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Subdivision schemes for surfaces with boundaries

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1 Introduction

Subdivision is a way to construct smooth surfaces out of polygonal meshes used in a variety of computer graphics and geometric modeling applications. Two features of subdivision algorithms are particularly important for applications. The first is the ability to handle a large variety of input meshes, including meshes with boundary. The second is the ease of modification of subdivision rules, which makes it possible to generate different surfaces (e.g. surfaces with sharp or soft creases) out of the same input mesh.

Importance of special boundary and crease rules was recognized for some time [12, 13, 8, 19]. However, most of the theoretical analysis of subdivision [18, 16, 26, 25] focused on the case of surfaces without boundaries and schemes invariant with respect to rotations. The goal of this paper is to develop the necessary theoretical foundations for analysis of subdivision rules for meshes with boundary, and present analysis for boundary rules extending several well-known subdivision schemes, described in [1].

In this paper, we consider surfaces with piecewise-smooth boundary. This class readily extends to a broader class of piecewise-smooth surfaces with crease curves and corner points. We demonstrate how the standard constructions of subdivision theory (subdivision matrices, characteristic maps etc.) generalize to surfaces with piecewise-smooth boundary. We demonstrate that convex and concave boundary corners inherenty require separate subdivision rules for the surfaces to have well-defined normals in both cases.

We proceed to extend the techniques for analysis of C^1-continuity developed in [25] to the case of piecewise-smooth surfaces with boundary. While we briefly consider C^k-continuity, we focus on C^1-continuity conditions.

The result allowing one to analyze C^1-continuity of most subdivision schemes for surfaces without boundaries is the sufficient condition of Reif [18]. This condition reduces the analysis of stationary subdivision to the analysis of a single map, called the *characteristic map*, uniquely defined for each valence of vertices in the mesh. The analysis of C^1-continuity is performed in three steps for each valence:

1. compute the control net of the characteristic map;
2. prove that the characteristic map is regular;
3. prove that the characteristic map is injective.

We show that similar conditions hold for surfaces with boundary, and under commonly satisfied assumptions injectivity of the characteristic map for surfaces with boundary can be inferred from regularity.

Finally, we use the theory that we have developed to derive and analyze several specific boundary subdivision rules, initially proposed in [1].

[*]An incomplete version of this work was a part of the first author's PhD thesis, which was unfinished at the time of his untimely passing from cancer

Previous work The theory presented in this paper is based on the theory developed for closed surfaces in [18, 26, 25], which was recently extended to subdivision on manifolds in [22, 21, 23]. Subdivision schemes for closed surfaces were analyzed in [16, 25]. Most of the standard theory is also summarized in the Book by Reif and Peters [17].

As far as we know, analysis of C^1-continuity of subdivision rules for surfaces with boundary was performed only in [19], where a particular choice of rules extending Loop subdivision was analyzed.

At the same time, substantial number of papers proposed various boundary rules starting with the first papers on subdivision by Doo and Sabin, and Catmull and Clark [2, 6, 12, 14, 8]. Most recently, a method for generating soft creases was proposed in [5] and a complexity analysis was done in [15]

2 Surfaces with Piecewise-smooth Boundary

2.1 Definitions

In this section we define surfaces with piecewise-smooth boundaries. Unlike the case of open surfaces, there is no commonly accepted definition that would be suitable for our purposes. We consider several definitions of surfaces with boundaries and motivate the choice that we make (Definition 2.3).

The least restrictive definition of a closed surface with boundary is a closed part of an open surface. This definition is too general for our purposes but provides a useful starting point.

Definition 2.1. *Let M be a closed topological space with boundary, and f a map from M to $\mathbf{R}^p$. We say that (M, f) is a* **closed C^k surface with boundary**, *if there is an open C^k-continuous surface (M', f') and an injective inclusion map $\iota : M \to M'$, such that $f' \circ \iota = f$.*

The boundary is often restricted to be a union of nonintersecting C^k-continuous curves (cf. [11],[7]) Assuming that the domains of these curves are separated in M', this type of surfaces can be defined using two local charts, the open unit disk D and the half-disk $Q_2 = H \cap D$, where H is the closed halfplane defined by $H = \{(x, y)|y \geq 0\}$.

This definition is too narrow for geometric modeling applications, as surfaces with corners (e.g. surfaces obtained by smooth deformations of a rectangle) are quite common. To include corners, we have to allow isolated singularities for the boundary curves. We consider a broader class of surfaces, which we call C^k-continuous surfaces with piecewise C^k-continuous boundary.

Definition 2.2. *Let (M, f), $f : M \to \mathbf{R}^p$ be a closed C^k-continuous surface with boundary as defined above, Let $\gamma_i : [0, 1] \to M$, $i \in I$, where I is finite, be a set of curve segments, such that each endpoint is shared by exactly 2 segments, and the curve segments intersect only at endpoints. Suppose the boundary of M coincides with the intersection of the images of the curve segments $\cap_i \mathrm{Im}\gamma_i$, the curves $f \circ \gamma_i$ are C^k-continuous. Then we call (M, f) a* **C^k-continuous surface with piecewise C^k-continuous boundary.**

The definition implies existence of the tangents to the boundary curves at the endpoints. If the tangents are collinear, two adjacent curves, may either meet in a cusp or a C^m-continuous joint for $0 < m \leq k$. In either case, k different charts are required to parametrize the surface to distinguish the different types, as two curves with a contact point of order m are clearly not C^k-diffeomorphic to two curves with a contact point of order $n \neq m$, for $n, m \leq k$.

Transversality assumption. We assume that the adjacent boundary curve segments intersect transversely, that is, their tangents are different at the shared endpoint. We call such endpoints of boundary curve segments **nondegenerate corners.** Thus, the surfaces that we consider do not contain cusps or C^m-continuous joints for $0 < m < k$,

Once we exclude the higher-order contact cases, we can use a more constructive equivalent definition of surfaces with piecewise C^k-continuous boundary with nondegenerate corners. We use four charts, for all possible types of points of the surface (Figure 1). In addition to the disk D and the halfdisk Q_2, we use a quarter of the disk Q_1 and three quarters of the disk Q_3. The domains Q_i $i = 1, 3$ are defined as follows: $Q_1 = \{(x, y)|y \geq 0 \text{ and } x \geq 0\} \cap D$, $Q_3 = \{(x, y)|y \geq 0 \text{ or } x \geq 0\} \cap D$.

Now we can give an alternative definition of a C^k-continuous surface with piecewise smooth boundary with nondegenerate corners:

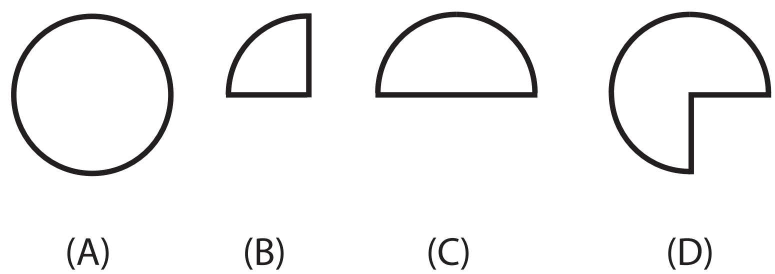


Figure 1: Types of local chart (A) is the disk D, (B) is he quarter disk Q_1, (C) is the half disk Q_2 and (D) is the three-quarter disk Q_3

Definition 2.3. *Consider a surface (M, f) where M is a topological space, and f is a map $f : M \to \mathbf{R}^p$. The surface (M, f) is called* **C^k-continuous with piecewise C^k-continuous boundary with non-degenerate corners** *if for any x there is a neighborhood U_x and a regular C^k-continuous parametrization of $f(U_x)$ over one of the domains Q_i, $i = 1, \ldots 3$, or over the disk D. In the first case, we call the point x a boundary point, in the second case an interior point. We distinguish two main types of boundary points: if U_x is diffeomorphic to Q_2, the boundary point is called smooth; otherwise it is called a corner. There are 2 types of corners:*

- *convex corners (U_x is diffeomorphic to Q_1);*
- *concave corners (U_x is diffeomorphic to Q_3);*

For brevity, we will also use the term C^k-continuous surface with piecewise-smooth boundary.

Equivalence of definitions 2.3 and 2.2 with degenerate corners excluded is straightforward to show using the well-known facts about existence of the extensions of functions defined on Lipschitz domains to the plane.

Surfaces satisfying Definition 2.3 can be used to model a large variety of features; for example, by joining the surfaces along boundary lines we can obtain surfaces with creases. However, in addition to boundary cusps, a number of useful features such as cones cannot be modeled, unless degenerate configurations of control points are used.

2.2 Tangent Plane Continuity and C^1-continuity

As we will see in Section 3, analysis of subdivision focuses on the behavior of surfaces which are known to be at least C^1-continuous in a neighborhood of a point, but nothing is known about the behavior at the point. In this case, it is convenient to split the analysis into several steps, the first being **tangent plane continuity**. In the definition below, we use $\wedge$ to denote the exterior product (vector product for $p = 3$) and $[\cdot]_+$ to denote normalization of a vector.

Definition 2.4. *Let D be the unit disk in the plane. Suppose a surface (M, f) in a neighborhood of a point $x \in M$ is parametrized by $h : U \to \mathbf{R}^p$, where U is a subset of the unit disk D containing 0, which is regular everywhere except 0, and $h(0) = f(x)$. Let $\pi(y) = [\partial_1 h \wedge \partial_2 h]_+$. where $\partial_1 h$ and $\partial_2 h$ are derivatives with respect to the coordinates in the plane of the disk D. The surface is* **tangent plane continuous** *at x if the limit $\lim_{y \to 0} \pi(y)$ exists.*

For an interior point x for which the surface is known to be C^1-continuous in a neighborhood of the point x excluding x, the surface is C^1-continuous at x if and only if it is tangent plane continuous and the projection of the surface into the tangent plane is injective ([26], Proposition 1.2). The proof of this proposition does not assume that the surface is defined on an open neighborhood of x. C^1 continuity for an interior point x is inferred from existence and C^1 continuity of two independent derivatives of reparametrization of the surface over the tangent plane. This fact alone is not sufficient to guarantee that the surface has piecewise continuous boundary with nondegenerate corners: we need to impose an additional condition on the boundary curve.

