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Regelungstechnisches Seminar
Technische Universität München

Numerical Solution of large and sparse Matrix Equations in Model Order Reduction

Numerische Lösung großer, dünn besetzter Matrix-Gleichungen zur Modellordnungsreduktion

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Contents

Model Order Reduction

Model Order Reduction

The System Description

Consider the

linear time invariant (LTI) System

$$\begin{aligned} E\dot{x}(t) &= Ax(t) + Bu(t), \\ y(t) &= Cx(t), \end{aligned} \qquad (\Sigma(E;A,B,C))$$

where

- $x \in \mathbb{R}^n$ state,
- $u \in \mathbb{R}^m$ input, or control,
- $y \in \mathbb{R}^p$ output, or measurement,

and

$$E, A \in \mathbb{R}^{n\times n}, \qquad B \in \mathbb{R}^{n\times m}, \qquad C \in \mathbb{R}^{p\times n}.$$

Model Order Reduction

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Example "FEM for the heat equation"

- E FEM mass matrix,
- A discretized spatial differential operator,
- x temperatures in degrees of freedom.

Model Order Reduction

Basic Idea of MOR

LTI System $\Sigma = (E; A, B, C)$:

$$\boxed{E}\,\dot{x}(t) = \boxed{A}\,x(t) + \boxed{B}\,u(t), \qquad y(t) = \boxed{C}\,x(t)$$

Model Order Reduction

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Model Order Reduction

Find Projection Matrices

$$\boxed{T_r} \in \mathbb{R}^{n\times k} \text{ and } \boxed{T_l} \in \mathbb{R}^{n\times k},$$

$$k \ll n.$$

Model Order Reduction

Basic Idea of MOR

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Model Order Reduction

Find Projection Matrices

$$T_r \in \mathbb{R}^{n\times k} \text{ and } T_l \in \mathbb{R}^{n\times k},$$

$$k \ll n.$$

$$T_l^T E T_r \dot{\tilde{x}}(t) = T_l^T A T_r \tilde{x}(t) + T_l^T B u(t), \quad \tilde{y}(t) = C T_r$$

Model Order Reduction

Basic Idea of MOR

LTI System $\Sigma = (E; A, B, C)$:

$$E\,\dot{x}(t) = A\,x(t) + B\,u(t), \qquad y(t) = C\,x(t)$$

Model Order Reduction

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Model Order Reduction

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Model Order Reduction

Find Projection Matrices

$T_r \in \mathbb{R}^{n\times k}$ and $T_l \in \mathbb{R}^{n\times k}$,

$k \ll n$.

$$\tilde{E}\,\dot{\tilde{x}}(t) = \tilde{A}\,\tilde{x}(t) + \tilde{B}\,u(t), \qquad \tilde{y}(t) = \tilde{C}\,\tilde{x}(t)$$

Goal: $\tilde{y}(t) \approx y(t)$

Balanced Truncation

1. Model Order Reduction
2. Balanced Truncation
   - BT Basics
   - Implementation
3. Large Scale Lyapunov Equations
4. Recent Contributions

Balanced Truncation

BT Basics

Idea:

- The system Σ, in realization $(E; A, B, C)$, is called balanced, if the solutions P, Q of the Lyapunov equations

$$APE^T + EPA^T + BB^T = 0, \qquad A^TQE + E^TQA + C^TC = 0,$$

satisfy: $P = E^TQE = \mathrm{diag}(\sigma_1, \ldots, \sigma_n)$,
where $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_n > 0$.

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where $\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_n > 0$.

- $\{\sigma_1, \dots, \sigma_n\}$ are the Hankel singular values (HSVs) of Σ.
- A balanced realization is computed via state space transformation

$$\begin{aligned}\mathcal{T} : (E; A, B, C) &\mapsto (TET^{-1}; TAT^{-1}, TB, CT^{-1}) \\ &= \left(\begin{bmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{bmatrix}; \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \begin{bmatrix} C_1 & C_2 \end{bmatrix} \right).\end{aligned}$$

Balanced Truncation

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- Truncation $\rightsquigarrow$ reduced order model: $(\hat{E}; \hat{A}, \hat{B}, \hat{C}) = (E_{11}; A_{11}, B_1, C_1)$.

Balanced Truncation

Implementation

The SR Method

1. Compute (Cholesky)factors of the solutions to the Lyapunov equation,

$$P = S^T S, \quad Q = R^T R.$$

Balanced Truncation

Implementation

The SR Method

1. Compute (Cholesky)factors of the solutions to the Lyapunov equation,
$$P = S^T S, \quad Q = R^T R.$$
2. Compute singular value decomposition
$$SER^T = [\, U_1, \; U_2 \,] \begin{bmatrix} \Sigma_1 & \\ & \Sigma_2 \end{bmatrix} \begin{bmatrix} V_1^T \\ V_2^T \end{bmatrix}.$$

Balanced Truncation

Implementation

The SR Method

1. Compute (Cholesky)factors of the solutions to the Lyapunov equation,
$$P = S^T S, \quad Q = R^T R.$$
2. Compute singular value decomposition
$$SER^T = [\, U_1,\ U_2 \,] \begin{bmatrix} \Sigma_1 & \\ & \Sigma_2 \end{bmatrix} \begin{bmatrix} V_1^T \\ V_2^T \end{bmatrix}.$$
3. Define
$$T_l := R^T V_1 \Sigma_1^{-1/2}, \qquad T_r := S^T U_1 \Sigma_1^{-1/2}.$$
4. Then the reduced order model is $(I;\, T_l^T A T_r,\, T_l^T B,\, C T_r)$.

Large Scale Lyapunov Equations

Large Scale Lyapunov Equations

LRCF-ADI

Lyapunov Equation

Consider

$$FX + XF^T = -GG^T$$

with $F \in \mathbb{R}^{n\times n}$ Hurwitz, $G \in \mathbb{R}^{n\times p}$, $p \ll n$.

Observation in practice:

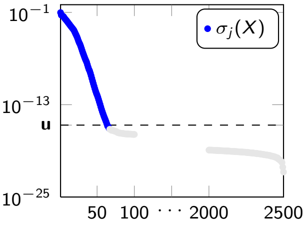
[Penzl '99, Ant./Sor./Zhou '02, Grasedyck '04, Truhar/Veselic '07]

$$\mathrm{rank}(X,\tau) = r \ll n$$

$\Rightarrow$ Compute *low-rank solution factor* $\tilde{Z} \in \mathbb{R}^{n\times r}$, $r \ll n$.

$$X \approx \tilde{Z}\tilde{Z}^T.$$

singular values for $n = 2\,500$ example



Large Scale Lyapunov Equations

LRCF-ADI

e.g.,[Benner/Li/Penzl '08]

Consider $FX + XF^T = -GG^T$ $F \in \mathbb{R}^{n \times n}, G \in \mathbb{R}^{n \times p}$

Task Find $Z \in \mathbb{C}^{n,nz}$, such that $nz \ll n$ and $X \approx ZZ^H$

Large Scale Lyapunov Equations
LRCF-ADI

Available low rank solvers in the literature:

- LR-ADI
  [Penzl '00], [Li/White '02], [Benner/Li/Penzl '08], [Benner/S.'05-'10], [Benner/Kürschner/S. '11-'12], [Stykel '04-], [Rommes/Freitas/Martins '08], [Heinkenschloss/Sorensen/Sun '08]
- Smith
  [Penzl '00], [Antoulas/Gugercin/Sor. '03]
- Krylov
  [Jaimoukha/Kasenally '94], [Saad '90], [Simoncini '07-]
- Sign function method
  [Benner/Quintana-Ortí 99],[Benner/Baur 06]
- SDA [Chu et al. '11-]

Large Scale Lyapunov Equations

LRCF-ADI

Alternating directions implicit (ADI) iteration for Lyapunov equations [Wachspress '88/'95]

$$
\begin{aligned}
X_0 &= 0 \\
(F + \overline{p_j} I) X_{j-\frac{1}{2}} &= -GG^T - X_{j-1}(F^T - \overline{p_j} I) \\
(F + p_j I) X_j &= -GG^T - X_{j-\frac{1}{2}}^H (F^T - p_j I)
\end{aligned}
$$

Large Scale Lyapunov Equations

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(F + p_j I) X_j &= -GG^T - X_{j-\frac{1}{2}}^H (F^T - p_j I)
\end{aligned}
$$

$\rightsquigarrow$ Rewrite as one step iteration and factorize $X_i = Z_i Z_i^H$, $i = 0, \ldots, J$

$$
\begin{aligned}
Z_0 Z_0^H &= 0 \\
Z_j Z_j^H &= -2\,\mathrm{Re}\,(p_j)(F + p_j I)^{-1} G G^T (F + \overline{p_j} I)^{-T} \\
&\quad + (F + p_j I)^{-1}(F - p_j I) Z_{j-1} Z_{j-1}^T (F - \overline{p_j} I)^T (F + \overline{p_j} I)^{-T}
\end{aligned}
$$

$$
\Rightarrow \quad Z_j = [\sqrt{-2p_j}(F + p_j I)^{-1} G,\ (F + p_j I)^{-1}(F - p_j I) Z_{j-1}]
$$

Large Scale Lyapunov Equations

LRCF-ADI

Using

$$P_i := \frac{\sqrt{\mathrm{Re}\,(p_i)}}{\sqrt{\mathrm{Re}\,(p_{i+1})}} \left[I_n - (p_{i+1} + \overline{p_i})(F + p_i I)^{-1} \right].$$

and $z_J = \sqrt{-2\,\mathrm{Re}\,(p_J)}(F + p_J I)^{-1} G$, the iterate Z_J can be rewritten as

$$Z_J = [z_J,\ P_{J-1} z_J,\ P_{J-2}(P_{J-1} z_J),\ \ldots,\ P_1(P_2 \cdots P_{J-1} z_J)]$$

