y = \mathcal{C}z$$

Second Order Projected Systems

$$W^T M V \ddot{\hat{x}} + W^T D V \dot{\hat{x}} + W^T K V \hat{x} = W^T H u$$
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Balanced Truncation for Large Scale Systems

Second Order Systems

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$$\hat{M}\ddot{\hat{x}} + \hat{D}\dot{\hat{x}} + \hat{K}\hat{x} = \hat{H}u, \quad \hat{y} = \hat{L}\hat{x}$$

First Order Form

$$\mathcal{E}\dot{z} = \mathcal{A}z + \mathcal{B}u, \; y = \mathcal{C}z$$

$$\mathcal{E} = \begin{bmatrix} 0 & M \\ M & D \end{bmatrix}, \; \mathcal{B} = \begin{bmatrix} 0 \\ H \end{bmatrix},$$

$$\mathcal{A} = \begin{bmatrix} M & 0 \\ 0 & -K \end{bmatrix}, \; \mathcal{C} = \begin{bmatrix} 0 \\ L \end{bmatrix}^T$$

Balanced Truncation for Large Scale Systems

Second Order Systems

Second Order Systems

$$M\ddot{x} + D\dot{x} + Kx = Hu, \quad y = Lx$$

Second Order Projected Systems

$$W^T M V \ddot{\hat{x}} + W^T D V \dot{\hat{x}} + W^T K V \hat{x} = W^T H u$$
$$\hat{y} = L V \hat{x}$$

Second Order Reduced Systems

$$\hat{M}\ddot{\hat{x}} + \hat{D}\dot{\hat{x}} + \hat{K}\hat{x} = \hat{H}u, \quad \hat{y} = \hat{L}\hat{x}$$

First Order Form

$$\mathcal{E}\dot{z} = \mathcal{A}z + \mathcal{B}u, \; y = \mathcal{C}z$$

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$$\mathcal{A} = \begin{bmatrix} M & 0 \\ 0 & -K \end{bmatrix}, \; \mathcal{C} = \begin{bmatrix} 0 \\ L \end{bmatrix}^T$$

Gramians

$$P = \begin{bmatrix} P_v & P_0 \\ P_0^T & P_p \end{bmatrix} \approx S^T S = \begin{bmatrix} S_v^T \\ S_p^T \end{bmatrix} \begin{bmatrix} S_v & S_p \end{bmatrix}$$

Balanced Truncation for Large Scale Systems

Second Order Systems

Second Order Systems

$$M\ddot{x} + D\dot{x} + Kx = Hu, \quad y = Lx$$

Second Order Projected Systems

$$W^T M V \ddot{\hat{x}} + W^T D V \dot{\hat{x}} + W^T K V \hat{x} = W^T H u$$
$$\hat{y} = L V \hat{x}$$

Second Order Reduced Systems

$$\hat{M}\ddot{\hat{x}} + \hat{D}\dot{\hat{x}} + \hat{K}\hat{x} = \hat{H}u, \quad \hat{y} = \hat{L}\hat{x}$$

First Order Form

$$\mathcal{E}\dot{z} = \mathcal{A}z + \mathcal{B}u, \; y = \mathcal{C}z$$

$$\mathcal{E} = \begin{bmatrix} 0 & M \\ M & D \end{bmatrix}, \; \mathcal{B} = \begin{bmatrix} 0 \\ H \end{bmatrix},$$
$$\mathcal{A} = \begin{bmatrix} M & 0 \\ 0 & -K \end{bmatrix}, \; \mathcal{C} = \begin{bmatrix} 0 \\ L \end{bmatrix}^T$$

Gramians

$$Q = \begin{bmatrix} Q_v & Q_0 \\ Q_0^T & Q_p \end{bmatrix} \approx R^T R = \begin{bmatrix} R_v^T \\ R_p^T \end{bmatrix} \begin{bmatrix} R_v & R_p \end{bmatrix}$$

