Optimal Control-Based Feedback Stabilization in Multi-Field Flow Problems

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Motivation

Problem Definition

- Derive and investigate numerical algorithms for optimal control-based boundary feedback stabilization of multi-field flow problems.
- Explore the potentials and limitations of feedback-based (Riccati) stabilization techniques.
- Employ recent advances in reducing complexity of Riccati solvers to achieve stabilization with cost proportional to the simulation of the forward problem.

Previous Work

- Stabilization of flows described by Navier-Stokes equations (NSE)

$$\left.\begin{aligned}\frac{\partial}{\partial t}\mathbf{v} - \frac{1}{\mathrm{Re}}\Delta\mathbf{v} + \mathbf{v}\cdot\nabla\mathbf{v} + \nabla p &= \mathbf{0}\\ \operatorname{div}\mathbf{v} &= \mathbf{0}\end{aligned}\right\} \text{ in } (0,\infty)\times\Omega, \quad (1)$$

  to steady-state solution, with $\Omega \subset \mathbb{R}^d, d = 2,3$, the velocity field $\mathbf{v}(t,\mathbf{x}) \in \mathbb{R}^d$, the pressure $p(t,\mathbf{x}) \in \mathbb{R}$, the time $t \in (0,\infty)$, the spatial variable $\mathbf{x} \in \Omega$, and the Reynolds number $\mathrm{Re} \in \mathbb{R}^+$.
- Construction based on associated linear quadratic control problem (LQR) for boundary control [4].
- Numerical treatment for 2D case with linearized NSE described in [1].

Discretization Scheme

- Standard finite element discretization of linearized version of (1) yields

$$\begin{aligned} M\dot{\mathbf{z}} &= A\mathbf{z} + G\mathbf{p} + B\mathbf{u},\\ \mathbf{0} &= G^T\mathbf{z},\\ \mathbf{y} &= C\mathbf{z}, \end{aligned} \qquad (2)$$

with the discretized velocity $\mathbf{z}(t) \in \mathbb{R}^{n_v}$ and pressure $\mathbf{p}(t) \in \mathbb{R}^{n_p}$, the symmetric positive definite mass matrix $M \in \mathbb{R}^{n_v\times n_v}$, the system matrix $A \in \mathbb{R}^{n_v\times n_v}$ and the discretized gradient $G \in \mathbb{R}^{n_v\times n_p}$ of rank n_p. To put this in the context of an LQR problem one additionally gets the feedback matrix $B \in \mathbb{R}^{n_v\times n_r}$ and the input $\mathbf{u}(t) \in \mathbb{R}^{n_r}$, which describes the boundary control. Because we can observe only parts of the velocity there arises an output equation with the output $\mathbf{y}(t) \in \mathbb{R}^{n_a}$ and the output matrix $C \in \mathbb{R}^{n_a\times n_v}$.

Feedback Control Approach

Projection Method

To rewrite the DAE system (2) with differential index two as a generalized state space system, we use the projector

$$\Pi = I - G(G^TM^{-1}G)G^TM^{-1},$$

defined in [3]. The projected system is of the form

$$\begin{aligned}\mathcal{M}\dot{\tilde{\mathbf{z}}} &= \mathcal{A}\tilde{\mathbf{z}} + \mathcal{B}\mathbf{u},\\ \mathbf{y} &= \mathcal{C}\tilde{\mathbf{z}},\end{aligned} \qquad (3)$$

with $\mathcal{M} = \mathcal{M}^T \succ 0$ and $\tilde{\mathbf{z}}(t) \in \mathbb{R}^{n_v - n_p}$.
To solve the algebraic Riccati equation associated to the system (3) we use a Newton-ADI-method. Instead of solving the projected dense Lyapunov equations in the innermost loop, we use [3, Lemma 5.2] and have to solve the saddle point system

$$\begin{bmatrix} A^T + p_iM^T & G\\ G^T & 0\end{bmatrix}\begin{bmatrix}\Lambda\\ *\end{bmatrix} = \begin{bmatrix} Y\\ 0\end{bmatrix}, \qquad (4)$$

for a couple of right hand sides Y and a different shift p_i in each ADI step during each Newton step.

Application Tasks

- Feedback control approach applied to multi-field flow problems.
- Coupling of flow with other physical field equations.
- Suggest different scenarios with increasing difficulty.
- Adapt the solvers for the different structures that arise for the scenarios.

Kármán Vortex Street

- First scenario for proof of concept.
- Feedback input: inject or exhaust fluid on two ports on the backside of the obstacle.
- Goal: get laminar flow behind the obstacle.
- Note: at the moment non-optimal feedback.

