Periodic Control Systems: Transient Analysis and Efficient Model Reduction

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Motivation

Periodic systems and control theory is of interest in various scientific fields such as aerospace realm, control of industrial processes, signal processing, and resonance in forced oscillators.
The main goals of this project include:

- analyze stability and compute bounds for linear periodic time-varying (LPTV) continuous-time systems,
- analyze the LPTV discrete-time descriptor systems and develop numerical algorithms of efficient model order reduction (MOR).

Problem Formulation

A small-signal circuit problem can be described by nonlinear ordinary differential equations (ODEs)

$$\frac{dq(x(t))}{dt} + f(x(t)) = u_L(t) + Bu(t), \qquad y(t) = C^T(t)x(t),$$

with $u_L(t), u(t) \in \mathbb{R}^m$ the vectors of large and small signal inputs, $x(t) \in \mathbb{R}^n$ the node voltages ($m \ll n$), $f(\cdot)$ and $q(\cdot)$ are nonlinear functions which describe the charge and resistance of the circuit.

Linearizing around the large signal $u_L(t)$ results in

$$E(t)\dot{x} = A(t)x + Bu(t), \tag{1}$$

where $A(t) = -\frac{\partial f(x)}{\partial x}|_{x(t)} - \frac{d}{dt}\frac{\partial q(x)}{\partial x}|_{x(t)}$, $E(t) = \frac{\partial q(x)}{\partial x}|_{x(t)}$, and $A(t), E(t) \in \mathbb{R}^{n\times n}$ are T-periodic.

Periodic ODEs

Analysis of Periodic ODEs

Dynamics of multibody systems such as resonance in an oscillator can be described by a system of 2nd order ODEs

$$M\ddot{y} + D\dot{y} + K(t)y = 0, \tag{2}$$

where $M, D, K(t) = K(t+T) \in \mathbb{R}^{\frac{n}{2}\times\frac{n}{2}}$ are mass, damping, stiffness matrices and $y(t) \in \mathbb{R}^{\frac{n}{2}}$ is the displacement vector. Rewriting (2) as a first order system ($x = (y, \dot{y})^T$) and applying **Floquet** theory yields the solution

$$x(t) = U(t)e^{Lt}x_0, \quad x(0) = x_0, \tag{3}$$

where $L \in \mathbb{R}^{n\times n}$ is a constant and $U(t) \in \mathbb{R}^{n\times n}$ a T-periodic matrix function. Let $\lambda_j \in \Lambda(L)$ be the eigenvalues, u_j the associated generalized left eigenvectors and $\nu(L)$ the spectral abscissa of L. For any $\varepsilon > 0$ there exists C_1^ε such that

$$\|x(t)\| \le C_1^\varepsilon e^{(\nu(L)+\varepsilon)t} \tag{4}$$

and with $\psi_j(t) = (x_0, u_j^*)e^{\Re\lambda_j t}$ for $t \ge 0$ and $j = 1, \dots, n$ an upper bound [4] for the solution (3) is defined by

$$\|x(t)\| \le C_2\|\psi(t)\|. \tag{5}$$

The upper bounds (4) and (5) for a multiple oscillator are shown in Figure 1. While the bound depending on the spectral abscissa (4) suggests an instability of the solution, the bound (5) tightens the solution accurately.

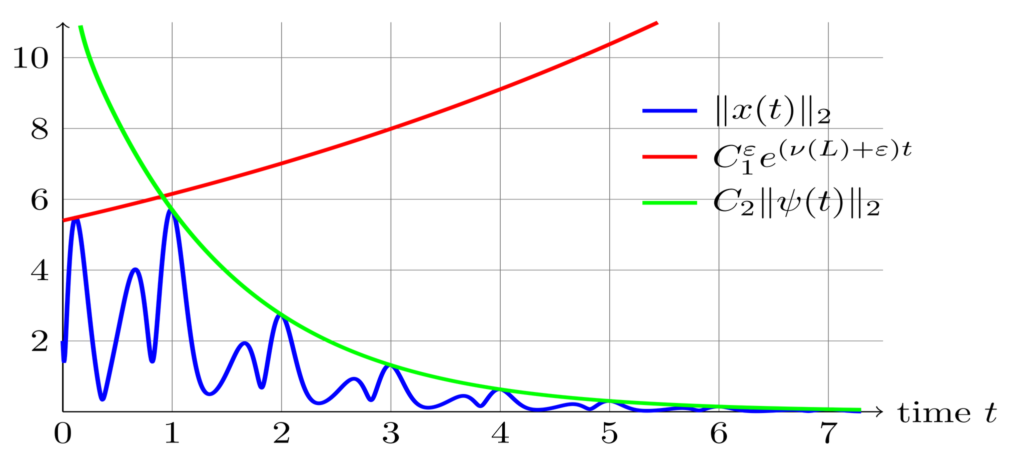


Figure 1: Solution and upper bounds

Future Work for Periodic ODEs

- Expand bounds to norms such that the solution has a specific structure, e.g. monotonicity.