Balanced Truncation for Large Scale Systems

Second Order Systems

Four Types of Left and Right Projectors

type	SVD	left proj. W	right proj. V
vv	$S_v M R_v^T = U_{vv}\Sigma_{vv}V_{vv}^T$	$R_v^T V_{vv,r}\Sigma_{vv,1}^{-\frac{1}{2}}$	$S_v^T U_{vv,1}\Sigma_{vv,1}^{-\frac{1}{2}}$
pp	$S_p M R_p^T = U_{pp}\Sigma_{pp}V_{pp}^T$	$R_p^T V_{pp,r}\Sigma_{pp,r}^{-\frac{1}{2}}$	$S_p^T U_{pp,r}\Sigma_{pp,r}^{-\frac{1}{2}}$
vp	$S_v M R_p^T = U_{vp}\Sigma_{vp}V_{vp}^T$	$R_p^T V_{vp,r}\Sigma_{vp,r}^{-\frac{1}{2}}$	$S_v^T U_{vp,r}\Sigma_{vp,r}^{-\frac{1}{2}}$
pv	$S_p M R_v^T = U_{pv}\Sigma_{pv}V_{pv}^T$	$R_v^T V_{pv,r}\Sigma_{pv,r}^{-\frac{1}{2}}$	$S_p^T U_{pv,r}\Sigma_{pv,r}^{-\frac{1}{2}}$

vv = velocity-velocity, pp = position-position,

vp = velocity-position, pv = position-velocity,

r = first r columns of the respective matrix

Form four types of reduced order model:

$$\hat{M} = W^T M V, \qquad \hat{D} = W^T D V, \qquad \hat{K} = W^T K V,$$

$$\hat{H} = W^T H, \qquad \hat{L} = LV$$

Solving Large Lyapunov Equations

LRCF-ADI [Benner/Li/Penzl '08]

Given $FX + XF^T = -GG^T$ $F \in \mathbb{R}^{n \times n}, G \in \mathbb{R}^{n \times p}$

- To solve Controllability LE : $F = A$ and $G = B$
- To solve Observability LE : $F = A^T$ and $G = C^T$

Task Find $Z \in \mathbb{C}^{n \times nz}$, such that $nz \ll n$ and $X \approx ZZ^H$

Solving Large Lyapunov Equations

LRCF-ADI

[Benner/Li/Penzl '08]

Given $FX + XF^T = -GG^T \qquad F \in \mathbb{R}^{n\times n}, G \in \mathbb{R}^{n\times p}$

- To solve Controllability LE : $F = A$ and $G = B$
- To solve Observability LE : $F = A^T$ and $G = C^T$

Task: Find $Z \in \mathbb{C}^{n\times nz}$, such that $nz \ll n$ and $X \approx ZZ^H$

Algorithm

$$V_1 = \sqrt{-2\,\mathrm{Re}\,(p_1)}(F + p_1 I)^{-1}G, \qquad Z_1 = V_1$$

$$V_i = \frac{\sqrt{\mathrm{Re}\,(p_i)}}{\sqrt{\mathrm{Re}\,(p_{i-1})}}\left[V_{i-1} - (p_i + \overline{p_{i-1}})(F + p_i I)^{-1}V_{i-1}\right], \qquad Z_i = [Z_{i-1}\, V_i]$$

- For certain shift parameters $\{p_1, ..., p_l\} \subset \mathbb{C}^-$
- Stop the algorithm if $\|FZ_iZ_i^H + Z_iZ_i^HF^T + GG^T\|$ is small

Solving Large Lyapunov Equations

G-LRCF-ADI (E invertible) [Saak '09]

Given $FXE^T + EXF^T = -GG^T$ $\quad E, F \in \mathbb{R}^{n\times n}, G \in \mathbb{R}^{n\times p}$

- To solve Controllability LE : $F = A$ and $G = B$
- To solve Observability LE : $F = A^T$ and $G = C^T$

Task Find $Z \in \mathbb{C}^{n\times nz}$, such that $nz \ll n$ and $X \approx ZZ^H$

Algorithm

$$V_1 = \sqrt{-2\,\mathrm{Re}\,(p_1)}(F + p_1 E)^{-1}G, \qquad Z_1 = V_1$$

$$V_i = \frac{\sqrt{\mathrm{Re}\,(p_i)}}{\sqrt{\mathrm{Re}\,(p_{i-1})}}\left[V_{i-1} - (p_i + \overline{p_{i-1}})(F + p_i E)^{-1} E V_{i-1}\right], \qquad Z_i = [Z_{i-1} V_i]$$