We can see that the boundary of (M, f) has a nondegenerate corner at x if there is a neighborhood U_x such that $f(U_x) \cap f(\partial M)$ admits a parametrization by two C^1-continuous curves $\gamma_i : (0,1] \to \mathbf{R}^p$, $i = 1,2$, such that $\gamma_1(1) = \gamma_2(1) = f(x)$, and the tangents to the curves are different at the common endpoint x.

Proposition 2.1. *Suppose a surface (M, f) is C^1-continuous with C^1-continuous boundary in a neighborhood U_x of a boundary point x excluding x. The surface is C^1-continuous at x with piecewise C^1-continuous boundary if and only if it is*

1. *tangent plane continuous,*
2. *the projection of the surface into the tangent plane is injective,*
3. *the boundary either has a nondegenerate corner at x or is C^1-continuous at x.*

Proof. Necessity of these conditions is straightforward. Most of the proof of sufficiency coincides with the proof of Proposition 1.2 from [26]: if we assume only that the surface is tangent plane continuous and the projection into the tangent plane is injective, we can show that the derivatives in two independent directions of π, the inverse of a projection of the surface into the tangent plane, exist and are continuous at point x.

It remains to show that the surface is C^1-diffeomorphic to one of the domains Q_i, $i = 1,2,3$.

As the boundary curves γ_i are C^1-continuous, and their tangents are in the tangent plane to the surface at all points, their projections $P\gamma_i$ into the tangent plane at x are also C^1-continuous. At the point x the tangents to the curves are in the tangent plane at x, and coincide with the tangents to the projections. By construction, the domain of π, the image of the projection of the surface into the tangent plane, is homeomorphic to a halfdisk. We have shown that the image of the boundary diameter of the halfdisk is C^1-continuous or C^1-continuous with a nondegenerate corner at x. The neighborhood U_x can be chosen in such a way that $Pf(\partial U_x \setminus \partial M)$, the image of the part of the boundary of U_x which is not the boundary of M, is a semicircle centered at x and intersects the curves $P\gamma_i$ only at a single point each. (We omit somewhat tedious but straightforward proof of this fact.)

Thus, our surface can be parametrized in a neighborhood of x over a planar domain $Pf(U_x)$ which is a subset of an open disk $D_{Pf(x)}$ bounded by two C^1 curve segments connecting the center $Pf(x)$ to the boundary. Let l_1 and l_2 be the rays along tangent directions to γ_1 and γ_2 at x (possibly collinear). Then for sufficiently small radius of the neighborhood, we can assume that orthogonal projections of γ_i to l_i is injective. Note that $l_1 \cap l_2$ split the disk $D_{Pf(x)}$ into two parts D_1 and D_2; either both parts are half-disks, or one part is convex and the other concave. Now we can directly construct a C^1-diffeomorphism of the domain $Pf(U_x)$ to one of the domains D_1 and D_2. For example, in the simplest case of l_1 and l_2 being collinear, we can use a coordinate system (s,t) in which l_1 and l_2 form the s axis, and γ_1 and γ_2 form a graph of a function $\gamma(s)$. Assuming that the disk $D_{Pf(x)}$ has radius 1 the formula

$$(s,t) \to \left(s, \sqrt{1-s^2}\frac{t-\gamma(s)}{\sqrt{(1-s^2)}-\gamma(s)} \right)$$

defines the desired diffeomorphism.

We have shown that the surface has a parametrization g over one of the domains Q_i $i = 1,2,3$ in the neighborhood of x, which has C^1-continuous derivatives everywhere on Q_i with nowhere degenerate Jacobian. □

This proposition provides a general strategy for establishing C^1-continuity of surfaces, which is particularly convenient for subdivision surfaces. Moreover, as we shall see, in most cases of practical importance we can infer the injectivity of the projection from the other two conditions, so only local tests need to be performed.

3 Subdivision Schemes on Complexes with Boundary

In this section we summarize the main definitions and facts about subdivision on complexes that we use; more details for the case of surfaces without boundaries can be found in [26, 24]. The changes that have to be made to make the constructions work for the boundary case are relatively small. We restrict the presentation to the case of schemes for triangle meshes to avoid making the notation excessively complex. However, the results equally apply to quadrilateral schemes; only minor changes in notation are necessary.

3.1 Definitions

Simplicial complexes. Subdivision surfaces are naturally defined as functions on two-dimensional polygonal complexes. A simplicial complex K is a set of vertices, edges and planar simple polygons (faces) in $\mathbf{R}^N$, such that for any face its edges are in K, and for any edge its vertices are in K. We assume that there are no isolated vertices or edges. $|K|$ denotes the union of faces of the complex regarded as a subset of $\mathbf{R}^N$ with induced metric. We say that two complexes K_1 and K_2 are *isomorphic* if there is a homeomorphism between $|K_1|$ and $|K_2|$ that maps vertices to vertices, edges to edges and faces to faces.

A *subcomplex* of a complex K is a subset of K that is a complex. A 1-neighborhood $N_1(v, K)$ of a vertex v in a complex K is the subcomplex formed by all faces that have v as a vertex. The m-neighborhood of a vertex v is defined recursively as a union of all 1-neighborhoods of vertices in the $(m-1)$-neighborhood of v. We omit K in the notation for neighborhoods when it is clear what complex we refer to.

Recall that a *link* of a vertex is the set of edges of $N_1(v, K)$ that do not contain v. We consider only complexes with all vertices having links that are connected simple polygonal lines, open or closed. If the link of a vertex is an open polygonal line, this vertex is a boundary vertex, otherwise it is an internal vertex.

In the analysis of schemes for surfaces without boundary the regular complex $\mathcal{R}$ and k-regular complexes $\mathcal{R}_k$ are are commonly used [26]. We are primarily interested in schemes that work on quadrilateral and triangle meshes, and we consider k-regular complexes with all faces being identical triangles or quads; however, similar complexes can be defined for the remaining regular tiling, with all faces being hexagons, and more generally for any Laves tiling. For schemes acting on meshes with boundary we use regular and k-regular complexes with boundary. A regular complex with boundary is isomorphic to a regular tiling of the upper half-plane. A k-regular complex $\mathcal{R}_k^\alpha$ with apex angle α is isomorphic to the regular tiling of a sector with apex angle α, consisting of identical polygons, with all internal vertices of equal valence and all vertices on the boundary of equal valence, excluding the vertex C at the apex which has valence $k+1$. For triangle meshes the valence of regular interior vertices is six, and for boundary vertices it is three.

Note that the complex is called k-regular, because the number of faces sharing the vertex C is k, not the number of edges. In the case of closed surfaces these numbers are equal.

Tagged complexes. The vertices, edges or faces of a complex can be assigned tags, or more formally, a map can be defined from the sets of vertices, edges or faces to a finite set of tags. These tags can be used to choose a type of subdivision rules applied at a vertex. In this paper, we use tags in a very limited way: specifically, a boundary vertex can be tagged as a *convex* or *concave* corner, or a smooth boundary vertex. However, as it is discussed below, the tags can be used to create creases in the interior of meshes and for other purposes. Subdivision on tagged complexes merits a separate detailed consideration in a future paper.

Isomorphisms of tagged complexes with identical tag sets can be defined as isomorphisms of complexes which preserve tags, i.e. if a vertex has a tag τ its image also has a tag τ.

Subdivision of simplicial complexes. We can construct a new complex $D(K)$ from a complex K by subdivision. For a triangle scheme, $D(K)$ is constructed by adding a new vertex for each edge of the complex and replacing each old triangular face with four new triangles. If some faces of the initial complex are not triangular, they have to be split into triangles first. For a quadrilateral scheme, $D(K)$ is constructed by adding a vertex for each edge and face, and replacing each n-gonal face with n quadrilateral faces. Note that k-regular complexes and k-regular complexes with boundary are self-similar, that is, $D(\mathcal{R}_k)$ and $\mathcal{R}_k$, as well as $D(\mathcal{R}_k^\alpha)$ and $\mathcal{R}_k^\alpha$, are isomorphic.

We use notation K^j for j times subdivided complex $D^j(K)$ and V^j for the set of vertices of K^j. Note that the sets of vertices are nested: $V^0 \subset V^1 \subset \ldots$.

If a complex is tagged, it is also necessary to define rules for assigning tags to the new edges, vertices and faces. For our vertex tags, we use a trivial rule: all newly inserted boundary vertices are tagged as smooth boundary.

Subdivision schemes. Next, we attach values to the vertices of the complex; in other words, we consider the space of functions $V \to B$, where B is a vector space over $\mathbf{R}$. The range B is typically $\mathbf{R}^l$ or $\mathbf{C}^l$ for some l. We denote this space $\mathcal{P}(V, B)$, or $\mathcal{P}(V)$, if the choice of B is not important.

A *subdivision scheme* for any function $p^j(v)$ on vertices V^j of the complex K^j computes a function $p^{j+1}(v)$ on the vertices of the subdivided complex $D(K) = K^1$. More formally, a subdivision scheme is a collection of operators $S[K]$ defined for every complex K, mapping $\mathcal{P}(K)$ to $\mathcal{P}(K^1)$. We consider only subdivision schemes that are linear, that is, the operators $S[K]$ are linear functions on $\mathcal{P}(K)$. In this case the subdivision operators are defined by equations

$$p^1(v) = \sum_{w \in V} a_{vw} p^0(w)$$

for all $v \in V^1$. The coefficients a_{vw} may depend on K.

We restrict our attention to subdivision schemes which are finitely supported, locally invariant with respect to a set of isomorphisms of tagged complexes and affinely invariant.