Discrete LPTV

Model Problem

- The LPTV discrete-time descriptor system is obtained, e.g., via sampling or time-discretization of (1).
- Discretization of (1) over time-domain $[0, K]$ results in

$$E_k x_{k+1} = A_k x_k + B_k u_k, \qquad y_k = C_k^T x_k, \quad k \in [0, K], \tag{6}$$

where $E_k, A_k \in \mathbb{R}^{n_k\times n_k}$, $B_k \in \mathbb{R}^{n_k\times m_k}$, $C_k \in \mathbb{R}^{p_k\times n_k}$ are periodic with $K > 1$.

Analysis of Discrete LPTV Systems

Stability analysis and MOR for (6) are strongly related to the generalized projected periodic discrete-time algebraic Lyapunov equations (PPDALEs) [3]

$$E_k G_{k+1}^{cr} E_k^T - A_k G_k^{cr} A_k^T = P_l(k)B_k B_k^T P_l(k)^T, \qquad G_k^{cr} = P_r(k)G_k^{cr}P_r(k)^T, \tag{7}$$

where G_k^{cr} are the causal reachability Gramians of (6) for $k = 0, 1, \dots, K-1$.

Spectral projectors:

$$P_l(k) = U_k^{-1}\begin{bmatrix} I_{n^f} & 0 \\ 0 & 0\end{bmatrix}U_k, \quad P_r(k) = V_k\begin{bmatrix} I_{n^f} & 0 \\ 0 & 0\end{bmatrix}V_k^{-1},$$

with U_k, V_k nonsingular, and n^f defines the number of finite eigenvalues of the periodic matrix pairs.

Remarks: Similar PPDALEs appear for causal observability Gramians $\{G_k^{co}\}_{k=0}^{K-1}$. The noncausal cases are also considered.

Lifted Representation

The matrices E_k and A_k in (7) can be singular. Hence, we solve an alternative form of (7), known as **lifted form** [2] of (7), which we denote by PLDALEs

$$\mathcal{E}\mathcal{G}^{cr}\mathcal{E}^T - \mathcal{A}\mathcal{G}^{cr}\mathcal{A}^T = \mathcal{P}_l\mathcal{B}\mathcal{B}^T\mathcal{P}_l^T, \quad \mathcal{G}^{cr} = \mathcal{P}_r\mathcal{G}^{cr}\mathcal{P}_r^T, \tag{8}$$

where

- $\mathcal{E} = \text{diag}(E_0, E_1, \dots, E_{K-1})$, $\mathcal{B} = \text{diag}(B_0, B_1, \dots, B_{K-1})$,

$$\mathcal{A} = \begin{bmatrix} 0 & \cdots & 0 & A_0 \\ A_1 & & & 0 \\ & \ddots & & \vdots \\ 0 & & A_{K-1} & 0 \end{bmatrix},$$