- For certain shift parameters $\{p_1, ..., p_l\} \subset \mathbb{C}^-$
- Stop the algorithm if $\|FZ_iZ_i^H E^T + EZ_iZ_i^H F^T + GG^T\|$ is small

Solving Large Lyapunov Equations

S-LRCF-ADI (index-1) [Rommes/Freitas/Martins '08]

Index-1 system

$$\begin{bmatrix} E_{11} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u(t)$$

Given $\tilde{F} X E_{11}^T + E_{11} X \tilde{F}^T = -\tilde{G}\tilde{G}^T, \qquad E_{11}, \tilde{F} \in \mathbb{R}^{n\times n}, \tilde{G} \in \mathbb{R}^{n\times p}$

$$\tilde{F} = F_{11} - F_{12}F_{22}{}^{-1}F_{21}, \quad \tilde{G} = B_1 - F_{12}F_{22}{}^{-1}B_2$$

Task Find $Z \in \mathbb{C}^{n\times nz}$, such that $nz \ll n$ and $X \approx ZZ^H$

Algorithm

$$\begin{bmatrix} V_1 \\ * \end{bmatrix} = \sqrt{-2\,\mathrm{Re}\,(p_1)} \begin{bmatrix} F_{11} + p_1 E_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix}^{-1} \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \qquad Z_1 = V_1$$

$$\begin{bmatrix} V_i \\ * \end{bmatrix} = \frac{\sqrt{\mathrm{Re}\,(p_i)}}{\sqrt{\mathrm{Re}\,(p_{i-1})}} \left[V_{i-1} - (p_i + \overline{p_{i-1}}) \begin{bmatrix} F_{11} + p_i E_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix}^{-1} \begin{bmatrix} E_{11} V_{i-1} \\ 0 \end{bmatrix} \right], \qquad Z_i = [Z_{i-1} V_i]$$

Solving Large Lyapunov Equations

S-LRCF-ADI (Real low-rank factor) [Benner/Kürschner/Saak '12]

Given $\tilde{F}XE_{11}^T + E_{11}X\tilde{F} = -\tilde{G}\tilde{G}^T, \qquad E_{11}, \tilde{F} \in \mathbb{R}^{n\times n}, \tilde{G} \in \mathbb{R}^{n\times p}$

$$\tilde{F} = F_{11} - F_{12}F_{22}^{-1}F_{21}, \quad \tilde{G} = B_1 - F_{12}F_{22}^{-1}B_2$$

Task Find $Z \in \mathbb{R}^{n\times nz}$, such that $nz \ll n$ and $X \approx ZZ^H$

Observation:

If $p_{i+1} = \overline{p_i}$

$$V_{i+1} = \overline{V_i} + \beta \mathrm{Im}(V_i)$$

$$Z_{i+1} = [Z_{i-1}, \sqrt{2}\mathrm{Re}(V_i) + \frac{\beta}{\sqrt{2}}\mathrm{Im}(V_i), \sqrt{\frac{\beta^2}{2} + 2}\,\mathrm{Im}(V_i)]$$

where $\beta = 2\frac{\mathrm{Re}(p_i)}{\mathrm{Im}(p_i)}$

Solving Large Lyapunov Equations

Efficient Solvers for Second Order Systems [Benner/Kürschner/Saak '12]

Second Order Form

$$M\ddot{x} + D\dot{x} + Kx = Hu$$

- x displacements
- $z = \left[\dot{x}^T, x^T\right]^T$
- M,D,K invertible

First Order Form

$$\mathcal{E}\dot{z} = \mathcal{A}z + \mathcal{B}u$$

$$\mathcal{E} = \begin{bmatrix} 0 & M \\ M & D \end{bmatrix}, \quad \mathcal{A} = \begin{bmatrix} M & 0 \\ 0 & -K \end{bmatrix},$$

$$\mathcal{B} = \begin{bmatrix} 0 \\ H \end{bmatrix}$$

Solving Large Lyapunov Equations

Efficient Solvers for Second Order Systems [Benner/Kürschner/Saak '12]