[J. R. Li/White '02]

Large Scale Lyapunov Equations

LRCF-ADI

Using

$$P_i := \frac{\sqrt{\mathrm{Re}\,(p_i)}}{\sqrt{\mathrm{Re}\,(p_{i+1})}} \left[I_n - (p_{i+1} + \overline{p_i})(F + p_i I)^{-1} \right].$$

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[J. R. Li/White '02]

$\rightsquigarrow$ Rearranging the order of shifts

$\Rightarrow$ Low-rank Cholesky factor ADI iteration (LRCF-ADI)

[Penzl '99, Benner/Li/Penzl '08]

$$V_1 = \sqrt{-2\,\mathrm{Re}\,(p_1)}(F + p_1 I)^{-1} G, \qquad Z_1 := V_1$$

$$V_j = \frac{\sqrt{\mathrm{Re}\,(p_j)}}{\sqrt{\mathrm{Re}\,(p_{j-1})}} \left[I - (p_j + \overline{p_{j-1}})(F + p_j I)^{-1} \right] V_{i-1}, \quad Z_j := [Z_{j-1},\ V_j].$$

Large Scale Lyapunov Equations
LRCF-ADI

Simplified schematic illustration of LRCF-ADI:

Iteration 1:

$Z_1 = V_1$

Computation

Solve $(F + p_1 I_n)\, V_1 = \sqrt{-2\,\mathrm{Re}\,(p_1)}G$ for V_1

Large Scale Lyapunov Equations

LRCF-ADI

Simplified schematic illustration of LRCF-ADI:

Iteration 2:

$$Z_2 = \begin{bmatrix} V_1 & V_2 \end{bmatrix}$$

Computation

Solve $(F + p_2 I_n)\,\tilde{V} = V_1$ for $\tilde{V}$

$$V_2 = \sqrt{\frac{\mathrm{Re}\,(p_2)}{\mathrm{Re}\,(p_1)}}\left(V_1 - (p_2 + \overline{p_1})\tilde{V}\right)$$

Large Scale Lyapunov Equations

LRCF-ADI

Simplified schematic illustration of LRCF-ADI:

Iteration 3:

$$Z_3 = \begin{bmatrix} V_1 & V_2 & V_3 \end{bmatrix}$$

Computation

Solve $(F + p_3 I_n)\,\tilde{V} = V_2$ for $\tilde{V}$

$$V_3 = \sqrt{\frac{\mathrm{Re}\,(p_3)}{\mathrm{Re}\,(p_2)}}\left(V_2 - (p_3 + \overline{p_2})\tilde{V}\right)$$

Large Scale Lyapunov Equations

LRCF-ADI

Simplified schematic illustration of LRCF-ADI:

Iteration k:

$$Z_k = \begin{bmatrix} V_1 & V_2 & V_3 & \cdots & V_k \end{bmatrix}$$

Computation

Solve $(F + p_k I_n)\,\tilde{V} = V_{k-1}$ for $\tilde{V}$

$$V_k = \sqrt{\frac{\mathrm{Re}\,(p_k)}{\mathrm{Re}\,(p_{k-1})}}\left(V_{k-1} - (p_k + \overline{p_{k-1}})\tilde{V}\right)$$

Large Scale Lyapunov Equations

LRCF-ADI

e.g.,[Benner/Li/Penzl '08]

Consider $FX + XF^T = -GG^T \qquad F \in \mathbb{R}^{n\times n}, G \in \mathbb{R}^{n\times p}$

Task Find $Z \in \mathbb{C}^{n,nz}$, such that $nz \ll n$ and $X \approx ZZ^H$

Large Scale Lyapunov Equations

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Consider $\quad FX + XF^T = -GG^T \quad\quad F \in \mathbb{R}^{n\times n}, G \in \mathbb{R}^{n\times p}$

Task $\quad$ Find $Z \in \mathbb{C}^{n,nz}$, such that $nz \ll n$ and $X \approx ZZ^H$

Algorithm

$$V_1 = \sqrt{-2\,\mathrm{Re}\,(p_1)}(F + p_1 I)^{-1}G, \qquad Z_1 = V_1$$

$$V_i = \frac{\sqrt{\mathrm{Re}\,(p_i)}}{\sqrt{\mathrm{Re}\,(p_{i-1})}}\left[I - (p_i + \overline{p_{i-1}})(F + p_i I)^{-1}\right] V_{i-1} \qquad Z_i = [Z_{i-1} V_i]$$

For certain shift parameters $\{p_1, \ldots, p_J\} \subset \mathbb{C}_{<0}$.

Stop if

- $\|V_i V_i^H\|$ is small, or
- $\|FZ_iZ_i^H + Z_iZ_i^HF^T + GG^T\|$ is small.

Large Scale Lyapunov Equations

G-LRCF-ADI (E invertible)

e.g.,[S. '09]

Consider $FXE^T + EXF^T = -GG^T \qquad E, F \in \mathbb{R}^{n\times n}, G \in \mathbb{R}^{n\times p}$

Task Find $Z \in \mathbb{C}^{n,nz}$, such that $nz \ll n$ and $X \approx ZZ^H$

Algorithm

$$V_1 = \sqrt{-2\,\mathrm{Re}\,(p_1)}(F + p_1 E)^{-1} G, \qquad Z_1 = V_1$$

$$V_i = \frac{\sqrt{\mathrm{Re}\,(p_i)}}{\sqrt{\mathrm{Re}\,(p_{i-1})}} \left[I - (p_i + \overline{p_{i-1}})(F + p_i E)^{-1} \right] E V_{i-1} \qquad Z_i = [Z_{i-1} V_i]$$

For certain shift parameters $\{p_1, \ldots, p_J\} \subset \mathbb{C}_{<0}$.

Stop if

- $\|V_i V_i^H\|$ is small, or
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Large Scale Lyapunov Equations

S-LRCF-ADI ((E, F) index 1) e.g.,[Rommes/Freitas/Martins '08]

Consider $\tilde{F} X E_{11}^T + E_{11} X \tilde{F}^T = -\tilde{G}\tilde{G}^T \quad E_{11}, \tilde{F} \in \mathbb{R}^{n\times n}, \tilde{G} \in \mathbb{R}^{n\times p}$

Task Find $Z \in \mathbb{C}^{n,nz}$, such that $nz \ll n$ and $X \approx ZZ^H$

Algorithm

$$\begin{bmatrix} V_1 \\ * \end{bmatrix} = \sqrt{-2\,\mathrm{Re}\,(p_1)} \begin{bmatrix} F_{11} + p_1 E_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix}^{-1} \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \qquad Z_1 = V_1$$

$$\begin{bmatrix} V_i \\ * \end{bmatrix} = \frac{\sqrt{\mathrm{Re}\,(p_i)}}{\sqrt{\mathrm{Re}\,(p_{i-1})}} \left[I - (p_i + \overline{p_{i-1}}) \begin{bmatrix} F_{11} + p_i E_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix}^{-1} \right] \begin{bmatrix} E_{11} V_{i-1} \\ 0 \end{bmatrix} \qquad Z_i = [Z_{i-1} V_i]$$

For certain shift parameters $\{p_1, \dots, p_J\} \subset \mathbb{C}_{<0}$.

Stop if

- $\|V_i V_i^H\|$ is small, or
- $\|\tilde{F} Z_i Z_i^H E_{11}^T + E_{11} Z_i Z_i^H \tilde{F}^T + \tilde{G}\tilde{G}^T\|$ is small.

Large Scale Lyapunov Equations

LRCF-ADI for higher index DAEs

arbitrary index DAEs [Mehrmann/Stykel '04]

- Require spectral projectors P_l, P_r onto the left and right deflating subspaces corresponding to the finite eigenvalues,
- and projected Lyapunov equations

$$\begin{aligned} F^T X E + E^T X F &= -P_r^T G G^T P_r, \\ X &= P_l^T X P_l. \end{aligned}$$

Large Scale Lyapunov Equations

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- Drawback: Usually P_l, P_r not easily accessible.

Large Scale Lyapunov Equations

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Stokes-like DAEs (index 2) [Heinkenschloss/Sorensen/Sun '08]

- Structure exploitation $\rightsquigarrow$ avoid P_l, P_r.
- Similar to S-LRCF-ADI.

Large Scale Lyapunov Equations

Shift Parameters

Optimal shift parameters

For J ADI iterations, the optimal shift parameters solve the rational minmax problem

$$\min_{p_1,\ldots,p_J \subset \mathbb{C}_-} \left(\max_{1\leq \ell \leq n} \left| \prod_{i=1}^{J} \frac{\overline{p_i} - \lambda_\ell}{p_i + \lambda_\ell} \right| \right), \ \lambda_\ell \in \Lambda(F).$$

Large Scale Lyapunov Equations

Shift Parameters

Optimal shift parameters

For J ADI iterations, the optimal shift parameters solve the rational minmax problem

$$\min_{p_1,\ldots,p_J\subset\mathbb{C}_-}\left(\max_{1\leq\ell\leq n}\left|\prod_{i=1}^{J}\frac{\overline{p_i}-\lambda_\ell}{p_i+\lambda_\ell}\right|\right),\ \ \lambda_\ell\in\Lambda(F).$$

Heuristic *Penzl* shifts [Penzl '99]

Since λ_ℓ not easily available in large-scale setting, take small numbers of Ritz values of F and F^{-1} (generated with Arnoldi) instead.