A subdivision scheme is *finitely supported* if there is an integer M such that $a_{vw} \neq 0$ only if $w \in N_M(v, K)$ for any complex K (note that the neighborhood is taken in the complex K^{j+1}). We call the minimal possible M the *support size* of the scheme.

We assume our schemes to be *locally defined* and *invariant with respect to isomorphisms of tagged complexes*. Together these two requirements can be defined as follows: there is a constant L such that if for two complexes K_1 and K_2 and two vertices $v_1 \in V_1$ and $v_2 \in V_2$ there is a tag-preserving isomorphism $\rho : N_L(v_1, K_1) \to N_L(v_2, K_2)$, such that $\rho(v_1) = v_2$, then $a_{v_1 w} = a_{v_2 \rho(w)}$. In most cases, the *localization size* $L = M$.

The final requirement that we impose on subdivision schemes is *affine invariance*: if T is a linear transformation $B \to B$, then for any v $\; Tp^{j+1}(v) = \sum a_{vw} Tp^j(v)$. This is equivalent to requiring that all coefficients a_{vw} for a fixed v sum up to 1.

For each vertex $v \in \cup_{j=0}^{\infty} V^j$ there is a sequence of values $p^i(v)$, $p^{i+1}(v)$,... where i is the minimal number such that V^i contains v.

Definition 3.1. *A subdivision scheme is called convergent on a complex K, if for any function $p \in \mathcal{P}(K, B)$ there is a continuous function f defined on $|K|$ with values in B, such that*

$$\lim_{j \to \infty} \sup_{v \in V^j} \left\| p^j(v) - f(v) \right\|_2 \to 0$$

The function f is called the limit function of subdivision.

Notation: $f[p]$ is the limit function generated by subdivision from the initial values $p \in \mathcal{P}(K)$.

It is easy to show that if a limit function exists, it is unique. A *subdivision surface* is the limit function of subdivision on a complex K with values in $\mathbf{R}^3$. In this case we call the initial values $p^0(v)$ the *control points* of the surface.

Assuming the trivial rule for assigning tags to the newly inserted boundary vertices, we observe that locally any surface generated by a subdivision scheme on an arbitrary complex can be thought of as a part of a subdivision surface defined on a k-regular complex or a k-regular complex with boundary.

Note that this fact alone does not guarantee that it is sufficient to study subdivision schemes only on k-regular complexes and k-regular complexes with boundary [26]. If the number of control points of the initial complex for a k-gonal patch is less than the number of control points of the central k-gonal patch in the k-regular complex, then only a proper subspace of all possible configurations of control points on the subdivided complexes can be realized. Although it is unlikely, it is possible that for such complexes almost all configurations of control points will lead to non-smooth surfaces, while the scheme is smooth on the k-regular complexes.

Subdivision matrices. Consider the part of a subdivision surface $f[y]$ with $y \in U_1^j = |N_1(0, \mathcal{R}_k^j)|$, defined on the domain formed by faces of the subdivided complex $\mathcal{R}_k^j$ adjacent to the central vertex. It is straightforward to show that the values at all dyadic points in $|N_1(0, \mathcal{R}_k^j)|$ can be computed given the initial values $p^j(v)$ for $v \in N_L(0, \mathcal{R}_k^j)$. In particular, the control points $p^{j+1}(v)$ for $v \in N_L(0, R_k^{j+1})$ can be computed using only control points $p^j(w)$ for $w \in N_L(0, \mathcal{R}_k^j)$. Let $\bar{p}^j$ be the vector of control points $p^j(v)$ for $v \in N_L(0, \mathcal{R}_k^j)$. Let $p+1$ be the number of vertices in $N_L(0, \mathcal{R}_k)$. As the subdivision operators are linear, $\bar{p}^{j+1}$ can be computed from $\bar{p}^j$ using a $(p+1) \times (p+1)$ matrix S^j: $\bar{p}^{j+1} = S^j \bar{p}^j$

If for some m and for all $j > m$, $S^j = S^m = S$, we say that the subdivision scheme is *stationary on the k-regular complex*, or simply stationary, and call S the *subdivision matrix* of the scheme.

Eigenbasis functions. let $\lambda_0 = 1, \lambda_i, \dots \lambda_J$ be different eigenvalues of the subdivision matrix in nonincreasing order, the condition $\lambda_0 > \lambda_1$ is necessary for convergence.

For any λ_i let J_j^i, $j = 1 \dots$ be the complex cyclic subspaces corresponding to this eigenvalue.

Let n_j^i be the *orders* of these cyclic subspaces; the order of a cyclic subspace is equal to its dimension minus one.

Let b_{jr}^i, $r = 0 \dots n_j^i$ be the complex generalized eigenvectors corresponding to the cyclic subspace J_j^i. The vectors b_{jr}^i satisfy

$$Sb_{jr}^i = \lambda_i b_{jr}^i + b_{j\,r-1}^i \quad \text{if } r > 0, \quad Sb_{j0}^i = \lambda_i b_{j0}^i \tag{3.1}$$

The complex *eigenbasis functions* are the limit functions defined by $f_{jr}^i = f[b_{jr}^i] : U_1 \to \mathbf{C}$

Any subdivision surface $f[p] : U_1 \to \mathbf{R}^3$ can be represented as

$$f[p](y) = \sum_{i,j,r} \beta_{jr}^i f_{jr}^i(y) \tag{3.2}$$

where $\beta_{jr}^i \in \mathbf{C}^3$, and if $b_{jr}^i = \overline{b_{lt}^k}$, $\beta_{jr}^i = \overline{\beta_{lt}^k}$, where the bar denotes complex conjugation.

One can show using the definition of limit functions of subdivision and (3.3) that the eigenbasis functions satisfy the following set of *scaling relations*:

$$f_{jr}^i(y/2) = \lambda_i f_{jr}^i(y) + f_{j\,r-1}^i(y) \quad \text{if } r > 0, \quad f_{j0}^i(y/2) = \lambda_i f_{j0}^i(y) \tag{3.3}$$

Real eigenbasis functions. As we consider real surfaces, it is often convenient to use real Jordan normal form of the matrix rather than the complex Jordan normal form. For any pair of the complex-conjugate eigenvalues λ_i, λ_k, we can choose the complex cyclic subspaces in such a way that they can be arranged into pairs J_j^i, J_j^k, and $b_{jr}^i = \overline{b_{jr}^k}$ for all j and r. Then we can introduce a single real subspace for each pair, with the basis c_{jr}^i, c_{jr}^k, $r = 0 \dots n_j^i$, where $c_{jr}^i = \Re b_{jr}^i$, and $c_{jr}^k = \Im b_{jr}^i$. We call such subspaces *Jordan subspaces*. Then we can introduce real eigenbasis functions $g_{jr}^i(y) = f_{jr}^i(y)$ for real λ_i, and $g_{jr}^i(y) = \Re f_{jr}^i(y)$, $g_{jr}^k(y) = \Im f_{jr}^i(y)$ for a pair of complex-conjugate eigenvalues (λ_i, λ_k). For a Jordan subspace corresponding to pairs of complex eigenvalues the order is the same as the order of one of the pair of cyclic subspaces corresponding to it.

Similar to (3.2) we can write for any surface generated by subdivision on U_1:

$$f[p](y) = \sum_{i,j,r} \alpha_{jr}^i g_{jr}^i(y) \tag{3.4}$$

Now all coefficients α_{jr}^i are real. Eigenbasis functions corresponding to the eigenvalue 0 have no effect on tangent plane continuity or C^k-continuity of the surface at zero. From now on we assume that $\lambda_i \neq 0$ for all i.

We can assume that the coordinate system in $\mathbf{R}^3$ is always chosen in such a way that the single component of $f[p]$ corresponding to eigenvalue 1 is zero. This allows us to reduce the number of terms in (3.4) to p.

3.2 Reduction to universal surfaces

In [26] we have shown that for surfaces without boundary the analysis of smoothness of subdivision can be reduced to the analysis of *universal surfaces*. Moreover, if a subdivision scheme is C^1, almost any surface produced by subdivision is diffeomorphic to the universal surface. In this section, we introduce the universal surfaces for neighborhoods of boundary vertices, and show that a similar reduction can be performed in this case.

This fact is of considerable practical importance for design of subdivision schemes for surfaces with piecewise-smooth boundary: as we have observed in Section 2, convex and concave corners are not diffeomorphic; therefore, a convex and and a concave corner in $\mathbf{R}^3$ cannot be diffeomorphic to the same universal surface, and cannot be generated by the same subdivision rule.

Universal map. The decomposition (3.4) can be written in vector form. Let h^i_{jr} be an orthonormal basis of $\mathbf{R}^p$. Let ψ be $\sum_{i,j,r} g^i_{jr} h^i_{jr}$; this is a map $U_1 \to \mathbf{R}^p$. Let $\alpha^1, \alpha^2, \alpha^3 \in \mathbf{R}^p$ be the vectors composed of components of coefficients α^i_{jr} from (3.4) (each of these coefficients is a vector in $\mathbf{R}^3$). Then (3.4) can be rewritten as

$$f[p](y) = \big((\psi, \alpha^1),\, (\psi, \alpha^2),\, (\psi, \alpha^3)\big) \tag{3.5}$$

This equation indicates that all surfaces generated by a subdivision scheme on U_1 can be viewed as projections of a single surface in $\mathbf{R}^p$. We call ψ the *universal map,* and the surface specified by ψ the *universal surface.* In [26], it was demonstrated that the analysis of tangent plane continuity and C^k continuity of subdivision can be reduced to analysis of the universal surface. Not surprisingly, we will see that this also holds for subdivision schemes with boundary.

In the chosen basis the matrix S is in the real Jordan normal form. Note that by definition of S for any $a \in \mathbf{R}^p$

$$(a, \psi(y/2)) = (Sa, \psi(y))$$

Using the well-known formula for inner products $(Su, v) = (u, S^T v)$, we get

$$(x, \psi(y/2)) = (x, S^T \psi(y)), \quad \text{for any } x$$

This means that the scaling relations can be jointly written as

$$\psi(y/2) = S^T \psi(y) \tag{3.6}$$

The universal map ψ is only piecewise C^k, even if we assume that subdivision produces C^k limit function on regular complexes and regular complexes with boundary: derivatives have discontinuity at the boundaries of polygons of U_1. However, one can easily construct a map κ (see [26]) such that $\varphi = \psi \circ \kappa^{-1}$ is C^1-continuous away from the center.