Second Order Form

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- $z = \left[\dot{x}^T, x^T\right]^T$
- M,D,K invertible

$$\mathcal{E} = \begin{bmatrix} 0 & M \\ M & D \end{bmatrix}, \quad \mathcal{A} = \begin{bmatrix} M & 0 \\ 0 & -K \end{bmatrix},$$

$$\mathcal{B} = \begin{bmatrix} 0 \\ H \end{bmatrix}$$

G-LRCF-ADI

main task per step: $(\mathcal{A} + p_i\mathcal{E})x = \mathcal{E}f, \quad x = [x_1^T, x_2^T]^T$

Solving Large Lyapunov Equations

Efficient Solvers for Second Order Systems [Benner/Kürschner/Saak '12]

Second Order Form

$$M\ddot{x} + D\dot{x} + Kx = Hu$$

First Order Form

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- $z = \left[\dot{x}^T, x^T\right]^T$
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G-LRCF-ADI

main task per step: $(\mathcal{A} + p_i\mathcal{E})x = \mathcal{E}f, \quad x = [x_1^T, x_2^T]^T$

SO-LRCF-ADI

$$(p_i^2 M - p_i D + K)x_2 = (p_i M - D)f_2 - Mf_1, \qquad x_1 = f_2 - p_i x_2$$

MOR for Second Order Index-1 Systems

Problem Structure

Second Order Index-1 Systems

$$\underbrace{\begin{bmatrix} M_1 & 0 \\ 0 & 0 \end{bmatrix}}_{\mathcal{M}} \begin{bmatrix} \ddot{z}(t) \\ \ddot{\varphi}(t) \end{bmatrix} + \underbrace{\begin{bmatrix} D_1 & 0 \\ 0 & 0 \end{bmatrix}}_{\mathcal{D}} \begin{bmatrix} \dot{z}(t) \\ \dot{\varphi}(t) \end{bmatrix} + \underbrace{\begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix}}_{\mathcal{K}} \begin{bmatrix} z(t) \\ \varphi(t) \end{bmatrix} = \underbrace{\begin{bmatrix} B_1 \\ B_2 \end{bmatrix}}_{\mathcal{H}} u(t)$$

$$y = \underbrace{\begin{bmatrix} C_1 & C_2 \end{bmatrix}}_{\mathcal{L}} \begin{bmatrix} z(t) \\ \varphi(t) \end{bmatrix}$$

where $\mathcal{M}, \mathcal{D}, \mathcal{K} \in \mathbb{R}^{n_1 \times n_1}$, $\mathcal{H} \in \mathbb{R}^{n_1 \times p}$ and $\mathcal{L} \in \mathbb{R}^{m \times n_1}$

Properties

- All matrices are sparse
- $\mathcal{M}$ and $\mathcal{D}$ are singular, K_{22} is invertible

MOR for Second Order Index-1 Systems

Problem Structure

Compact Form of the Systems

$$\underbrace{M_1}_{M}\ddot{z}(t) + \underbrace{D_1}_{D}\dot{z}(t) + (\underbrace{K_{11} - K_{12}K_{22}^{-1}K_{21}}_{K})z(t) = (\underbrace{B_1 - K_{12}K_{22}^{-1}B_2}_{H})u(t)$$

$$y = (\underbrace{C_1 - C_2K_{22}^{-1}K_{21}}_{L})z(t) + \underbrace{C_2K_{22}^{-1}B_2}_{D_a}u(t)$$

MOR for Second Order Index-1 Systems

Problem Structure

ROM of Compact Form (applying W, V)

$$\underbrace{\hat{M}_1}_{\hat{M}}\ddot{\hat{z}}(t) + \underbrace{\hat{D}_1}_{\hat{D}}\dot{\hat{z}}(t) + (\underbrace{\hat{K}_{11} - \hat{K}_{12}K_{22}^{-1}\hat{K}_{21}}_{\hat{K}})\hat{z}(t) = (\underbrace{\hat{B}_1 - \hat{K}_{12}K_{22}^{-1}B_2}_{\hat{H}})u(t)$$

$$\hat{y} = (\underbrace{\hat{C}_1 - C_2K_{22}^{-1}\hat{K}_{21}}_{\hat{L}})\hat{z}(t) + \underbrace{C_2K_{22}^{-1}B_2}_{D_a}u(t)$$

$$\hat{M}_1 = W^T M_1 V, \quad \hat{D}_1 = W^T D_1 V, \quad \hat{K}_{11} = W^T K_{11} V, \quad \hat{K}_{12} = W^T K_{12}$$
$$\hat{K}_{21} = K_{21} V, \quad \hat{B}_1 = W^T B_1, \quad \hat{C}_1 = C_1 V$$