Large Scale Lyapunov Equations

Shift Parameters

Required by all LRCF-ADI variants:

Proper shifts

A set of the form

$$\{p_1, \ldots, p_J\} = \{\nu_1, \ldots, \nu_L\} \subset \mathbb{C}_-,$$

where either $\nu_i = p_i \in \mathbb{R}_-$ or $\nu_i = \{p_i,\ \overline{p_i}\} \subset \mathbb{C}_-$ is referred to as *proper set of shift parameters*.

That means:

- both p_i and $\overline{p_i}$ are shifts and follow each other,
- ZZ^H is real, although Z is complex.

Recent Contributions

1. Model Order Reduction
2. Balanced Truncation
3. Large Scale Lyapunov Equations
4. Recent Contributions
   - Real Solution Fators for Complex Shifts
   - Efficient LRCF-ADI for Second Order Systems
   - A Dual LRCF-ADI Iteration for Balanced Truncation

Recent Contributions

Generation of purely real solution factors in the presence of complex ADI shift parameters.

Recent Contributions

Real Solution Fators for Complex Shifts

Theorem [BENNER/KÜRSCHNER/S. '11]

The LRCF-ADI iterates associated to $\nu_j = \{p_j,\ p_{j+1} := \overline{p_j}\}$ are constructed by

$$V_j = \sqrt{\frac{\mathrm{Re}\,(p_j)}{\mathrm{Re}\,(p_{j-1})}}\left(V_{j-1} - (p_j + \overline{p_{j-1}})(F + p_j I_n)^{-1} V_{j-1}\right),$$

and $\quad V_{j+1} = V_j - 2\overline{p_j}(F + \overline{p_j} I_n)^{-1} V_j.$

Recent Contributions

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and $\quad V_{j+1} = \overline{V_j} + \beta_j \,\mathrm{Im}\,(V_j), \qquad \beta_j := 2\frac{\mathrm{Re}\,(p_j)}{\mathrm{Im}\,(p_j)}.$

Recent Contributions

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and $$V_{j+1} = \overline{V_j} + \beta_j \,\mathrm{Im}\,(V_j), \qquad \beta_j := 2\frac{\mathrm{Re}\,(p_j)}{\mathrm{Im}\,(p_j)}.$$

Moreover, if $\nu_j = p_j \in \mathbb{R}_-$ it holds $\mathrm{Im}\,(V_j) = 0$.

Recent Contributions

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Theorem [Benner/Kürschner/S. '11]

The LRCF-ADI iterates associated to $\nu_j = \{p_j,\ p_{j+1} := \overline{p_j}\}$ are constructed by

$$V_j = \sqrt{\frac{\mathrm{Re}\,(p_j)}{\mathrm{Re}\,(p_{j-1})}}\left(V_{j-1} - (p_j + \overline{p_{j-1}})(F + p_j I_n)^{-1} V_{j-1}\right),$$

and $$V_{j+1} = \overline{V_j} + \beta_j \,\mathrm{Im}\,(V_j), \qquad \beta_j := 2\frac{\mathrm{Re}\,(p_j)}{\mathrm{Im}\,(p_j)}.$$

Moreover, if $\nu_j = p_j \in \mathbb{R}_-$ it holds $\mathrm{Im}\,(V_j) = 0$.

$\Rightarrow$ **the second linear system involving $\overline{p_j}$ becomes redundant.**

Recent Contributions

Real Solution Fators for Complex Shifts

Consequence of the theorem

Augmentation of Z_{j-1} by the results of the iterations j and $j+1$, s.t., $Z_{j+1} = [Z_{j-1},\ \hat{Z}]$ with $\hat{Z} := [V_j,\ V_{j+1}]$. It holds

$$\hat{Z} = [V_j,\ V_{j+1}] = [\mathrm{Re}\,(V_j) + \mathrm{i}\,\mathrm{Im}\,(V_j),\ \ \mathrm{Re}\,(V_{j+1}) + \mathrm{i}\,\mathrm{Im}\,(V_{j+1})]$$

Recent Contributions

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Recent Contributions

Real Solution Fators for Complex Shifts

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$$= \underbrace{[\mathrm{Re}\,(V_j),\ \ \mathrm{Im}\,(V_j)]}_{=:\tilde{Z}} \begin{bmatrix} I_m & I_m \\ \mathrm{i}I_m & (\beta_j - \mathrm{i})I_m \end{bmatrix}$$

Recent Contributions

Real Solution Fators for Complex Shifts

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$$\Rightarrow\ \ \hat{Z}\hat{Z}^H = \tilde{Z} \begin{bmatrix} 2I_m & \beta_j I_m \\ \beta_j I_m & (\beta_j^2 + 2)I_m \end{bmatrix} \tilde{Z}^T$$

Recent Contributions

Real Solution Fators for Complex Shifts

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$$= \underbrace{\tilde{Z} \begin{bmatrix} \sqrt{2}I_m & 0 \\ \frac{\beta_j}{\sqrt{2}} I_m & \sqrt{\frac{1}{2}\beta_j^2 + 2}I_m \end{bmatrix} \begin{bmatrix} \sqrt{2}I_m & 0 \\ \frac{\beta_j}{\sqrt{2}} I_m & \sqrt{\frac{1}{2}\beta_j^2 + 2}I_m \end{bmatrix}^T \tilde{Z}^T}_{=\tilde{Z}LL^T\tilde{Z}^T}$$

Recent Contributions

Real Solution Fators for Complex Shifts

$$\Rightarrow \quad \hat{Z}\hat{Z}^H = Z_{\mathbb{R}} Z_{\mathbb{R}}^T$$

$$\text{with } Z_{\mathbb{R}} := \tilde{Z}L = \left[\sqrt{2}\,\mathrm{Re}\,(V_j) + \frac{\beta_j}{\sqrt{2}}\,\mathrm{Im}\,(V_j),\ \sqrt{\frac{\beta_j^2}{2} + 2} \cdot \mathrm{Im}\,(V_j)\right] \in \mathbb{R}^{n \times 2m}.$$

Recent Contributions

Real Solution Fators for Complex Shifts

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Advantages:

- computation of real LRCFs,
- **only one** complex linear system needed for each complex pair,
- original sparsity preserved.

Disadvantages:

- still one complex linear system per complex pair,
- intermediate V_j is complex.

Recent Contributions

Real Solution Fators for Complex Shifts

Simplified schematic illustration of this strategy in LRCF-ADI:

Iteration j, j + 1:

$$Z_{j+1} = V_1 \; V_2 \; V_3 \; \cdots$$

Computation

Solve $(F + p_j I_n)\,\tilde{V} = V_{j-1}$ for $\tilde{V}$

$$V_j = \sqrt{\frac{\mathrm{Re}\,(p_j)}{\mathrm{Re}\,(p_{j-1})}}\left(V_{j-1} - (p_j + \overline{p_{j-1}})\tilde{V}\right)$$

Recent Contributions

Real Solution Fators for Complex Shifts

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Recent Contributions

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$$\tilde{Z} = [\mathrm{Re}\,(V_j), \;\; \mathrm{Im}\,(V_j)]$$

$$L = \begin{bmatrix} \sqrt{2} I_m & 0 \\ \frac{\beta_j}{\sqrt{2}} I_m & \sqrt{\frac{1}{2}\beta_j^2 + 2} I_m \end{bmatrix}$$

Recent Contributions

Real Solution Fators for Complex Shifts

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$$Z_{j+1} = \begin{bmatrix} V_1 & V_2 & V_3 & \cdots & \boxed{\tilde{Z}L} \end{bmatrix}$$

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Recent Contributions

Efficient balancing based MOR for damped vibrational Models.

Recent Contributions

Efficient LRCF-ADI for Second Order Systems [Benner/S. '09, Benner/Kürschner/S. '12]

Second Order Form

$$M\ddot{x} + D\dot{x} + Kx = Bu$$

- x displacements,
- $z = (\dot{x}^T, x^T)^T$
- M,D,K invertible,
- F arbitrary but invertible.

First Order Form

$$\mathcal{E}\dot{z} = \mathcal{A}z + \mathcal{B}u$$

$$\mathcal{E} = \begin{bmatrix} 0 & F \\ M & D \end{bmatrix}, \quad \mathcal{A} = \begin{bmatrix} F & 0 \\ 0 & -K \end{bmatrix},$$

$$\mathcal{B} = \begin{bmatrix} 0 \\ B \end{bmatrix}.$$

Recent Contributions

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G-LRCF-ADI

main task per step: $(\mathcal{A} + p_i\mathcal{E})x = \mathcal{E}f$

Recent Contributions

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main task per step: $(\mathcal{A} + p_i\mathcal{E})x = \mathcal{E}f$

SO-LRCF-ADI

$$(p_i^2 M - p_i D + K)x_2 = (p_i M - D)f_2 - Mf_1, \qquad x_1 = f_2 - p_i x_2.$$

Recent Contributions

ADI stopping criteria,
quality of reduced order models
and
efficient solution of the two dual Lyapunov equations.