We will impose the following condition on the subdivision schemes which we call **Condition A.** For any $y \in U_1$

$$\partial_1 \psi(y) \wedge \partial_2 \psi(y) \neq 0 \quad \text{for all } y \in U_1,\, y \neq 0.$$

This condition holds for all known practical schemes.

Reduction theorem. Our goal is to relate tangent plane continuity and C^k-continuity of the universal surface in $\mathbf{R}^p$ and tangent plane continuity of the subdivision scheme. The following theorem holds under our assumptions:

Theorem 3.1. *For a subdivision scheme satisfying Condition A to be tangent plane continuous on a k-regular complex with boundary, it is necessary and sufficient that the universal surface be tangent plane continuous; for the subdivision scheme to be C^k-continuous with p.w. C^k-continuous boundary, it is necessary and sufficient that the universal surface is C^k-continuous with p.w. C^k-continuous boundary. Almost all surfaces generated by a subdivision scheme on a k-regular complex with boundary are locally diffeomorphic to the universal surface.*

Proof. Sufficiency is clear as any surface is a linear projection of the universal surface. To prove necessity, we use Proposition 2.1, and show that

- if the universal surface is not tangent plane continuous then a set of subdivision surfaces of non-zero measure is not tangent plane continuous;

- if the universal surface has non-injective projection into the tangent plane same is true for a set of subdivision surfaces of non-zero measure;
- if the projection of the universal surface into the tangent plane is not C^k, same is true for a set of subdivision surfaces of non-zero measure;
- if the boundary of the universal surface is not C^k-continuous, or is not C^k-continuous with nondegenerate corner, same is true for a set of subdivision surfaces of non-zero measure.

The proof of the first three statements coincides with the proof for the surface without boundary presented in [26].

We only need to consider the fourth statement. By assumption, the boundary of the surface is C^1-continuous away from zero. Let the two pieces of the boundary be $\gamma_i : (0, 1] \to \mathbf{R}^p$, $i = 1, 2$, with $\gamma_1(1) = \gamma_2(1)$. We can assume both pieces to be C^1-continuous away from one. Suppose γ_1 does not have a tangent at one; then there are at least two directions τ_1 and τ_2 which are limits of sequences of tangent directions to $\gamma_1(t)$ as t approaches one. There is a set of three-dimensional subspaces π of measure non-zero in the space of all three-dimensional subspaces, for which the projections of both vectors τ_1 and τ_2 to the subspace are not zero. If we project the universal surface to any of these subspaces, the boundary curve of the resulting surface will not be tangent continuous. For curves tangent continuity is equivalent to C^1-continuity. For C^k-continuity the proof for curves is identical to the proof for surfaces. We conclude that the curves γ_1 and γ_2 should be C^k-continuous. Similarly, if the curves are joined with continuity less than k, then almost all curves obtained by projection into $\mathbf{R}^3$ will have the same property. Finally, if the tangents to the curves coincide, same is true for almost all projections of the curves, which means that almost all projections do not have a non-degenerate corner. □

The following important corollary immediately follows from Theorem 3.1:

Corollary 3.2. *Almost all surfaces generated by a given C^k-continuous subdivision scheme on a k-regular complex are diffeomorphic.*

Indeed, as any subdivision surface $f : U_k \to \mathbf{R}^3$ is obtained as a projection of the universal surface, for almost any choice of projection it defines a diffeomorphism of the universal surface and f.

This corollary implies in particular that the same subdivision rule cannot generate convex and concave corners simultaneously in a stable way, and separate rules are required for these cases.

4 Criteria for tangent plane and C^1 continuity

Tangent plane continuity criteria of [26] do not use the fact that only interior points of a surface are considered. Similarly, C^1-continuity criteria use only the fact that C^1-continuity is equivalent to tangent plane continuity and injectivity of the projection into the tangent plane. Therefore, C^1-continuity criteria also hold for boundary points. We only need to establish the conditions that guarantee that the boundary curves are C^1-continuous, possibly with corners.

We focus on a sufficient condition for C^1-continuity ([26] Theorem 3.6 and Theorem 4.1), which is most relevant for applications. More general necessary and sufficient conditions can be extended in a similar way.

To state the sufficient condition, we need to define *characteristic maps*, which are commonly used to analyze C^1-continuity of subdivision surfaces. We use a definition somewhat different from the original definition of Reif [18].

4.1 Conditions on Characteristic maps

Definition 4.1. *The* **characteristic map** $\Phi : U_1 \to \mathbf{R}^2$ *is defined for a pair of cyclic subspaces J_b^a, J_d^c of the subdivision matrix as*

1. (f_{a0}, f_{a1}) *if* $J_b^a = J_d^c$, λ_a *is real,*
2. (f_{a0}, f_{c0}) *if* $J_b^a \neq J_d^c$, λ_a, λ_c *are real,*

3. $(\Re f_{a0}, \Im f_{a0})$ *if* $\lambda_a = \bar{\lambda_c}$, $b = d$.

Three types of characteristic maps are shown in Figure 2.

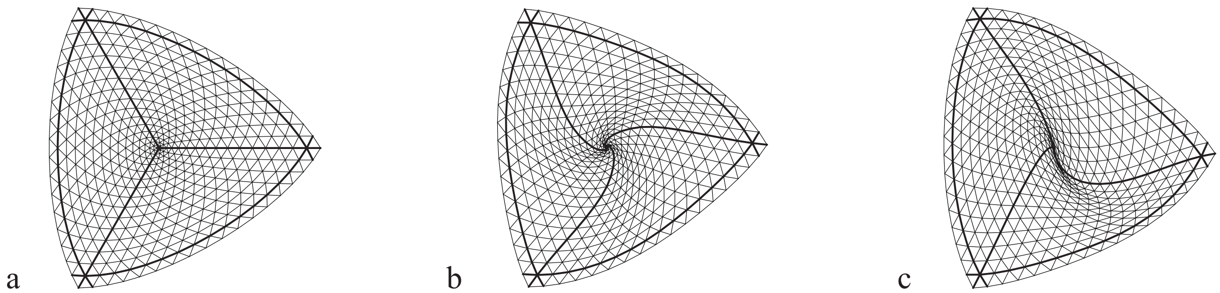


Figure 2: Three types of characteristic maps: control points after 4 subdivision steps are shown. a. Two real eigenvalues. b. A pair of complex-conjugate eigenvalues. c. single eigenvalue with Jordan block of size 2.

The domain of a characteristic map is the neighborhood U_1, consisting of k faces of the regular complex; we call these faces *segments*. We assume that the subdivision scheme generates C^1-continuous limit functions on regular complexes, and the characteristic map is C^1-continuous inside each segment and has continuous one-sided derivatives on the boundary.

Characteristic map satisfies the scaling relation $\Phi(t/2) = T\Phi(t)$, where T is one of the matrices

$$T_{\text{scale}} = \begin{pmatrix} \lambda_a & 0 \\ 0 & \lambda_c \end{pmatrix}, \quad T_{\text{skew}} = \begin{pmatrix} \lambda_a & 1 \\ 0 & \lambda_a \end{pmatrix}, \quad T_{\text{rot}} = |\lambda_a| \begin{pmatrix} cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix},$$

where φ is the argument of a complex λ_a.

Sufficient condition for C^1-continuity. The following sufficient condition is a special case of the condition that was proved in [26]. Although all our constructions apply in the more general case, we state only a simplified version of the criterion sufficient for most applications. This condition generalizes Reif's condition [18].

Define for any two cyclic subspaces $\operatorname{ord}\left(J_j^i, J_l^k\right)$ to be $n_j^i + n_l^k$, if $J_j^i \neq J_l^k$; let $\operatorname{ord}\left(J_j^i, J_j^i\right) = 2n_j^i - 2$; note that for $n_j^i = 0$, this is a negative number, and it is less than ord for any other pair. This number allows us to determine which components of the limit surface contribute to the limit normal (see [26, 24] for details). We say that a pair of cyclic subspaces J_b^a, J_d^c is *dominant* if for any other pair J_j^i, J_l^k we have either $|\lambda_a\lambda_c| > |\lambda_i\lambda_k|$, or $|\lambda_a\lambda_c| = |\lambda_i\lambda_k|$ and $\operatorname{ord}(J_b^a, J_d^c) > \operatorname{ord}\left(J_j^i, J_l^k\right)$. Note that the blocks of the dominant pair may coincide.

Theorem 4.1. *Let b_{jr}^i be a basis in which a subdivision matrix S has Jordan normal form. Suppose that there is a dominant pair J_b^a, J_d^c. If $\lambda_a\lambda_c$ positive real, and the Jacobian of the characteristic map of J_b^a, J_d^c has constant sign everywhere on U_1 except zero, then the subdivision scheme is tangent plane continuous on the k-regular complex.*

If the characteristic map is injective, the subdivision scheme is C^1-continuous.

In the special case when all Jordan blocks are trivial, this condition reduces to an analog of Reif's condition. Theorem 4.1 doesn't make any claim about the type of boundary however. It is therefore not enough for the analysis of the desired surfaces.

Criterion for piecewise C^1-continuity of the boundary. Assuming that the scheme at a boundary vertex satisfies the conditions of Theorem 4.1, we establish additional conditions which guarantee that the scheme for almost all control meshes generates C^1-continuous surfaces with piecewise C^1-continuous boundary with nondegenerate corners. The domain of the characteristic function is called U_1. We assume that the part of U_1 that corresponds with the boundary of the surface is a straight line. We call I_1 and

I_2 the two parts of this boundary line achieved by excluding the center vertex. When we talk about ∂_1 we mean the derivative in the direction of this boundary line. ∂_2 will be the orthogonal direction. We will call the two components of the characteristic map by f_1 and f_2 in the following theorem.