MOR for Second Order Index-1 Systems

Problem Structure

ROM of Compact Form (applying W, V)

$$\underbrace{\hat{M}_1}_{\hat{M}}\ddot{\hat{z}}(t) + \underbrace{\hat{D}_1}_{\hat{D}}\dot{\hat{z}}(t) + (\underbrace{\hat{K}_{11} - \hat{K}_{12}K_{22}^{-1}\hat{K}_{21}}_{\hat{K}})\hat{z}(t) = (\underbrace{\hat{B}_1 - \hat{K}_{12}K_{22}^{-1}B_2}_{\hat{H}})u(t)$$

$$\hat{y} = (\underbrace{\hat{C}_1 - C_2K_{22}^{-1}\hat{K}_{21}}_{\hat{L}})\hat{z}(t) + \underbrace{C_2K_{22}^{-1}B_2}_{D_a}u(t)$$

ROM of Index-1 Form

$$\underbrace{\begin{bmatrix}\hat{M}_1 & 0\\ 0 & 0\end{bmatrix}}_{\hat{\mathcal{M}}}\begin{bmatrix}\ddot{\hat{z}}(t)\\ \ddot{\varphi}(t)\end{bmatrix} + \underbrace{\begin{bmatrix}\hat{D}_1 & 0\\ 0 & 0\end{bmatrix}}_{\hat{\mathcal{D}}}\begin{bmatrix}\dot{\hat{z}}(t)\\ \dot{\varphi}(t)\end{bmatrix} + \underbrace{\begin{bmatrix}\hat{K}_{11} & \hat{K}_{12}\\ \hat{K}_{21} & K_{22}\end{bmatrix}}_{\hat{\mathcal{K}}}\begin{bmatrix}\hat{z}(t)\\ \varphi(t)\end{bmatrix} = \underbrace{\begin{bmatrix}\hat{B}_1\\ B_2\end{bmatrix}}_{\hat{\mathcal{H}}}u(t)$$

$$y = \underbrace{\begin{bmatrix}\hat{C}_1 & C_2\end{bmatrix}}_{\hat{\mathcal{L}}}\begin{bmatrix}\hat{z}(t)\\ \varphi(t)\end{bmatrix}$$

MOR for Second Order Index-1 Systems

Efficient LRCF-ADI Methods

Second Order Index-1 Systems

$$\underbrace{\begin{bmatrix} M_1 & 0 \\ 0 & 0 \end{bmatrix}}_{\mathcal{M}} \begin{bmatrix} \ddot{z}(t) \\ \ddot{\varphi}(t) \end{bmatrix} + \underbrace{\begin{bmatrix} D_1 & 0 \\ 0 & 0 \end{bmatrix}}_{\mathcal{D}} \begin{bmatrix} \dot{z}(t) \\ \dot{\varphi}(t) \end{bmatrix} + \underbrace{\begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix}}_{\mathcal{K}} \begin{bmatrix} z(t) \\ \varphi(t) \end{bmatrix} = \underbrace{\begin{bmatrix} B_1 \\ B_2 \end{bmatrix}}_{\mathcal{H}} u(t)$$

$$y = \underbrace{\begin{bmatrix} C_1 & C_2 \end{bmatrix}}_{\mathcal{L}} \begin{bmatrix} z(t) \\ \varphi(t) \end{bmatrix}$$