Recent Contributions

A Dual LRCF-ADI Iteration for Balanced Truncation

Key Target

Reuse LU-factorizations in a sparse direct solver framework.

$$V_i = \frac{\sqrt{\mathrm{Re}\,(p_i)}}{\sqrt{\mathrm{Re}\,(p_{i-1})}} \left[I - (p_i + \overline{p_{i-1}})(F + p_i E)^{-1} \right] E V_{i-1},$$

Recent Contributions

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$$LU := (F + p_i E) \quad \Rightarrow \quad U^H L^H = (F^T + \overline{p_i} E^T)$$

Recent Contributions

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Reuse LU-factorizations in a sparse direct solver framework.

$$V_i = \frac{\sqrt{\mathrm{Re}\,(p_i)}}{\sqrt{\mathrm{Re}\,(p_{i-1})}} \left[I - (p_i + \overline{p_{i-1}}) U^{-1} L^{-1} \right] E V_{i-1},$$

- complex shifts come in conjugate pair,
- they are used one after another,

$\rightsquigarrow$ reverse order of complex conjugate shifts for the second equation.

Recent Contributions

A Dual LRCF-ADI Iteration for Balanced Truncation

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Reuse LU-factorizations in a sparse direct solver framework.

$$V_i = \frac{\sqrt{\mathrm{Re}\,(p_i)}}{\sqrt{\mathrm{Re}\,(p_{i-1})}} \left[I - (p_i + \overline{p_{i-1}}) U^{-1} L^{-1} \right] E V_{i-1},$$

$$W_i = \frac{\sqrt{\mathrm{Re}\,(p_i)}}{\sqrt{\mathrm{Re}\,(p_{i-1})}} \left[I - (\overline{p_i} + p_{i-1}) L^{-H} U^{-H} \right] E^T W_{i-1}.$$

- complex shifts come in conjugate pair,
- they are used one after another,

$\leadsto$ reverse order of complex conjugate shifts for the second equation.

Recent Contributions

A Dual LRCF-ADI Iteration for Balanced Truncation: Stopping the iteration

Problem

- Number of columns in LRCFs limits ROM dimension.
- Lyapunov residuals usually totally unrelated to ROM quality.

Idea

Use goal oriented stopping criteria.

Recent Contributions

A Dual LRCF-ADI Iteration for Balanced Truncation: Stopping the iteration

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Identify and monitor the property of interest to underlying application.

Recent Contributions

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In Balanced Truncation MOR:

Assume we are interested in an order k ROM.

Recent Contributions

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Idea

Use goal oriented stopping criteria.

Identify and monitor the property of interest to underlying application.

In Balanced Truncation MOR:

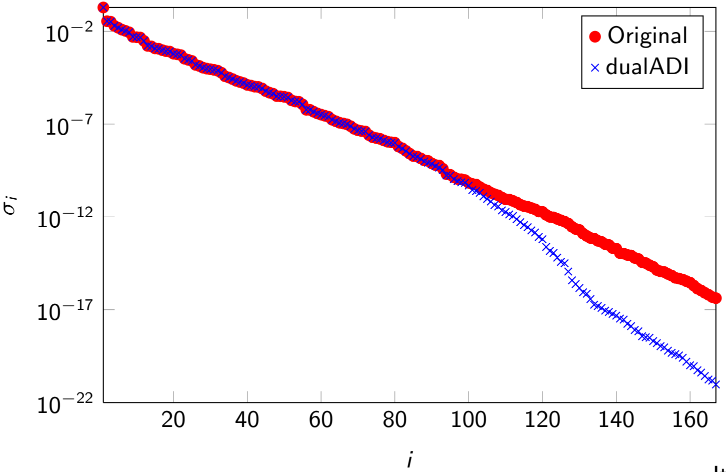
Assume we are interested in an order k ROM.

- Monitor the relative change of k leading HSVs.
- Stop when leading HSVs do no longer change.

Recent Contributions

A Dual LRCF-ADI Iteration for Balanced Truncation

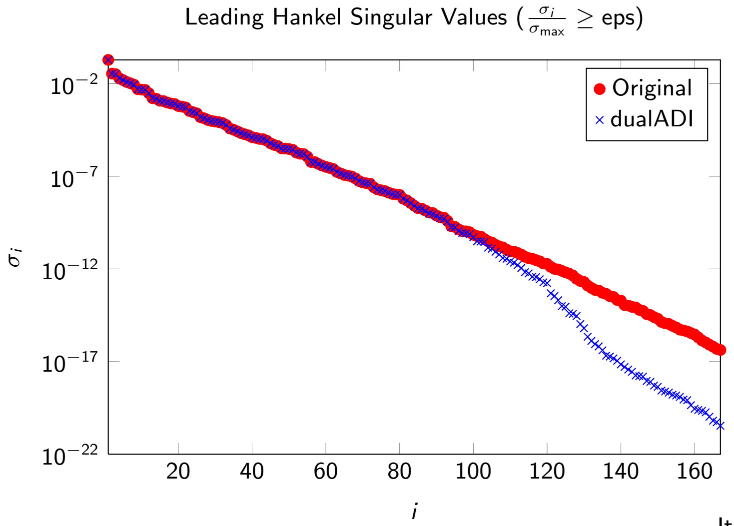
Leading Hankel Singular Values ($\frac{\sigma_i}{\sigma_{\max}} \geq$ eps)



Iteration: 28

Recent Contributions

A Dual LRCF-ADI Iteration for Balanced Truncation

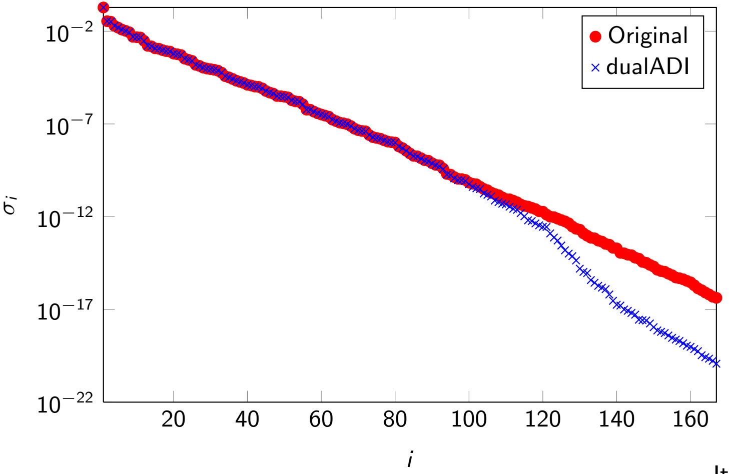


Iteration: 29

Recent Contributions

A Dual LRCF-ADI Iteration for Balanced Truncation

Leading Hankel Singular Values ($\frac{\sigma_i}{\sigma_{\max}} \geq$ eps)

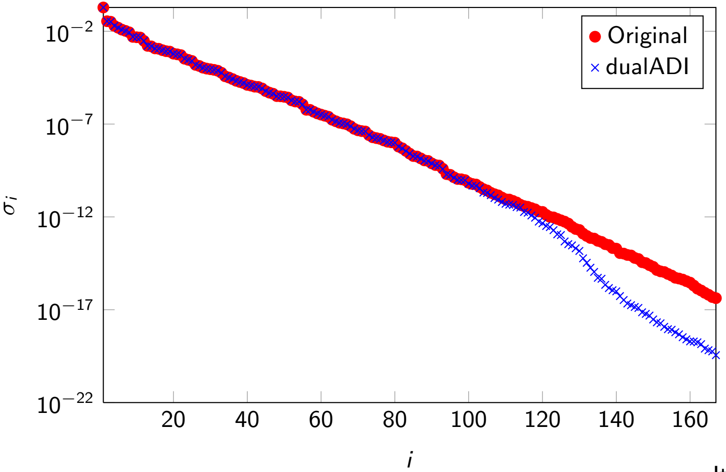


Iteration: 30

Recent Contributions

A Dual LRCF-ADI Iteration for Balanced Truncation

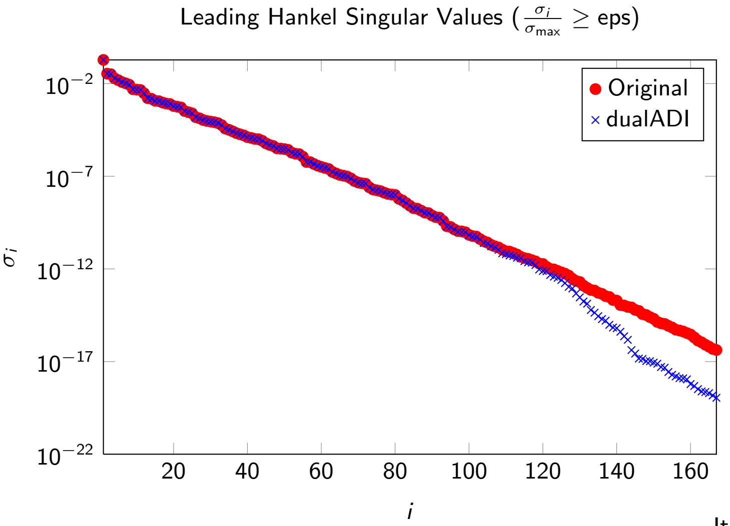
Leading Hankel Singular Values ($\frac{\sigma_i}{\sigma_{\max}} \geq$ eps)