Theorem 4.2. *Suppose a subdivision scheme satisfies the conditions of Theorem 4.1 for boundary vertices of valence k. Then the scheme is p.w. C^1-continuous with nondegenerate corners for boundary vertices of valence k if and only if the following conditions are satisfied.*

1. *λ_a and λ_c are positive real.*
2. *Suppose $\lambda_a > \lambda_c$, (diagonal scaling matrix, asymmetric scaling). Then the scheme is boundary C^1-continuous if and only if $\partial_1 f_1 \neq 0$ and has the same sign on I_1 and I_2 or $\partial_1 f_1 \equiv 0$ on I_1 and I_2.*

   The scheme is a nondegenerate corner scheme, if and only if $\partial_1 f_1 \neq 0$ on I_1 and $\partial_1 f_1 \equiv 0$ on I_2. Same is true if I_1 and I_2 are exchanged.
3. *Suppose $J_c^a = J_d^b$ (scaling matrix is a Jordan block of size 2), and $\partial_1 f_1$ does not vanish on I_1 and I_2. The scheme is boundary C^1-continuous if $\partial_1 f_2$ has the same sign everywhere on I_1 and I_2 and if $\partial_1 f_2(t_1) = 0$ for a $t_1 \in I_1 \cup I_2$ then $\partial_1 f_1(t_1)$ needs to have this sign as well. Nondegenerate corners cannot be generated by a scheme of this type.*
4. *Suppose $a = c$ (diagonal scaling matrix, symmetric scaling). The boundary is C^1-continuous if and only if there is a nontrivial linear combination $\alpha_1 \partial_1 f_1 + \alpha_2 \partial_1 f_2$ identically vanishing on I_1 and I_2, and any other independent linear combination has the same sign on I_1 and I_2. The scheme is a corner scheme if and only if there is a linear combination $\alpha_1 \partial_1 f_1 + \alpha_2 \partial_1 f_2$ identically vanishing on I_1 and a different linear combination $\beta_1 \partial_1 f_1 + \beta_2 \partial_1 f_2$ identically vanishing on I_2 with $[\alpha_1, \alpha_2]$ and $[\beta_1, \beta_2]$ linearly independent.*

Proof. For each of the boundary segments defined on I_1 and I_2 we need to show that the limit of the tangent exists at the common endpoint. If these limits coincide then the boundary curve of the universal surface is C^1-continuous; if the limits have different directions, then the universal surface has a nondegenerate corner.

First, we observe that by assumption the characteristic map has non-zero Jacobian on the boundary. This means that one of the components has nonzero derivative along the boundary $\partial_1 f_1(t) \neq 0$ or $\partial_1 f_2(t) \neq 0$ at any point $t \in I_1 \cup I_2$. Consider the tangent to the boundary of the surface defined by the characteristic map. It is a two-dimensional vector $v(t) = (\partial_1 f_1(t), \partial_1 f_2(t))$, where t is a point of I_1 or I_2. The tangent satisfies the scaling relation of the form $v(t/2) = 2Tv(t)$, where T is the scaling matrix for the characteristic map. The direction of the tangent has a limit if and only if T is either T_{scale} or T_{skew} and its eigenvalues are positive (Lemma 3.1, [26]). As the projection of the universal surface is arbitrarily well approximated by the characteristic map, or coincides with it for simple Jordan structures of the subdivision matrix, we conclude that for the universal surface boundary to have well-defined tangents at zero, the eigenvalues of the characteristic map have to be positive and real. However, this condition is not sufficient for existence of tangents.

Diagonal scaling matrix, asymmetric case. First we consider the case of dominant cyclic subspace pair J_b^a, J_d^c with $a \neq c$ (different eigenvalues). In this case the sequences $\partial_1 f_1(t/2^m)$ and $\partial_1 f_2(t/2^m)$, for $\partial_1 f_1(t), \partial_1 f_2(t) \neq 0$, change at a different rate. This can be easily seen from the scaling relation. Moreover, the ratio $\|\partial_1 f_2(t/2^m)\|/\|\partial_1 f_1(t/2^m)\|$ approaches zero as $m \to \infty$.

Suppose at some points t_1, t_2 of I $\partial_1 f_1(t_1) \neq 0$ and $\partial_1 f_1(t_2) = 0$. Then $\partial_1 f_2(t_2) \neq 0$ and the tangents at points $t_2/2^m$ all point in the direction $\pm e_2$, where e_2 is the unit vector along the coordinate axis corresponding to f_2. $\|\partial_1 f_2(t_1/2^m)\|/\|\partial_1 f_1(t_1/2^m)\| \to 0$ as $m \to \infty$, thus, at points $t_1/2^m$ the direction of the tangent approaches $\pm e_1$. We conclude that there is no limit, unless $\partial_1 f_1$ is either nowhere or everywhere zero I_1. Same applies to I_2. Conversely, if $\partial_1 f_1$ is nowhere zero, then the limit tangent direction at the center is $\pm e_1$. If it is zero everywhere, then by assumption about the characteristic map, $\partial_1 f_2$ is nowhere zero, and the limit tangent direction is $\pm e_2$. The choice of sign in each case depends on the sign of $\partial_1 f_1$ or $\partial_1 f_2$.

If $\partial_1 f_1$ is not zero and has the same sign on both I_1 and I_2 then the tangent is continuous, and the boundary curve is C^1-continuous. If $\partial_1 f_1 \equiv 0$ on I_1 and I_2 the images of I_1 and I_2 under the characteristic map are straight lines on the e_2 axis and therefore the boundary curve is C^1 continuous. If it is zero on I_1 and nonzero on I_2, then the tangents are not parallel, and the surface defined by the characteristic map has a corner; and the same for I_1 and I_2 interchanged which proves the second part.

Scaling matrix is a Jordan block of size 2. The second condition of the theorem applies if the characteristic map components correspond to a cyclic subspace of size 2, i.e. satisfy $f_1(t/2) = \lambda_a f_1(t) + f_2(t)$. Thus, $\partial_1 f_1 \equiv 0$ implies $\partial_1 f_2 \equiv 0$ on I_1 or I_2. Otherwise $v(t/2^m)$ converges to $\pm e_1$ for any t on I_1 as well as I_2. If $\partial_1 f_2(t) \neq 0$ its sign determines the sign of the limit tangent.

Diagonal scaling matrix, symmetric case. In the symmetric case where $a = b$ the sequences defined above change at the same rate, and any linear combination $\alpha_1 f_1 + \alpha_2 f_2$ is also an eigenbasis function. Suppose f_1 and f_2 come from different cyclic subspaces of the same eigenvalue which have the same size. Suppose $\alpha_1\partial_1 f_1 + \alpha_2\partial_1 f_2$ does not vanish identically on I_1 for any nontrivial choice of α_1 and α_2. Pick two linearly independent combinations $g_1 = \alpha_1\partial_1 f_1 + \partial_1\alpha_2 f_2$ and $g_2 = \beta_1\partial_1 f_1 + \beta_2\partial_1 f_2$ which do not vanish at points t_1 and t_2 of I_1 respectively. Then the vectors $v(t_i) = [\partial_1 f_1(t_i), \partial_1 f_2(t_i)]$ are linearly independent and the sequences $v(t_1/2^m)$ and $v(t_2/2^m)$ converge to different limit directions. Therefore, for the limit tangents at zero to exist, there should be a nontrivial linear combination of $\partial_1 f_1$ and $\partial_1 f_2$ which vanishes on I_1. If $\alpha_1\partial_1 f_1 + \alpha_2\partial_1 f_2$ is such combination, it is easy to see that the limit tangent direction is, up to the sign,the direction of the vector $[-\alpha_2, \alpha_1]$. For the boundary to be C^1-continuous, the direction should be the same on two sides. Finally, the tangents on two sides exist and do not coincide if the vectors (α_1, α_2) for I_1 and I_2 are linearly independent. $\square$

An interesting corollary of this theorem is that in the symmetric case it is necessary for p.w. C^1-continuity of the boundary that the images of I_1 and I_2 under the characteristic map are straight line segments. In this case we have that $\alpha_1\partial_1 f_1 + \alpha_2\partial_1 f_2 \equiv 0$ which means that $\alpha_1 f_1 + \alpha_2 f_2$ is constant and the image of $(f1, f2)$ is a straight line segment. Note that this is not necessary if the eigenvalues λ_a and λ_b are different.

4.2 Analysis of Characteristic Maps

To verify conditions of Theorem 4.1 we need to establish that the characteristic map is regular and injective, and verify that it has the expected behavior on the boundary. Typically, analysis of the boundary behavior is relatively easy, as in most cases the boundary curve is independent from the interior. In this section we focus on regularity and injectivity of the characteristic map.

Regularity of the characteristic map. Just as in the case of interior points we use self-similarity of the characteristic map to verify the regularity condition of Theorem 4.1: for any $t \in U_1$, the Jacobian satisfies $J[\Phi](t/2) = 4\lambda_a\lambda_b J[\Phi](t)$. It is immediately clear that to prove regularity of the characteristic map it is sufficient to consider the Jacobian on a single annular portion of U_1 as shown in Figure 3. As all vertices of such a ring are either regular or boundary regular, we can estimate the Jacobian of the characteristic map using tools developed for analysis of subdivision on regular grids. However, there is a significant difference from the case of interior vertices: to establish regularity on a single ring, in general, we have to consider subdivision schemes not just on regular meshes but on regular meshes with boundary, which makes the estimates for the Jacobians somewhat more complex.

Injectivity of the characteristic map. Even if the Jacobian of a map is nonzero everywhere, only local injectivity is guaranteed. However, for interior vertices, self-similarity of the characteristic maps allows one to reduce the injectivity test to computing the index of a closed curve around zero [25]. A closed curve with winding number ± 1 gives injectivity in a small neighborhood of zero. This is a relatively simple and fast operation: for example, the index can be computed counting the number of intersections of the curve with a line. This test cannot be applied for boundary points, as there are no closed curves around zero, since the boundary curve goes through zero.