MOR for Second Order Index-1 Systems

Efficient LRCF-ADI Methods

State-Space Form $(x_1 = \dot{z}, x_2 = z)$

$$\underbrace{\begin{bmatrix} 0 & J & 0 \\ M_1 & D_1 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{\mathcal{E}} \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{\varphi}(t) \end{bmatrix} = \underbrace{\begin{bmatrix} J & 0 & 0 \\ 0 & -K_{11} & -K_{12} \\ 0 & -K_{21} & -K_{22} \end{bmatrix}}_{\mathcal{A}} \begin{bmatrix} x_1(t) \\ x_2(t) \\ \varphi(t) \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ B_1 \\ B_2 \end{bmatrix}}_{\mathcal{B}} u(t)$$

$$y(t) = \underbrace{\begin{bmatrix} 0 & C_1 & C_2 \end{bmatrix}}_{\mathcal{C}} \begin{bmatrix} x_1(t) \\ x_2(t) \\ \varphi(t) \end{bmatrix}$$

MOR for Second Order Index-1 Systems

Efficient LRCF-ADI Methods

State-Space Form $(x_1 = \dot{z}, x_2 = z)$

$$\underbrace{\begin{bmatrix} 0 & J & 0 \\ M_1 & D_1 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{\mathcal{E}} \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{\varphi}(t) \end{bmatrix} = \underbrace{\begin{bmatrix} J & 0 & 0 \\ 0 & -K_{11} & -K_{12} \\ 0 & -K_{21} & -K_{22} \end{bmatrix}}_{\mathcal{A}} \begin{bmatrix} x_1(t) \\ x_2(t) \\ \varphi(t) \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ B_1 \\ B_2 \end{bmatrix}}_{\mathcal{B}} u(t)$$

$$y(t) = \underbrace{\begin{bmatrix} 0 & C_1 & C_2 \end{bmatrix}}_{\mathcal{C}} \begin{bmatrix} x_1(t) \\ x_2(t) \\ \varphi(t) \end{bmatrix}$$

Advantages

- $B_1 = C_1^T,\ B_2 = C_2^T \Rightarrow \mathcal{B} = \mathcal{C}^T$
- $J = M_1 \Rightarrow \mathcal{E}, \mathcal{A}$ symmetric

  $\Rightarrow$ System symmetric $\Rightarrow$ Lyapunov equations coincide

MOR for Second Order Index-1 Systems

Efficient LRCF-ADI Methods

Linear System Inside the ADI

$$\begin{bmatrix} M_1 & \mu_1 M_1 & 0 \\ \mu_1 M_1 & \mu_1(D_1 - K_{11}) & -K_{12} \\ 0 & -K_{21} & -K_{22} \end{bmatrix} \begin{bmatrix} x_1^{(i)} \\ x_2^{(i)} \\ \Lambda \end{bmatrix} = \begin{bmatrix} 0 \\ f_1 \\ f_2 \end{bmatrix}$$

- Solve $\begin{bmatrix} \mu_1^2 M_1 - \mu_1 D_1 + K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} \begin{bmatrix} x_2^{(i)} \\ \Lambda \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$ for $x_2^{(i)}$
- If $i = 1$, $f_1 = -B_1, f_2 = -B_2$ and $x_1^{(1)} = -\mu_1 x_2^{(1)}$
- Otherwise, $f_1 = (p_i V_{i-1}^{(2)} - V_{i-1}^{(1)})M_1 + D_1 V_{i-1}^{(2)}, f_2 = 0$ and
  $x_1^{(i)} = -p_i x_2^{(i)} + V_{i-1}^{(2)}$
- $V_{i-1} = \begin{bmatrix} V_{i-1}^{(1)} \\ V_{i-1}^{(2)} \end{bmatrix}$ (see ADI methods)

Numerical Results

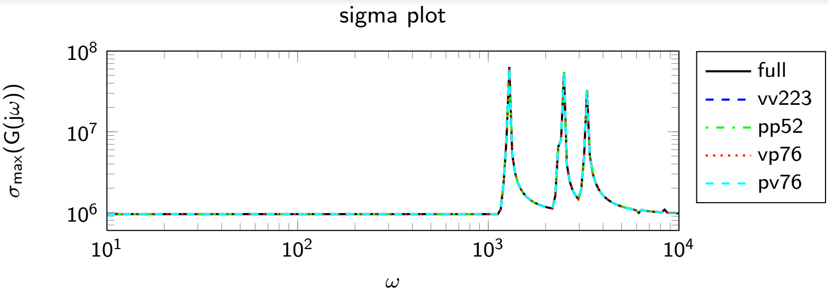
ASS Model (Fraunhofer IWU)