Iteration: 31

Recent Contributions

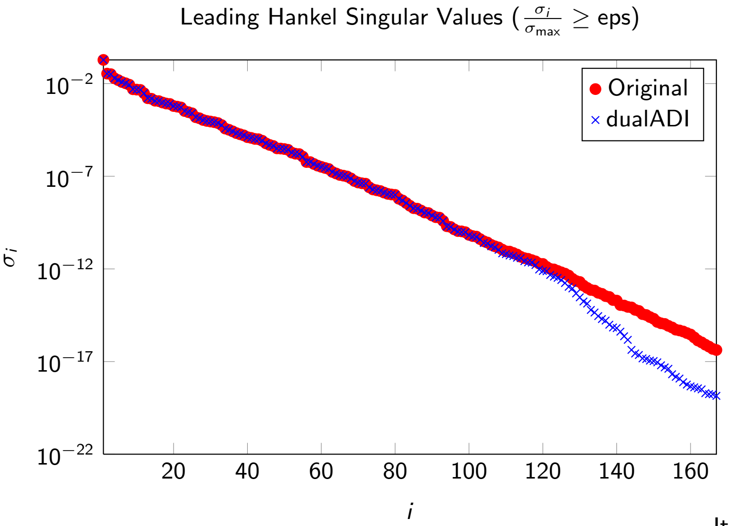
A Dual LRCF-ADI Iteration for Balanced Truncation



Iteration: 32

Recent Contributions

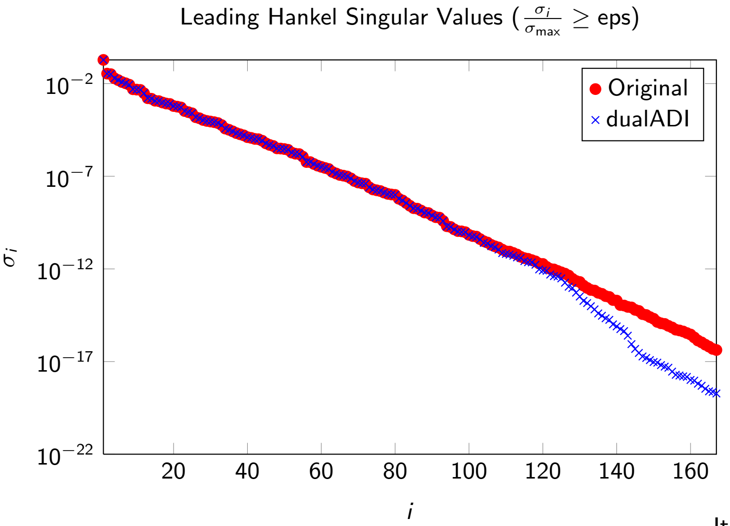
A Dual LRCF-ADI Iteration for Balanced Truncation



Iteration: 33

Recent Contributions

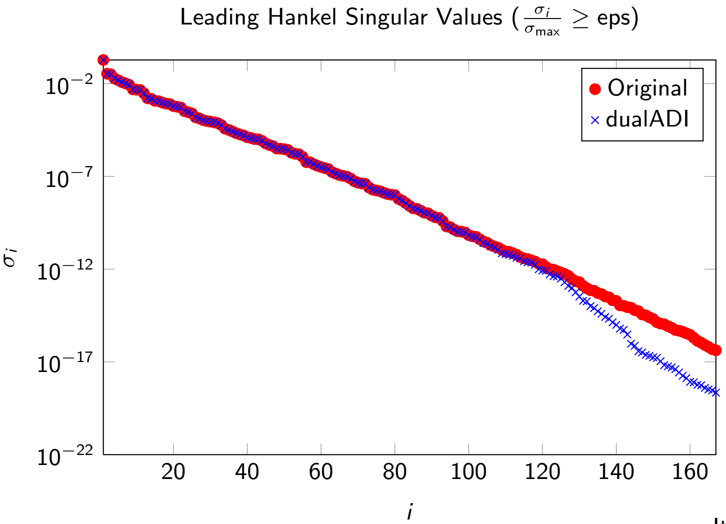
A Dual LRCF-ADI Iteration for Balanced Truncation



Iteration: 34

Recent Contributions

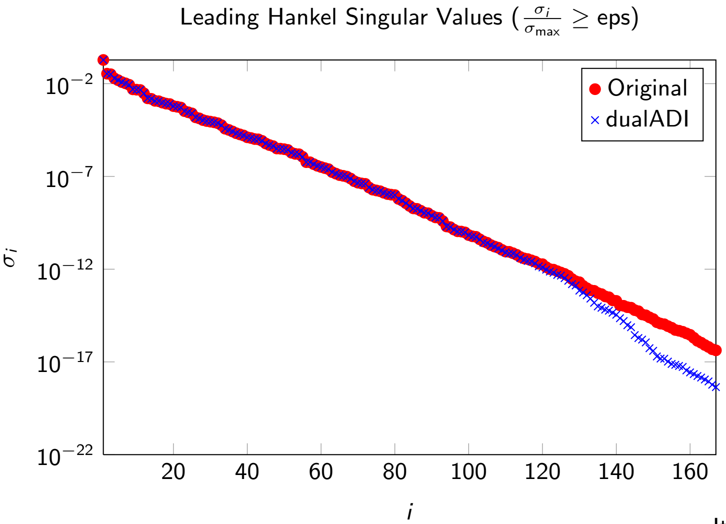
A Dual LRCF-ADI Iteration for Balanced Truncation



Iteration: 35

Recent Contributions

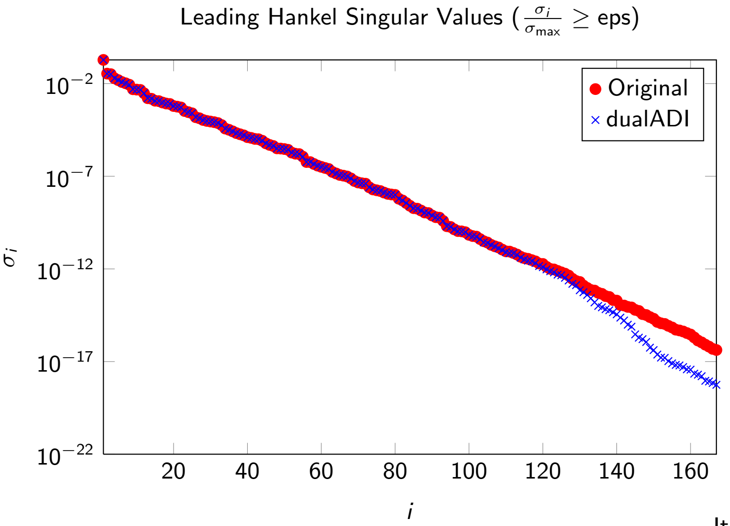
A Dual LRCF-ADI Iteration for Balanced Truncation



Iteration: 36

Recent Contributions

A Dual LRCF-ADI Iteration for Balanced Truncation

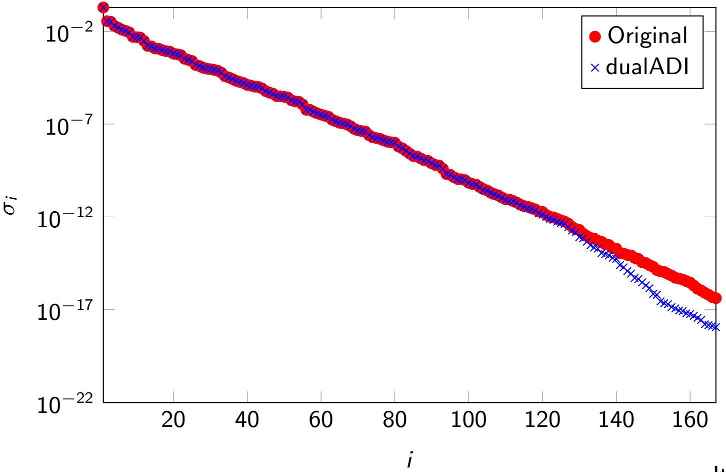


Iteration: 37

Recent Contributions

A Dual LRCF-ADI Iteration for Balanced Truncation

Leading Hankel Singular Values ($\frac{\sigma_i}{\sigma_{\max}} \geq$ eps)

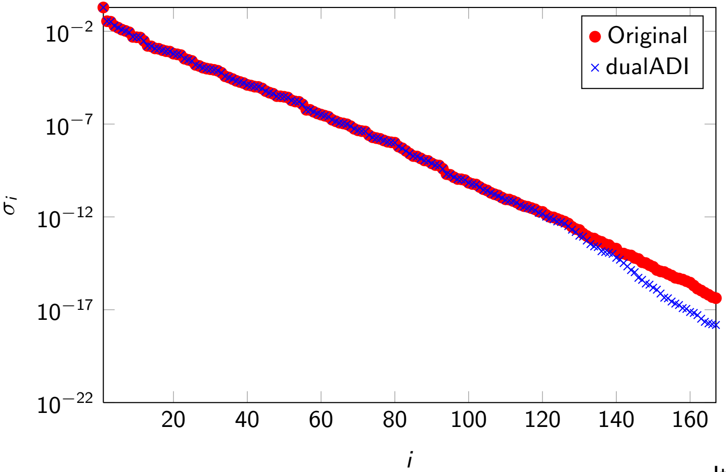


Iteration: 38

Recent Contributions

A Dual LRCF-ADI Iteration for Balanced Truncation

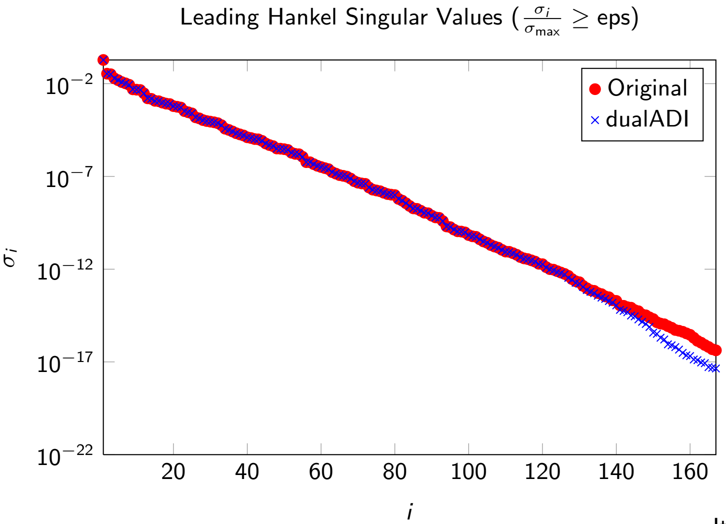
Leading Hankel Singular Values ($\frac{\sigma_i}{\sigma_{\max}} \geq$ eps)