For boundary points, a different simple test (Theorem 4.3) suffices, which in all cases that we have considered is even easier to apply. However, unlike the curve index test, it does not immediately yield a general computational algorithm.

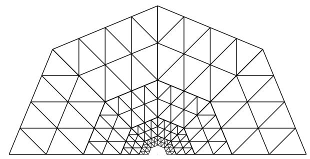


Figure 3: The k-gon without origin $U_1\backslash\{0\}$ can be decomposed into similar rings, each two times smaller than the previous ring. The size of the ring is chosen in such a way that the control set of any ring does not contain the extraordinary vertex. In this figure the control set is assumed to consist out of the vertices of the triangles of the ring itself, and of a single layer of vertices outside the ring.

The characteristic map can be extended using scaling relations to a complete k-regular complex with boundary. In the following theorem we assume that the characteristic map is defined on the whole complex $|\mathcal{R}_k^\alpha|$.

Theorem 4.3. *Suppose a characteristic map $\Phi = (f_a, f_c)$ satisfies the following conditions:*

1. *the preimage $\Phi^{-1}(0)$ contains only one element, 0;*
2. *the characteristic map has a Jacobian of constant sign at all points of the domain besides 0.*
3. *The image of the boundary of the characteristic map has no self-intersections;*
4. *the image of the characteristic map is not the whole plane.*

Then this characteristic map is injective.

Proof. We can show that the characteristic map is continuous at infinity, and if P is the stereographic projection of the sphere to the plane such that the south pole gets mapped to 0, $\tilde{\Phi} = P^{-1}\Phi P$ is a continuous mapping of a subset $D = P^{-1}(|\mathcal{R}_k^\alpha|)$ of the sphere into the sphere, with the south pole mapped to the south pole; $\tilde{\Phi}$ is a local homeomorphism away from the south pole.

We observe that the points of the boundary of the image $\tilde{\Phi}(D)$ can be images only of the boundary of D due to the properties of local homeomorphisms meaning $\partial\left(\tilde{\Phi}(D)\right) \subset \tilde{\Phi}(\partial D)$. Suppose the boundary of the image is not empty; we show that the image of the boundary curve $\tilde{\Phi}(\partial D)$ coincides with the boundary of the image $\partial\left(\tilde{\Phi}(D)\right)$.

The image of the boundary has no self intersections. It is easy to see that the boundary of the domain is a simple closed Jordan curve, and so is its image $\tilde{\Phi}(\partial D)$. Suppose $\partial\left(\tilde{\Phi}(D)\right) \neq \tilde{\Phi}(\partial D)$. Then there is a point y on the image of the boundary $\tilde{\Phi}(\partial D)$ which is an interior point of $\tilde{\Phi}(D)$. As $\tilde{\Phi}(\partial D)$ separates the sphere into two linearly connected domains, we can connect each point in either domain to point y with a continuous curve which does not intersect $\partial\left(\tilde{\Phi}(D)\right)$. Thus, any two points on the sphere can be connected by a continuous curve which does not intersect $\partial\tilde{\Phi}(D)$. We conclude that the image $\tilde{\Phi}(D)$ is the whole sphere. Therefore, either $\partial\left(\tilde{\Phi}(D)\right) = \tilde{\Phi}(\partial D)$, or the image is the whole sphere. The latter option contradicts the last condition of the theorem.

Now we need to use this to prove that the map is injective. If we exclude the south pole of the sphere, the mapping is a local homeomorphism of one simply connected domain to another. We can easily prove it is a covering: consider an interior point y of the image, and the set $\tilde{\Phi}^{-1}(y)$. Suppose it is infinite. Then it has a limit point, which cannot be an interior point of D (otherwise, $\tilde{\Phi}$ is not a local homeomorphism at that point). Similarly, it cannot be a boundary point, unless it is the south pole. It cannot be the south pole x_s for which $P(x_s) = 0$, because then $\tilde{\Phi}(x_s)$ has to be y which means that $y = 0$ which contradicts the assumption $\Phi^{-1}(0) = \{0\}$. We conclude that $\tilde{\Phi}^{-1}(y)$ is finite for each point y of the interior of the image. Similar arguments holds for boundary points away from the poles. $\tilde{\Phi}$ is a local homeomorphism and maps the boundary exactly to the boundary. Let y be a point of the image away from poles, and let $x_1, x_2, \ldots x_n$ be points of $\tilde{\Phi}^{-1}(y)$. Then for each x_i there is a sufficiently small neighborhood U_i which maps homeomorphically to a neighborhood of $x_i \in \tilde{\Phi}(D)$. Then the inverse image of $\cap_i \tilde{\Phi}(U_i)$ is a finite union of disjoint diffeomorphic subsets of D. But since y is in each of this sets it is only one set. We conclude that $\tilde{\Phi}$ is a covering on D with south pole excluded. However, we have observed that the image of D is simply connected. Therefore, the covering has to be injective. We conclude that the characteristic map is injective. □

5 Verification of C^1-continuity

5.1 Loop scheme

In this section we describe the structure of the boundary subdivision matrices for the Loop scheme. Some parts of our analysis are similar to the analysis performed by Jean Schweitzer [19].

The control mesh for a boundary patch surrounding an extraordinary vertex is shown in Figure 4. There are 3 different types of vertices in the control mesh, shown in the same figure. A different subdivision mask is used for each type. There are two masks for the vertices of types 1 and 3, one for boundary vertices and one for interior vertices. We consider these vertices to have the same type for notational convenience.

The figure also shows the masks of the rules that we consider. Our family of schemes includes all schemes satisfying the following conditions:

1. The support for each mask is the same as for the Loop scheme or for the cubic B-spline on the boundary;
2. The only masks that are modified are the masks for odd vertices adjacent to the central vertex, and for the central vertex itself (types 0,1).
3. The masks for interior edge vertices of type 1 are all identical and symmetric with respect to the edge connecting the vertex with the central vertex. The masks for two boundary vertices of type 1 are also identical.

We assume that all coefficients in the masks are positive. This choice is sufficiently general to construct a variety of schemes; on the other hand, complete eigenanalysis can be performed for all schemes from this family. We show that no scheme from this family can produce a rule for a concave corner. There are reasons to believe that this is true for any scheme with positive coefficients or small support.

For the specific schemes that we consider the boundaries do not depend on the control points in the interior. Potentially, the boundary can depend on the valence of a boundary vertex, this is the case with the scheme presented in [8]. However, we believe that this is best avoided, and present a set of schemes for which the boundary rules are simply cubic spline rules, except at vertices marked as corners, where interpolation is forced.

Subdivision matrix. We assume that $k > 1$; we will consider the case $k = 1$ separately. The subdivision matrix for a boundary vertex with k adjacent triangles has the following form:

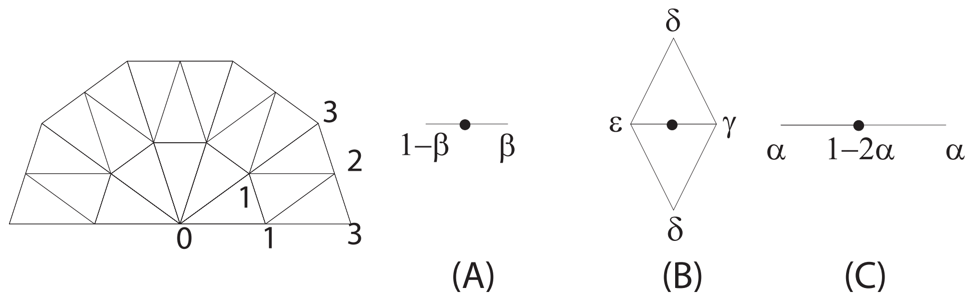


Figure 4: Control mesh for a boundary patch of a Loop subdivision surface and masks of the subdivision rules. (A) The rule for the odd vertices on the boundary adjacent to the central vertex (type 1). (B) The rule for the interior odd vertices adjacent to the central vertex (type 1). (C) The rule for the central vertex (type 0). The rules for vertices of type 2 and 3 (interior) are the standard Loop rules; the rule for the vertex of type 3 (boundary) is the standard one-dimensional cubic spline rule.

$$\left(\begin{array}{c|cc|c|c|cc|c}
1-2\alpha & \alpha & \alpha & & & & & \\
\hline
1-\beta & \beta & & & & & & \\
1-\beta & & \beta & & & & & \\
\hline
a_1 & \multicolumn{2}{c|}{A_{10}} & A_{11} & & & & \\
\hline
a_2 & \multicolumn{2}{c|}{A_{20}} & A_{21} & \frac{1}{8}I_k & & & \\
\hline
1/8 & 3/4 & & & & 1/8 & & \\
1/8 & & 3/4 & & & & 1/8 & \\
\hline
a_3 & & & A_{31} & A_{32} & & & \frac{1}{16}I_{k-1}
\end{array}\right) \tag{5.1}$$

The vectors a_1 and a_3 have length $k-1$, the vector a_2 has length k, I_k and I_{k-1} are unit matrices of sizes k and $k-1$. Note that the eigenvalues of the matrix are 1/8 1/16, the eigenvalues of the upper-left 3×3 block A_{00} and the eigenvalues of the matrix A_{11}. The matrix A_{11} is tridiagonal, of size $k-1 \times k-1$. The eigenvalues of A_{00} are $1, \beta, \beta - 2\alpha$ where the eigenvector to 1 is the vector $\mathbf{e} = [1, \ldots, 1]$. Following [19], we observe that $k-1 \times k-1$ tridiagonal symmetric matrices have the following eigenvectors, independent of the matrix, $j = 1 \ldots k-1$:

$$v^j = [\sin j\theta_k, \sin 2j\theta_k, \ldots \sin{(k-1)j\theta_k}] \tag{5.2}$$

where $\theta_k = \pi/k$. Multiplying the matrix A_{11} by the vectors, we see that the eigenvalues are $\lambda_j = 2\delta \cos j\theta_k + \gamma$.