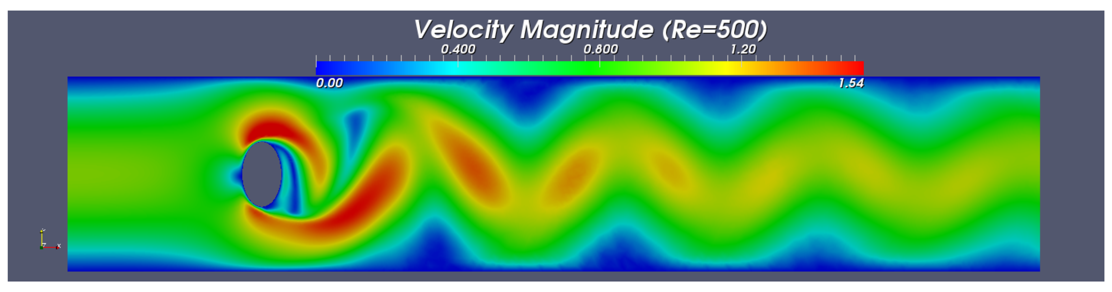

Figure 1: Unstable flow field

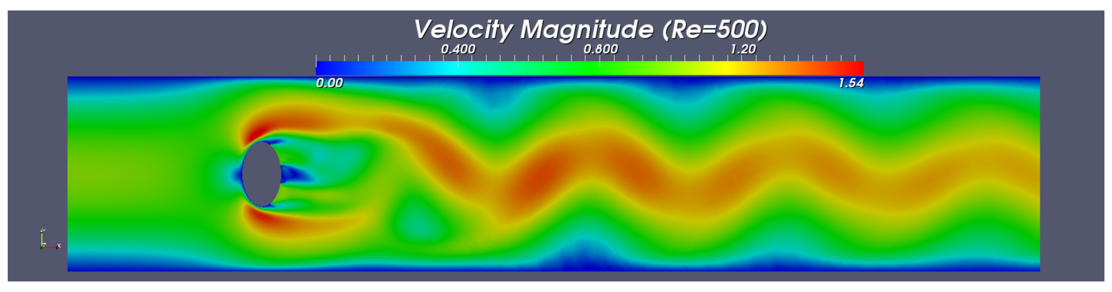

Figure 2: After 1 sec boundary feedback

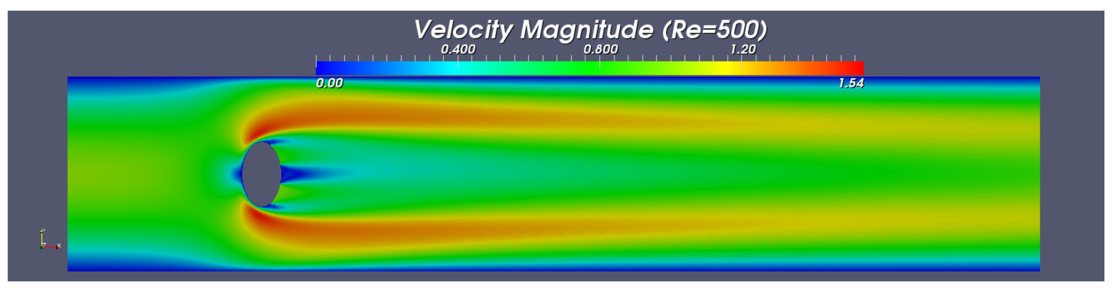

Figure 3: After 15 sec boundary feedback

Reactor Model

- Coupling of flow with spread of concentration.
- Control: influence the inflow of concentration
  ↪piecewise constant inflow of concentration.
- Goal: get a fixed rate of reaction on the obstacle.
- Note: at the moment just delayed control.
- Pictures: left without and right with control.

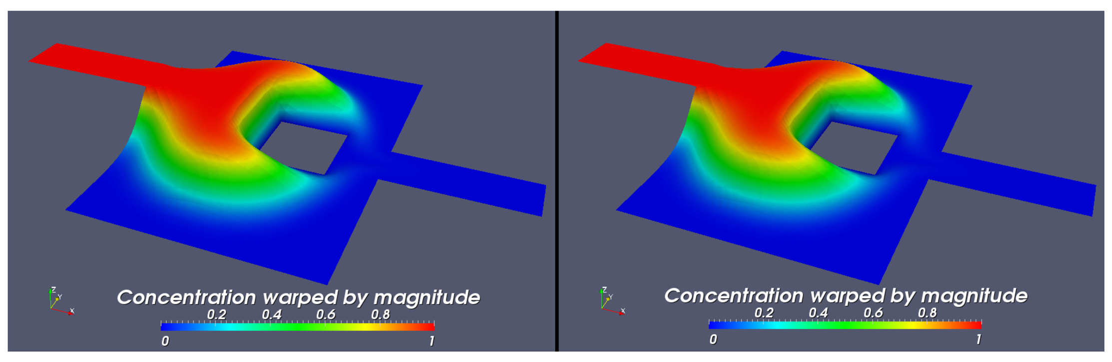

Figure 4: Before first correction (19.9 sec)