Iteration: 39

Recent Contributions

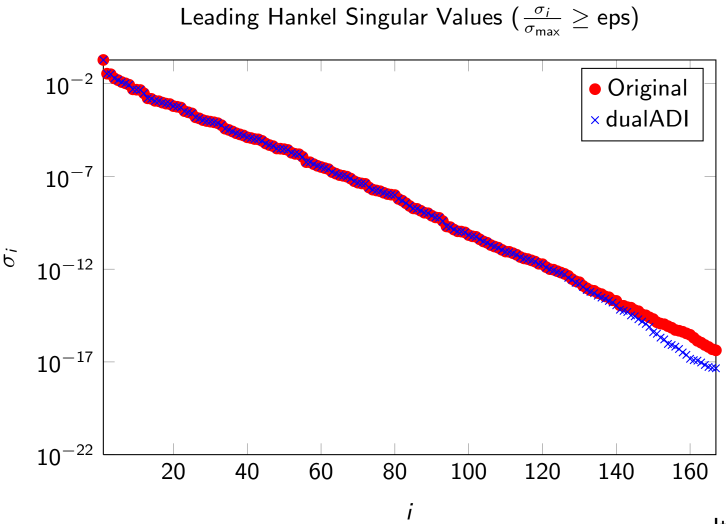
A Dual LRCF-ADI Iteration for Balanced Truncation



Iteration: 40

Recent Contributions

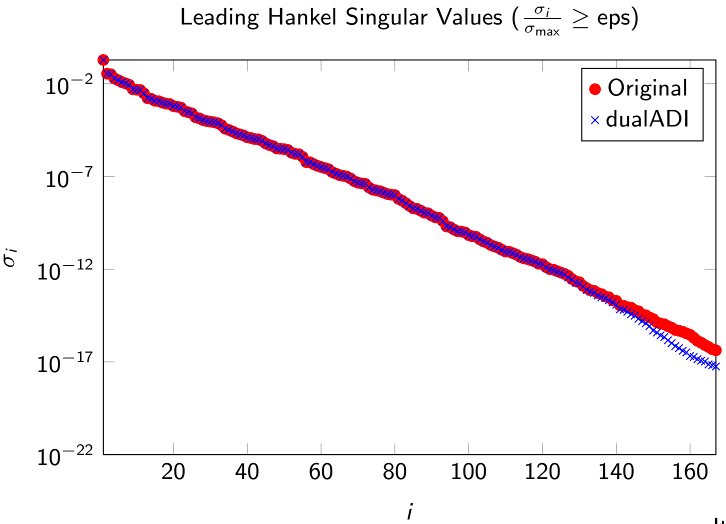
A Dual LRCF-ADI Iteration for Balanced Truncation



Iteration: 41

Recent Contributions

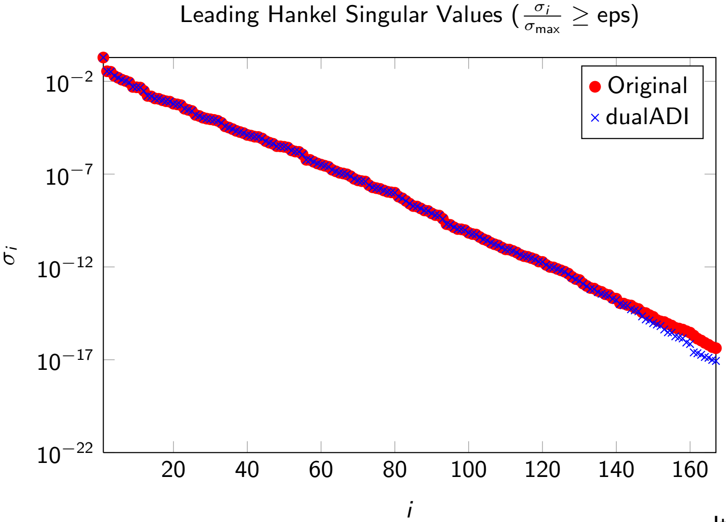
A Dual LRCF-ADI Iteration for Balanced Truncation



Iteration: 42

Recent Contributions

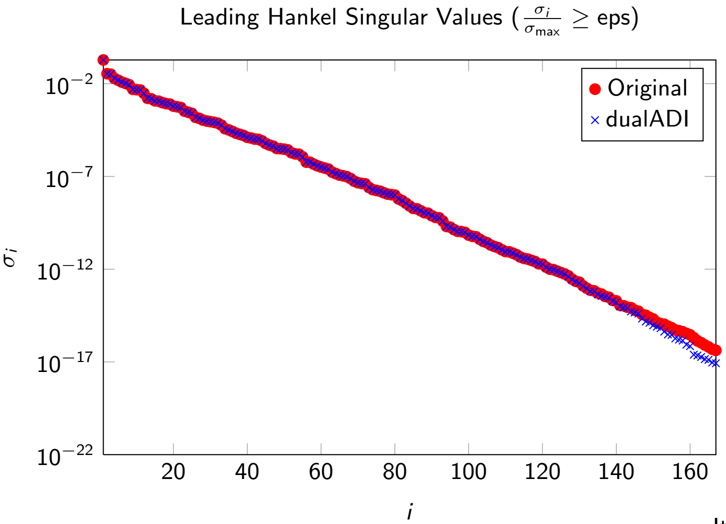
A Dual LRCF-ADI Iteration for Balanced Truncation



Iteration: 43

Recent Contributions

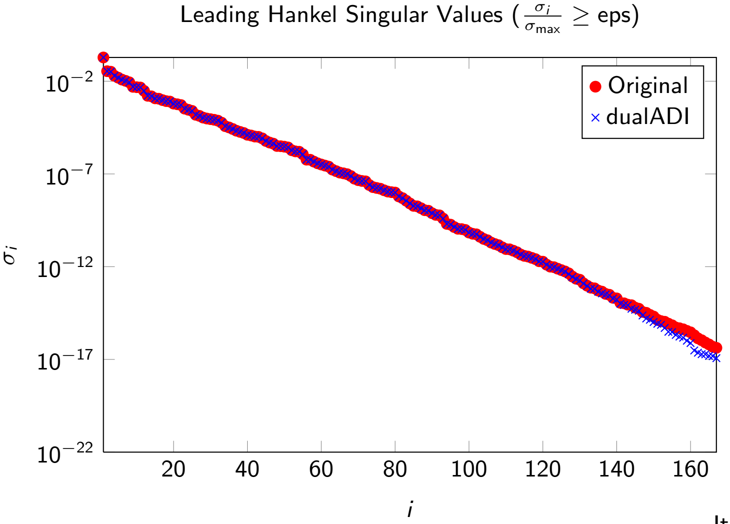
A Dual LRCF-ADI Iteration for Balanced Truncation



Iteration: 44

Recent Contributions

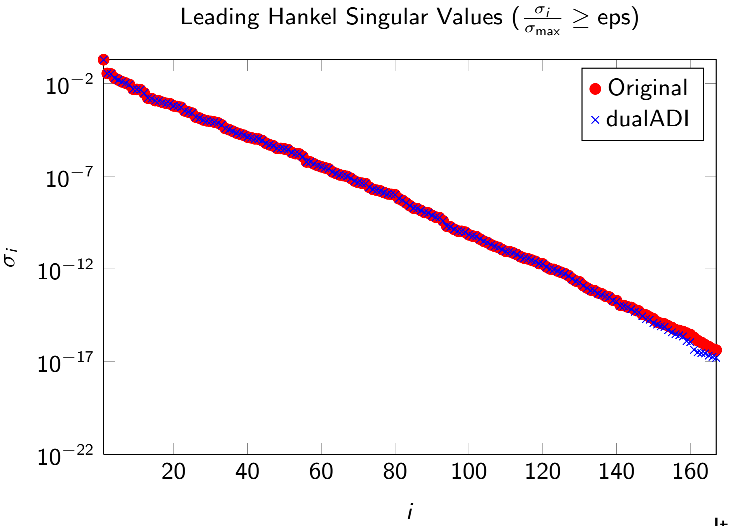
A Dual LRCF-ADI Iteration for Balanced Truncation



Iteration: 45

Recent Contributions

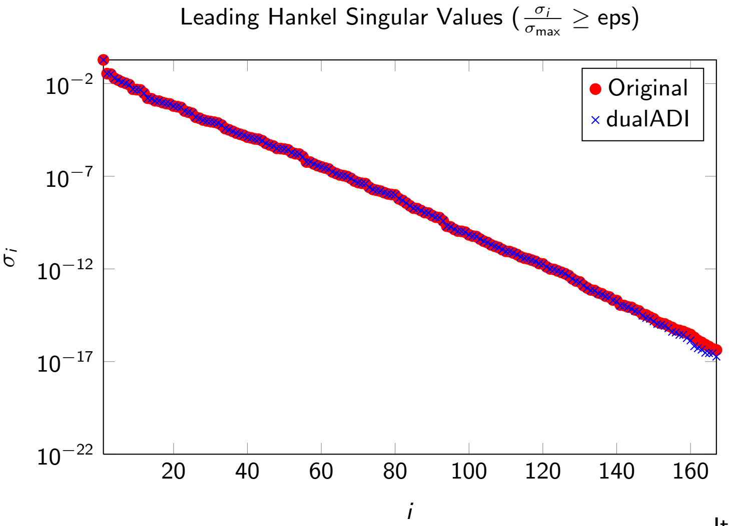
A Dual LRCF-ADI Iteration for Balanced Truncation