If $\alpha \neq 0$, out of two remaining eigenvectors, only the eigenvector v^β corresponding to β is typically of interest to us. It has the form $[0, 8C, -8C, \left(\beta I - A_{11}^{-1}\right)[C, 0 \ldots -C]]$, where C is a constant, if $\beta I - A_{11}$ is non-degenerate.

A more revealing expression for the components can be found if we regard the eigenvector as a solution to the recurrence

$$\delta\left(v_{i-1}^\beta + v_{i+1}^\beta\right) + (\gamma - \beta)v_i^\beta = 0, \quad i = 1 \ldots k-1$$

(the numbering of entries in v_β is such that $v^\beta = [0, v_0^\beta, v_k^\beta, v_1^\beta, ..., v_{k-1}^\beta]$ to make the equations uniform equations). In addition, we have the condition $v_0^\beta = -v_k^\beta$, to ensure that $\left[0, v_\beta^0, v_\beta^1\right]$ is the eigenvector of A_{00}.

The behavior of the solution of the recurrence depends on the ratio $r = (\gamma - \beta)/\delta$, assuming $\delta \neq 0$ (otherwise, A_{11} is diagonal with all eigenvalues equal to γ and the eigenvector with respect to β is found easily.). The additional condition $v_0^\beta = -v_k^\beta$ determines a unique solution up to a constant multiplier, even if the matrix $\beta I - A_{11}$ is degenerate.

If $\alpha = 0$, the eigenvalue β has a two-dimensional eigenspace. Two eigenvectors v^β and v'^β satisfying conditions $v_0^\beta = 0$ and $v'^\beta_k = 0$ can be computed explicitly, for the cases when the matrix $\beta I - A_{11}$ is not degenerate, i.e. when for all $1 \leq j \leq k-1$, $r \neq -2\cos j\theta_k$.

Finally, suppose $\alpha = 0$ and $r = -2\cos(j\theta_k)$ for some j. In this case $\beta = \gamma - \delta r$ is also an eigenvalue of A_{11}, and, therefore, has multiplicity 3. In this case it has a Jordan block of size 2, and only 2 eigenvectors which can be taken to be $v_i^\beta = \sin i\theta_k$ and $v'^\beta_i = \cos i\theta_k$, $i = 0 \ldots k$.

Summary of the eigenstructure. We have determined that the eigenvalues of the subdivision matrix are 1,β,$\beta - 2\alpha$, $1/8$, $1/16$, and $\lambda_j = 2\delta\cos j\theta_k + \gamma$, $j = 1 \ldots k-1$. The eigenvectors corresponding to the eigenvalues λ_j do not depend on the matrix and are given by (5.2). The eigenvectors corresponding to the eigenvalue β depends on the ratio $r = (\gamma - \beta)/\delta$; for $\alpha \neq 0$, there is a single eigenvector. For $\alpha = 0$, there is a pair of eigenvectors for the case when β is not an eigenvalue of A_{11}. If β is an eigenvalue of A_{11}, it has a nontrivial Jordan block of size 2.

The case $k = 1$. The matrix in this case has eigenvalues β, $\beta - 2\alpha$, and a triple eigenvalue $1/8$. The eigenvectors can be trivially computed.

Coefficients for smooth boundary vertices. One possible choice was given by Hoppe et al. [8] and examined in detail in [19]. In our notation, this choice corresponds to $\beta = 5/8$, $\alpha = 1/8$, $\gamma = 3/8$, $\delta = 1/8$. For extraordinary vertices, and $\beta = 1/2$ for other vertices. Remarkably, the ratio r is -2. The disadvantage of this choice is that the shape of the boundary curve depends on the valence of the vertices on the boundary, hence it becomes impossible to join two meshes continuously along a boundary if extraordinary vertices on two sides do not match.

If we require the boundary curve to be a cubic spline, β has to be $1/2$ and α has to be $1/8$. We have two degrees of freedom left: γ and δ. It turns out to be sufficient to use only one, and we fix δ at the value corresponding to the regular valence, i.e. $1/8$.

We consider the cases $k > 2$, $k = 2$ and $k = 1$ separately.

Case $k > 2$. Once α, β and δ are fixed, the eigenvalues of the subdivision matrix become 1, $\beta = 1/2$, $\beta - 2\alpha = 1/4$, $1/8$, $1/16$, and $\lambda_j = (1/4)\cos j\theta_k + \gamma$.

The tangent vector on the boundary of the surface corresponds to the eigenvector of the subdivision matrix with eigenvalue $\beta = 1/2$. This vector should be one of the subdominant eigenvectors. The second subdominant eigenvector is likely to correspond to the largest of the eigenvalues λ_j, i.e. to the eigenvalue $\lambda_1 = \gamma + (1/4)\cos\theta_k$. In order for the eigenvalue $1/2$ to be subdominant, we choose γ in such a way that $|\lambda_j| < 1/2$ for $j > 1$, i.e. $\lambda_2 < 1/2$ and $\lambda_{k-1} > -1/2$. For positive γ, the second condition is satisfied automatically. We also would like $\lambda_1 > \beta - 2\alpha = 1/4$. This leads to the following range for γ:

$$\frac{1}{4}(1 - \cos\theta_k) < \gamma < \frac{1}{2} - \frac{1}{4}\cos 2\theta_k \tag{5.3}$$

In this range we also have $|\lambda_1| > |\lambda_j|$ for $j > 1$. There are two choices of γ that we find particularly interesting: $\gamma = 1/4$ and $\gamma = 1/2 - 1/4\cos\theta_k$.

The first choice, $\gamma = 1/4$, is the maximal value of γ independent of k for which it is in the correct range for all $k > 2$. Note that in this case $r = -2$ again. The second choice, leads to equal subdominant eigenvalues $\beta = \lambda_1 = 1/2$. In this case, $r = -2\cos\theta_k$. The expressions for the subdominant eigenvectors are $v_j^1 = \sin j\theta_k$ and $v_j^\beta = \cos j\theta_k$, i.e. form a half of a regular $2k$-gon.

The choice of $\gamma = 1/2 - 1/4\cos\theta_k$, although being slightly more complex, appears to be more natural. It has the additional advantage of coinciding with the regular value $\gamma = 3/8$ for $k = 3$.

Case $k = 2$. In this case, the eigenvalues are 1, $1/2$, $1/4$, $1/8$, $1/16$, and $\lambda_1 = \gamma$. Thus, we need to pick $1 > \gamma > 1/4$, to get the same eigenvectors as in the case $k > 2$. It is interesting to note however, that the choice of $\gamma = 1/4$ also results in a C^1 surface, although the behavior of the scheme becomes less desirable.

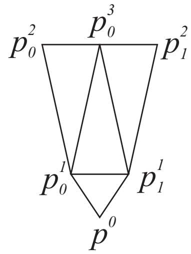
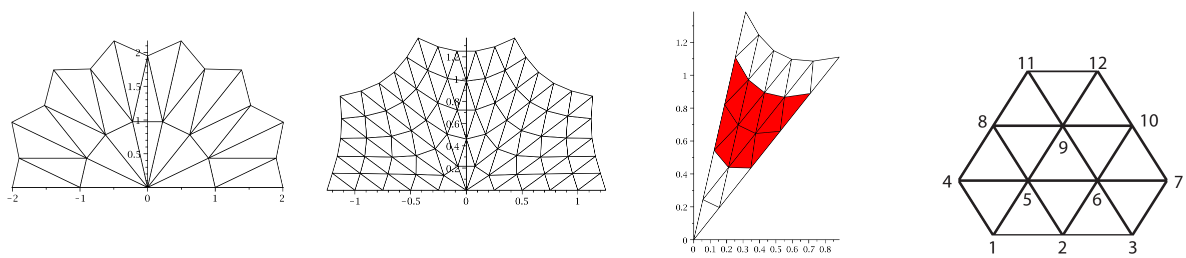


Figure 5: The control mesh for the characteristic map in the case $k = 1$.



(a) (b) (c) (d)

Figure 6: Fig (a) show the 2 ring control data for a mesh with 5 sectors which we subdivide twice. We get 5 rings show in Fig (b) for a mesh with 5 sectors. In Fig (c) we see one sector in which the relevant triangles are marked and Fig (d) shows a control net for a given triangle.

Case $k = 1$. The subdominant eigenvalues are 1/2 and 1/4. They define a configuration of eigenvectors shown in Figure 5.

Proposition 5.1. *Let $\beta = 1/2, \alpha = 1/8, \delta = 1/8$ and $\gamma = 1/2 - 1/4\cos\theta_k$ and Φ be the characteristic map which is defined by the eigenvectors to $\beta = 1/2$ and $\lambda_1 = 1/4\cos(\theta_k) + \gamma = 1/2$. Then*

1. *the preimage $\Phi^{-1}(0)$ contains only one element, 0;*
2. *the characteristic map has a Jacobian of constant sign at all points of the domain besides 0;*
3. *the image of the boundary of the characteristic map has no self-intersections;*
4. *the image of the characteristic map is not the whole plane.*

Proof. We consider the boundary k-regular 2 ring mesh with data given by the 2 eigenvectors described above (shown for $k = 7$ in Figure 6(a)). We subdivide this twice by our given rules. We then have 5 accurate rings of a k-regular mesh (Figure 6(b)). In the standard Loop scheme if a triangle is sourrounded by one ring of triangles and all subdivision at all those 12 points and points inserted on these edges and faces going forward are done by regular Loop subdivision the polynomial on the triangles in (u, v, w) Bezier coordinates $u + v + w = 1$ is given by (see [19]):

$$p(u, v, w) = B \cdot Q \cdot P$$

where

$$B = (u^4, 4u^3v, 4u^3w, 6u^2v^2, 12u^2vw, 6u^2w^2, 4uv^3, 12uv^2w, 12uvw^2, 4uw^3, v^4, 4v^3w, 6v^2w^2, 4vw^3, w^4)$$

and Q is a 15×12 matrix given in [19] and $P \in \mathbf{R}^{12\times n}$ such that $P_i \in \mathbf{R}^n$ is the data on the point i numbered as shown in Figure 6(d). The 12 triangles in the 3rd and 4th ring of our 5 ring mesh (see Figure 6(b) for $k = 5$) which are away from the boundary are regular. We can therefore compute the 12 different polynomials highlighted in Figure 6(c). We are able to compute the polynomials depending on the number of sectors $k > 3$ and which sector $i = 3, \dots k-2$. We have to treat the case $k \leq 3$ and the case where $i = 1, 2, k-1, k$ separately. The triangles on the boundary are not surrounded by a control net of regular vertices but since the boundary rules are regular cubic B-spline rules subdividing with boundary rules is equivalent to subdividing with regular rules a mesh that is extended by a mirror image over the boundary. The sectors 2 and $k-1$ have to be considered separately only in creating the 5 rings as they are more influenced by the extraordinary boundary rules then the other sectors. If $k \leq 3$ we check all the triangles directly. For any k however the eigenvector data has a symmetry across the y-axes and therefore the characteristic map has the same symmetry. For each different type we compute the polynomial $p = (f_1, f_2)$ on the triangle in Berstein-Bezier coordinates.