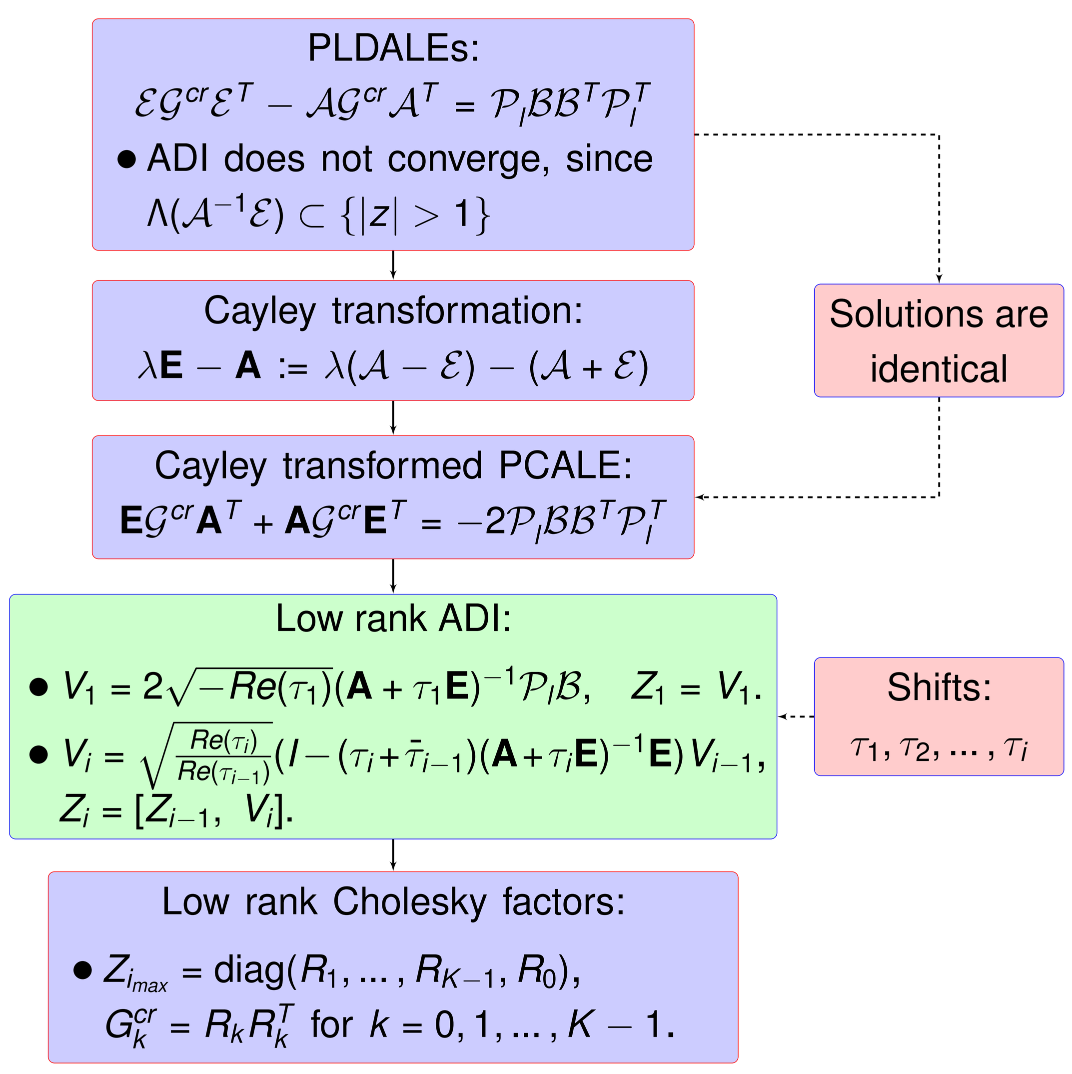
- $\mathcal{P}_l = \text{diag}(P_l(0), P_l(1), \dots, P_l(K-1))$, $\mathcal{P}_r = \text{diag}(P_r(1), \dots, P_r(K-1), P_r(0))$, and the solution $\mathcal{G}^{cr}$ is given by

$$\mathcal{G}^{cr} = \text{diag}(G_1^{cr}, \dots, G_{K-1}^{cr}, G_0^{cr}).$$

Iterative Solution of PPDALEs

The computational complexity for direct solvers of (8) is $\mathcal{O}(Kn_k^3)$. Hence, we propose LR-ADI to compute the low-rank Cholesky factor for solution of (8).

Algorithm: Low-rank CF-ADI iteration for (8) [1].



Application to MOR

Reduced system of dimension $r = \sum_{k=0}^{K-1} r_k$ is given by

$$\tilde{E}_k = S_{k,r}^T E_k T_{k+1,r}, \quad \tilde{A}_k = S_{k,r}^T A_k T_{k,r},$$
$$\tilde{B}_k = S_{k,r}^T B_k, \quad \tilde{C}_k = C_k T_{k,r}, \quad (r_k \ll n_k, r \ll n),$$

where matrices $S_{k,r}$ and $T_{k,r}$ are computed using the low-rank Cholesky factors of the approximated Gramians [1].

Numerical Results

Artificial problem with $n_k = 404$, $m_k = 2$, $p_k = 3$, and period $K = 10$. Figure 2 shows the decay of the normalized residual norms (with $tol = 10^{-10}$) at ADI iterations.

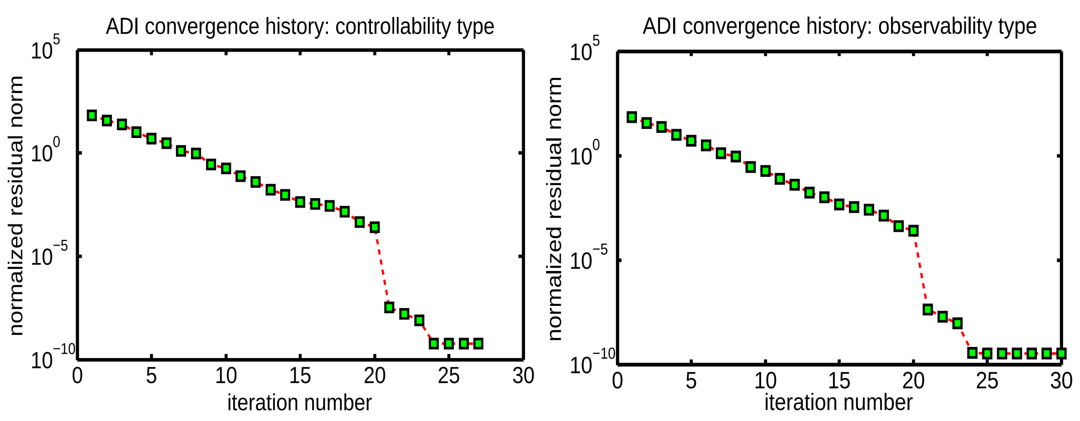


Figure 2: Normalized residual norms

The HSVs of the original, computed, and reduced-order models are shown in Figure 3. Here $r_k = (9, 10, 10, 11, 10, 9, 10, 11, 11, 11)$ with MOR tolerance 10^{-2}.