Iteration: 46

Recent Contributions

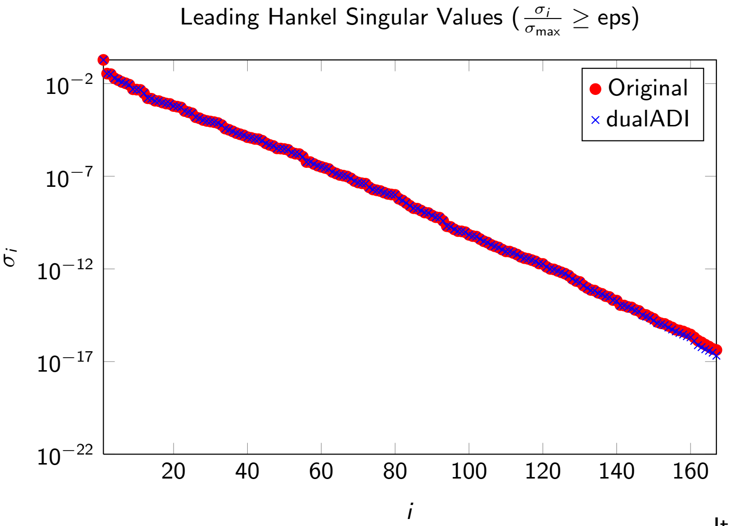
A Dual LRCF-ADI Iteration for Balanced Truncation



Iteration: 47

Recent Contributions

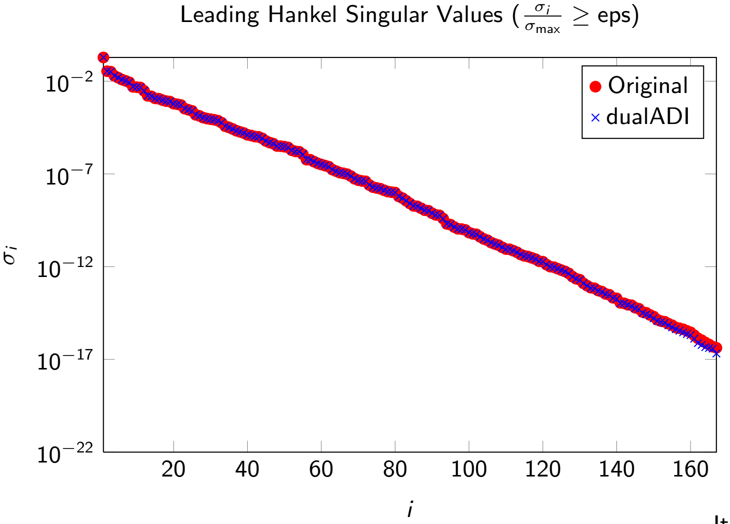
A Dual LRCF-ADI Iteration for Balanced Truncation



Iteration: 48

Recent Contributions

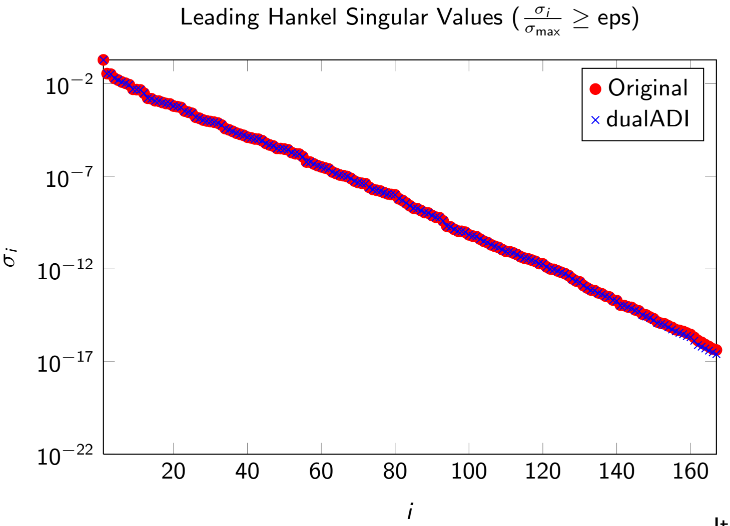
A Dual LRCF-ADI Iteration for Balanced Truncation



Iteration: 49

Recent Contributions

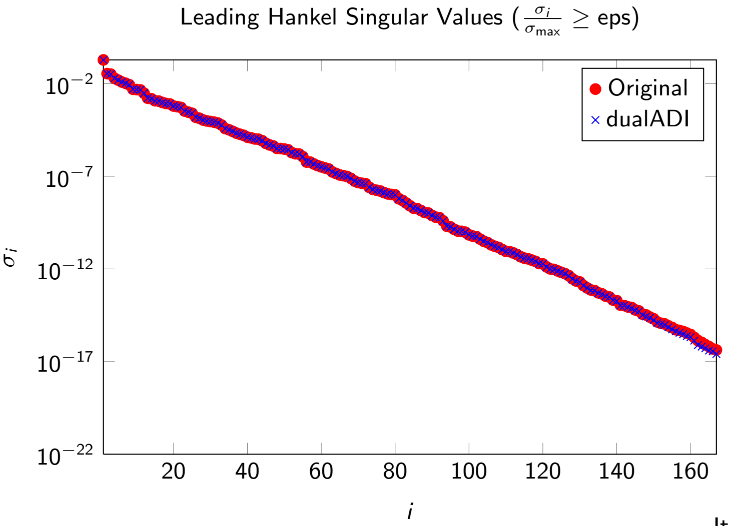
A Dual LRCF-ADI Iteration for Balanced Truncation



Iteration: 50

Recent Contributions

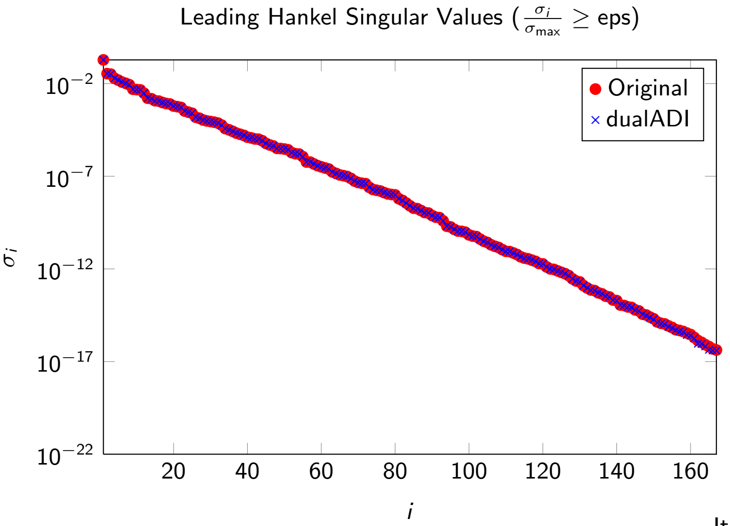
A Dual LRCF-ADI Iteration for Balanced Truncation



Iteration: 51

Recent Contributions

A Dual LRCF-ADI Iteration for Balanced Truncation

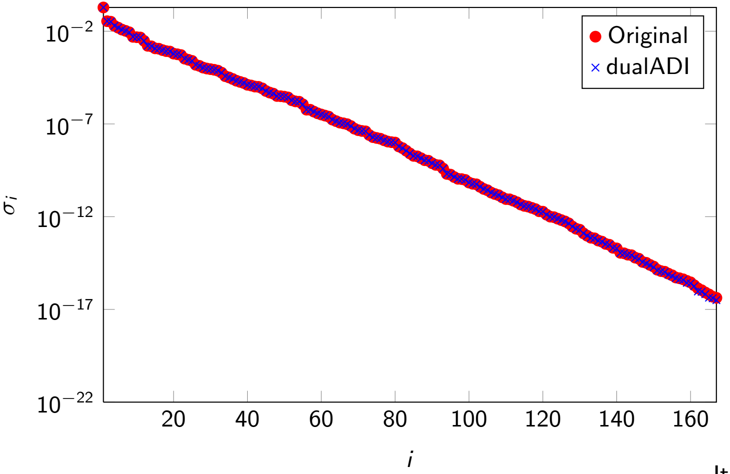


Iteration: 52

Recent Contributions

A Dual LRCF-ADI Iteration for Balanced Truncation

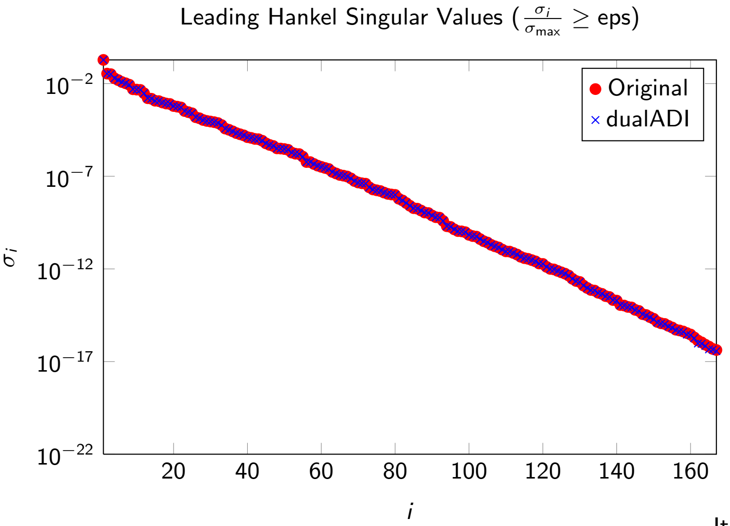
Leading Hankel Singular Values ($\frac{\sigma_i}{\sigma_{\max}} \geq$ eps)