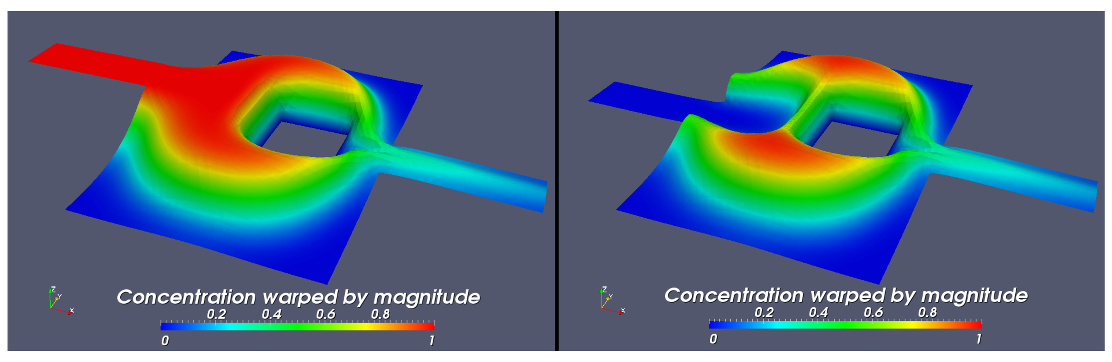

Figure 5: After 30.0 sec

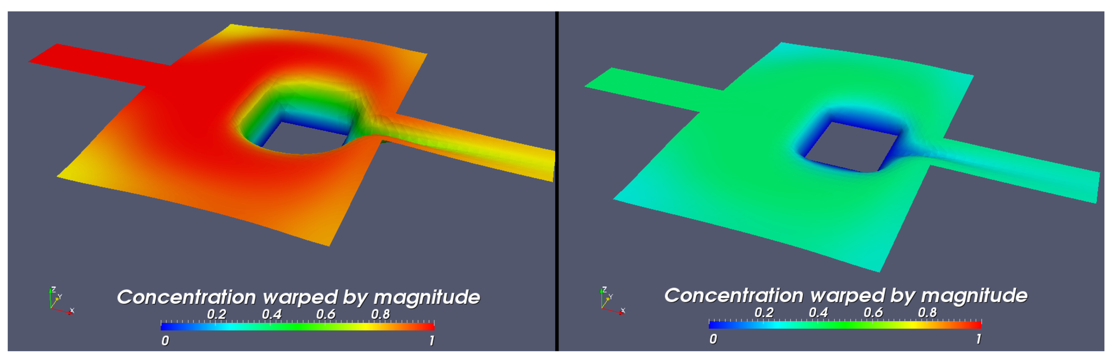

Figure 6: Achieve goal after 180.0 sec

Solving Large-Scale Saddle Point Systems

Preconditioning

- Using a block preconditioner **P** based on [2, Chapter 8] to solve (4) efficiently with an iterative solver.

$$\mathbf{P} = \begin{bmatrix} P_F & 0\\ G^T & -P_{SC}\end{bmatrix} \Rightarrow \mathbf{P}^{-1} = \begin{bmatrix} P_F^{-1} & 0\\ P_{SC}^{-1}G^TP_F^{-1} & -P_{SC}^{-1}\end{bmatrix}, \quad (5)$$

  with $P_F \approx F = A^T + p_iM^T$ and $P_{SC} = G^TF^{-1}G$ (*Schur complement*).
- Using MULTIGRID-methods to approximate $P_F \approx F$.
- Need an approximation of the dense P_{SC} as well.
- *Schur complement* approximation is derived from a least-squares commutator approach [2, Chapter 8.2].

$$P_{SC} \approx S_pF_p^{-1}M_p \Rightarrow P_{SC}^{-1} \approx M_p^{-1}F_pS_p^{-1},$$

  where S_p is the discretized Laplacian on the pressure space and F_p, M_p are the system and mass matrices as before, just defined on the pressure space as well.

Iterative Solver

- Using GMRES with preconditioner (5) to solve (4).
- Convergence of preconditioned GMRES is robust with respect to the mesh parameter.
- Reynolds number Re and ADI shifts p_i influence convergence rate.

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www.mpi-magdeburg.mpg.de/mpcsc/projekte/optconfeestabmultiflow/

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