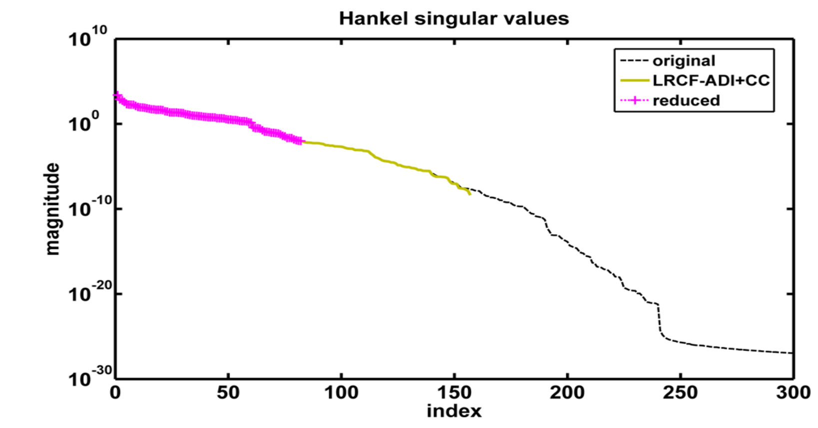


Figure 3: HSVs of the original, computed, and reduced order model

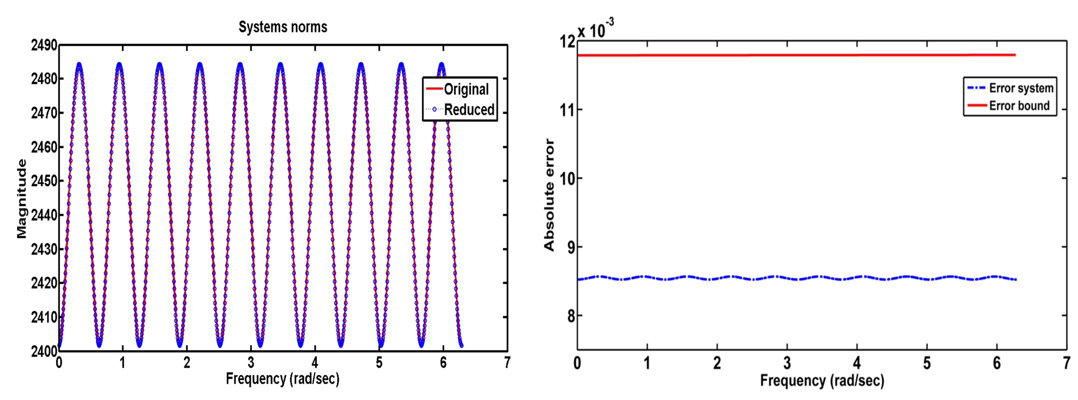


Figure 4: Norms of the frequency responses and absolute error of the original and the reduced order lifted systems

Future Work

- Structure preserving iterative solutions of PPDALEs using the generalized inverses of periodic matrix pairs,
- test the algorithms for real-world problems.

References

[1] P. Benner, M.-S. Hossain, and T. Stykel, *Low-rank iterative methods of periodic projected Lyapunov equations and their application in model reduction of periodic descriptor systems*, Chemnitz Scientific Computing Preprints 11–01, 2011.

[2] P. Benner, M.-S. Hossain, and T. Stykel, *Model reduction of periodic descriptor systems using balanced truncation*, in Model Reduction in Circuit Simulation, P. Benner, M. Hinze, and J. ter Maten, eds., vol. 74 of Lecture Notes in Electrical Engineering, Dordrecht, 2011, Springer-Verlag, pp. 187–200.

[3] E.-W. Chu, H.-Y. Fan, and W.-W. Lin, *Projected generalized discrete-time periodic Lyapunov equations and balanced realization of periodic descriptor systems*, SIAM J. Matrix Anal. Appl., 29 (2007), pp. 982–1006.

[4] L. Kohaupt, *Solution of the matrix eigenvalue problem with applications to the study of free linear dynamical systems*, J. Comp. Appl. Math., 213 (2008), pp. 142–165.