Iteration: 53

Recent Contributions

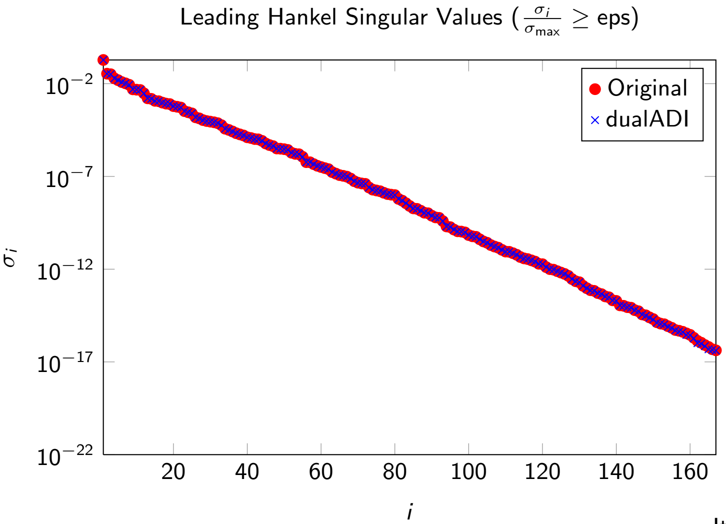
A Dual LRCF-ADI Iteration for Balanced Truncation



Iteration: 54

Recent Contributions

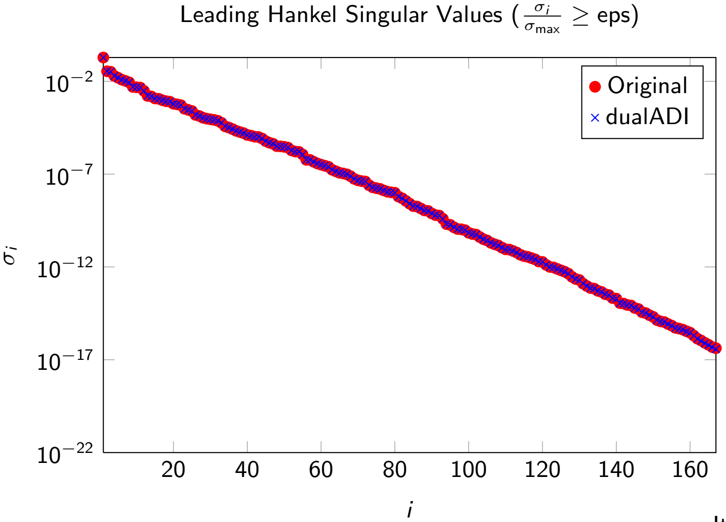
A Dual LRCF-ADI Iteration for Balanced Truncation



Iteration: 55

Recent Contributions

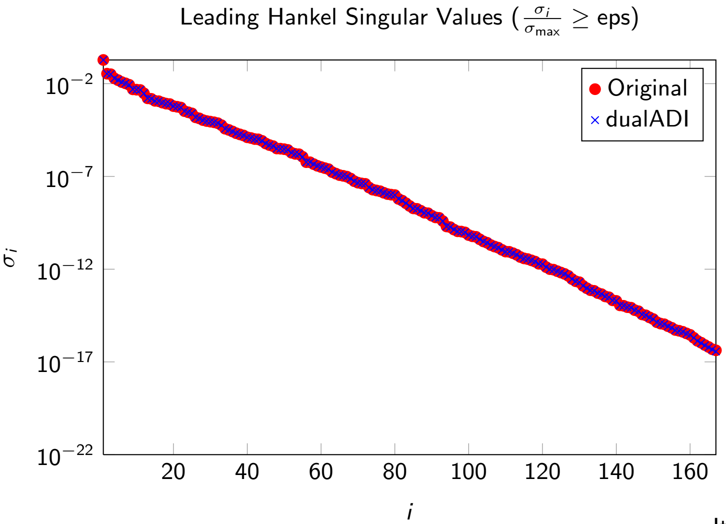
A Dual LRCF-ADI Iteration for Balanced Truncation



Iteration: 56

Recent Contributions

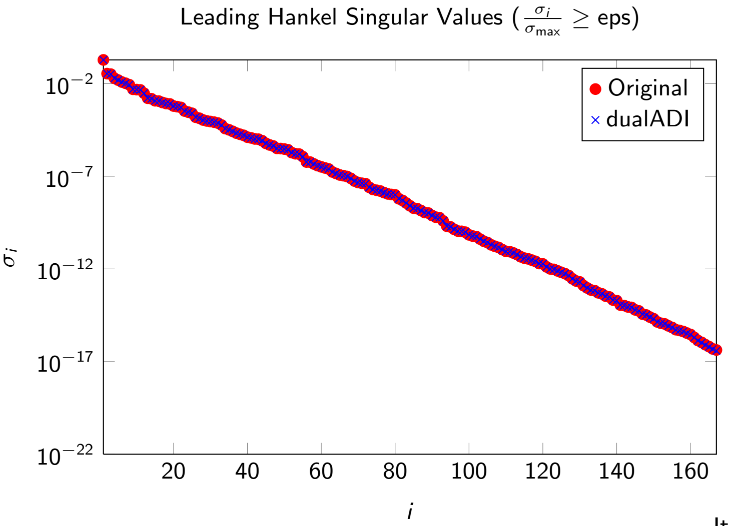
A Dual LRCF-ADI Iteration for Balanced Truncation



Iteration: 57

Recent Contributions

A Dual LRCF-ADI Iteration for Balanced Truncation

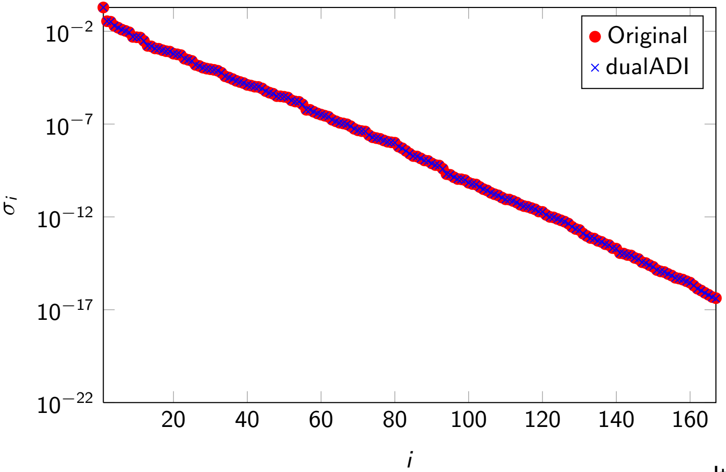


Iteration: 58

Recent Contributions

A Dual LRCF-ADI Iteration for Balanced Truncation

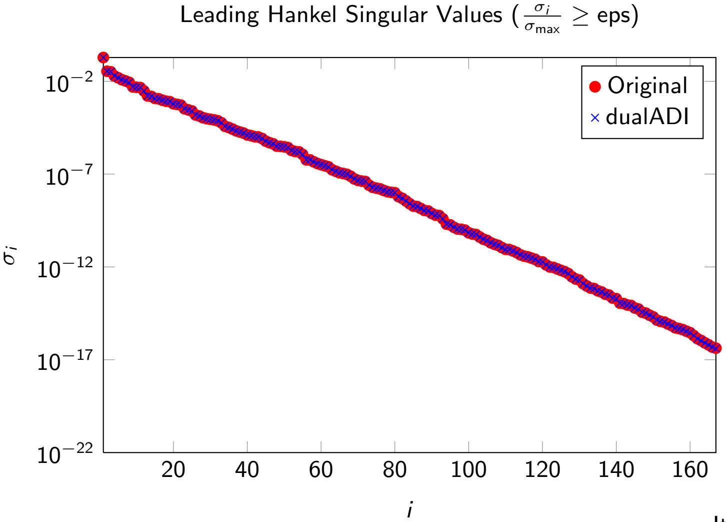
Leading Hankel Singular Values ($\frac{\sigma_i}{\sigma_{\max}} \geq$ eps)



Iteration: 59

Recent Contributions

A Dual LRCF-ADI Iteration for Balanced Truncation



Iteration: 60

Conclusions

- *-LRCF-ADI is an efficient solver for large and sparse Lyapunov equations of several kinds.
- It can be modified to always compute real solution factors $\rightsquigarrow$ real reduced order models in BT.
- Efficiently extensible to the second order case.
- Lyapunov residuals do not always give the right information for BT-MOR. HSV monitoring qualitatively gives more reliable results.
- Quantification of the previous is work in progress.
- Extension of the ideas [Rommes/Freitas/Martins '08] and [Heinkenschloss/Sorensen/Sun '08] to index three mechanical systems with holonomic constraints under investigation.

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Thank you for your attention.