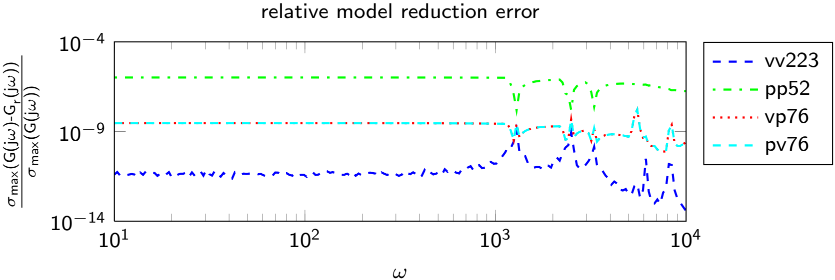
- Dimension of full model: 290 137
- ADI optimal shift parameters: 40
- ADI iteration steps 400 to compute $Z \in \mathbb{R}^{n \times nz}$
- Number of inputs/outputs: 9
- Tolerance for ROM: 10^{-3}
- Dimension of reduced order models in different balancing levels:

different types	ROM dimension
velocity-velocity (vv)	223
position-position (pp)	52
velocity-position (vp)	76
position-velocity (pv)	76

Numerical Results

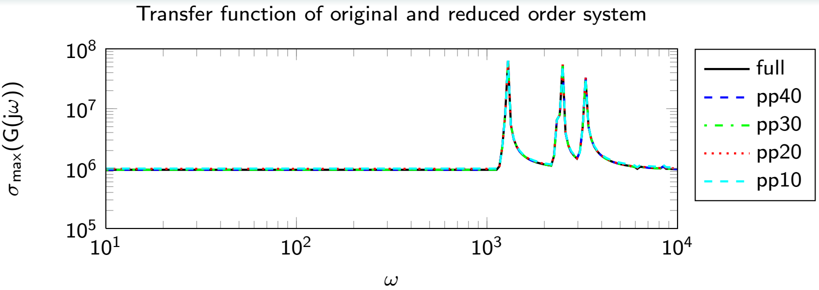
Sigma Plot

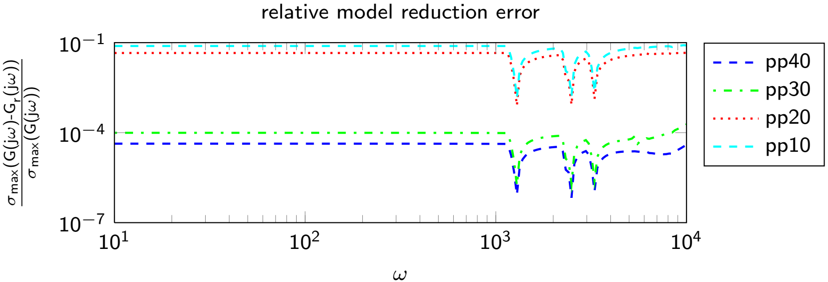




Numerical Results

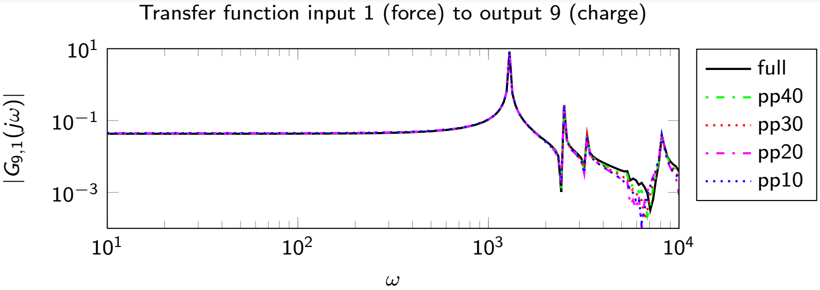
Sigma Plot

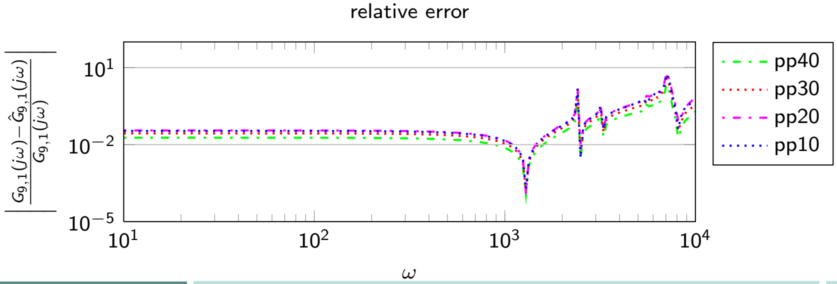




Numerical Results

Single Input to Single Output Relation





Numerical Results

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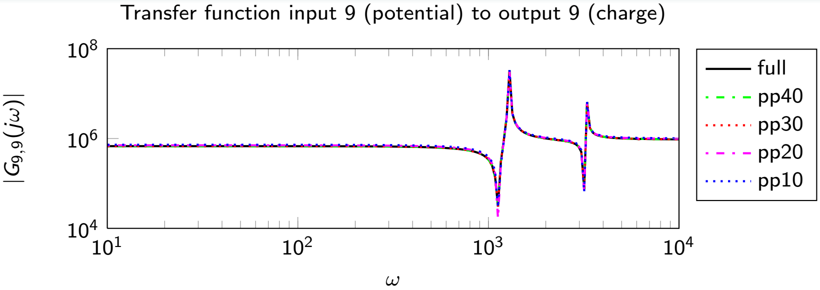

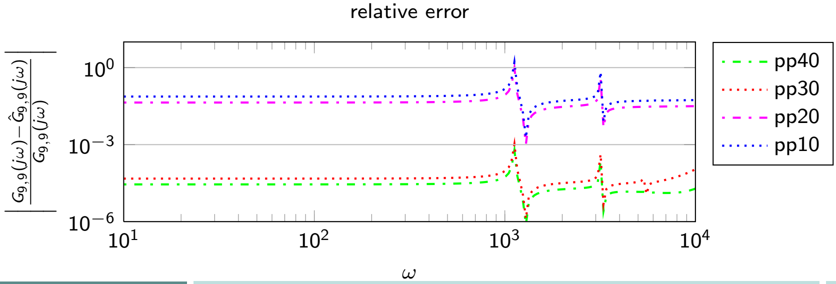

Conclusion and Outlook

Overview

- Second order to second order MOR techniques are shown for second order index-1 systems, and applied to ASS model
- The accuracy of the method is demonstrated by a frequency domain error analysis
- Even very low order surrogate models (10dof) preserve the main features of the transfer behavior of the full (290137dof) FEM model
- They are expected to perform well in controller design
- Our approach performs well on any computer that can solve the finite element model

Future Work

- Implicit handling of higher index, e.g., structural dynamics with holonomic constraints

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Overview

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Thank you for your attention!

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References

Benner, P., Kürschner, P., Saak, J.: Efficient handling of complex shift parameters in the low-rank Cholesky factor ADI method,
Numerical Algorithms, 1–27 (2012)

Freitas, F., Rommes, J., Martins, N.: Gramian-based reduction method applied to large sparse power system descriptor models,
IEEE Transactions on Power Systems **23**(3), 1258–1270 (2008)

Neugebauer, R., Drossel, W.G., Bucht, A., Kranz, B., Pagel, K.: Control design and experimental validation of an adaptive spindle support for enhanced cutting processes,
CIRP Annals - Manufacturing Technology **59**(1), 373–376 (2010)

Saak, J. : Efficent Numerical Solution of Large Scale Algebraic Matrix Equations in PDE Control and Model Order Reduction,
PhD thesis, Chemnitz University of Technology, Chemnitz, Germany,2009

Uddin, M. M., Saak, J., Kranz, B. and Benner, P.: Computation of a compact state space model for a spindle head configuration with Piezo actuators using balanced truncation,
Production Engineering Research and Development 6(6), 577–586 (2012)