We will prove 1.-4. for each of those polynomials and by the scaling property

$$\Phi(t/2) = T\Phi(t) \text{ where } T = \begin{pmatrix} \beta & 0 \\ 0 & \lambda_1 \end{pmatrix} = \frac{1}{2}I \tag{5.4}$$

we can then extend it to the whole sector. To prove that a polynomial in Bernstein-Bezier coordinates is positive on the given triangle we need to check that all the coefficients are positive.

1. In order to prove that there is no other element than 0 in the preimage $\Phi^{-1}(0)$ we check that $f_1^2 + f_2^2 > 0$ in each triangle of each sector. Then by the scaling property we know that

   $$f_1(t/2)^2 + f_2(t/2)^2 = \lambda_a^2 f_1(t)^2 + \lambda_b^2 f_2(t)^2 = (\lambda_a^2 - \lambda_b^2) f_1(t)^2 + \lambda_b (f_1(t)^2 + f_2(t)^2) > 0.$$

   Since $\|\Phi(t)\| > 0$ for all $t > 0$ we proved the first statement.

2. We compute the Jacobian

   $$J[\Phi] = \partial_x f_1 \partial_y f_2 + \partial_x f_2 \partial_y f_1 = (\partial_u f_1 - \partial_w f_1)(\partial_v f_2 - \partial_w f_2) + (\partial_u f_2 - \partial_w f_2)(\partial_v f_1 - \partial_w f_1)$$

   in each triangle and see that the coefficients of J (a polynomial in Bezier coordinates) are all of the same sign independent of k and i. Therefore the polynomial has the same sign everywhere. By the scaling property we can extend it from the ring to the sector. The scaling property for the Jacobian is

   $$J[\Phi](t/2) = 4\beta\lambda_1 J[\Phi](t) = J[\Phi](t)$$

3. We take the 2 triangles in the third ring that form the boundary to the second ring and find the expression of the polynomial that describes the boundary curve. We want to show that the angle grows monotonically and since the angle is given by $\arctan(f_1/f_2)$ it is enough to show that f_1/f_2 grows monotonically. We compute $f_1' f_2 - f_2' f_1$, the enumerater of the derivative of f_1/f_2 and see that all coefficients have the same sign. Since the denominator is a square, it is also positive. This means that f_1/f_2 is monoton, and therefore the angle is monoton. Therefore in each sector the curves can not intersect. There can not be intersection between sectors as the curves limit lies strictly within their sectors.

4. Box Spline surfaces lie strictly within the convex hull of their control net and therefore the image of the characteristic map has to lie in the upper half plane.

All the explicit checks were done in Maple. □

We can now conclude by Theorem 4.3 that the characteristic map is injective. It is also regular as the Jacobian of the characteristic map has constant sign everywhere. This means that in order for the scheme to be C^1 smooth with smooth boundary we have to check the 4th condition of Theorem 4.2, since the subdominant eigenvalues are equal and span a 2-dimensional eigenspace. Since the boundary curve is a B-Spline interpolating points on the x-axes we get that $\partial_1 f_1 > 0$ and $\partial_1 f_2 = 0$, giving us the condition for a scheme that is C^1 smooth with smooth boundary.

Lets now consider the corner case.

Coefficients for corner vertices. Separate rules have to be defined for corners. The interpolation conditions for corners require $\alpha = 0$. Therefore, the block A_{00} has a double eigenvalue β. For a corner, the tangent plane is defined by the two tangents at the non-C^1-continuous point of the boundary. Unlike the case of the smooth boundary points, there is no need to fix all rules on the boundary – parameter β still can be used to ensure smoothness of the limit surface. Hence the rules of Hoppe et al. [8] can be used. One can see [19] that the characteristic map has a convex corner. Therefore, this scheme cannot produce concave corners. *It turns out that in fact no scheme from the class that we have defined can produce smooth concave corners.*

The explicit knowledge of eigenvectors and the convex hull property allows us to determine quickly if a scheme can possibly produce convex or concave corner. If β has multiplicity 3 with Jordan blocks of size 2 and 1 which happens when it is an eigenvalue of A_{11}, the scheme is likely to be non tangent plane continuous; we assume that this is not the case. Then the eigenvectors of interest can be found explicitly for various values of $r = (\gamma - \beta)/\delta$.

It is easy to see that positive values of r are of little interest to us, because the components of the vectors alternate signs in these cases, and are likely to produce non-regular characteristic maps. Also, for $r \leq -2$ we are guaranteed to get a convex configuration of control points for the characteristic map. As the characteristic map interpolates the boundary curve, it cannot have a concave corner. We conclude that we have to use r from the range $(-2, 0)$. We have seen that in this case the eigenvectors corresponding to the eigenvalue β can be taken to be $\sin i\theta$, $\sin(i-k)\theta$, where θ is such that $r = -2\cos\theta$. *This means that the corner is convex if $\theta < \theta_k$, and concave otherwise.* In other words, $r = -2\cos\theta < -2\cos\theta_k$, or

$$\gamma < \beta - 2\delta\cos\theta_k \tag{5.5}$$

In addition, we need to ensure that the double eigenvalue β is actually subdominant. To achieve this, we choose δ and γ large enough so that $2\delta\cos j\theta_k + \gamma < \beta$, $j = 1\ldots k-1$. As $2\delta\cos j\theta_k + \gamma$ decreases as a function of j, and we assume that $\gamma > 0$, it is sufficient to require that $2\delta\cos\theta_k + \gamma < \beta$, which coincides with the convexity condition. We conclude that for $r < 0$ the subdivision scheme can generate only convex smooth corners.

One can show that this is true even if we do not assume that $\alpha = 0$.

In the case $k = 1$, one can also immediately see that the corner produced by subdivision is convex.

Concave corner vertices. We assume that $k > 1$. It is impossible to have stationary subdivision rules for a triangular mesh producing a concave corner for $k = 1$. As we have observed, concave corners cannot be produced simply by changing some of the coefficients using the same stencil. One can also show that no scheme with positive coefficients can produce interpolating smooth concave corners. It is possible to construct rules to produce C^1-continuous surfaces with concave corners, but negative coefficients and larger support have to be used.

Our approach to deriving the rules is based on the idea of reduction of the magnitudes of all eigenvalues, excluding 1 and $\beta = 1/2$. It turns out that this approach leads to particularly simple rules for subdivision.

For the scheme to produce smooth surfaces at a corner vertex the eigenvectors x^β, x'^β of the eigenvalue $\beta = 1/2$ should be subdominant. If we choose these eigenvectors to be $x^\beta = [0, 0, 1, v_1^\beta/\sin k\theta, \ldots]$, $x'^\beta = [0, 1, 0, v'^\beta_1/\sin k\theta \ldots]$, corresponding left eigenvectors are very simple: $l = [-1, 0, 1, 0, \ldots]$, $l' = [-1, 1, 0, 0, \ldots 0]$. The left eigenvector l^0 for the eigenvalue 1 is $[1, 0, \ldots 0]$. Consider the following modification of the vector of control points

$$\tilde{p} = (1-s)p + s\left((l^0, p)x^0 + (l, p)x^\beta + (l', p)x'^\beta\right)$$

where x_0 is the eigenvector $[1, \ldots 1]$ of the eigenvalue 1. Substituting expressions for the left eigenvectors we get

$$\tilde{p} = (1-s)p + s\left(p^0 x^0 + \left(p_0^1 - p^0\right)x^\beta + \left(p_k^1 - p^0\right)x'^\beta\right). \tag{5.6}$$

The effect of this transformation is to scale all components of p in the eigenbasis of the subdivision matrix by $(1-s)$ except those corresponding to the eigenvalues 1 and β. If repeated at each subdivision step, it is equivalent to scaling all eigenvalues except 1 and β by $(1-s)$.

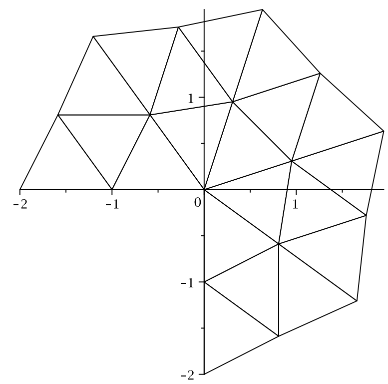


Figure 7: Control mesh for a boundary patch of a Loop subdivision surface with concave corner

To simplify the rules, we observe that it is unnecessary to scale multiple eigenvalues 1/16 and 1/8 of the lower-right blocks of the subdivision matrix. If we apply the rules (5.6), not to the whole vectors of control points p, but to a truncated part, modifying only control points of type 1, as a result, the eigenvalues 1/8 and 1/16 will not change. This observation leads us to the following choice of rules:

$$\tilde{p}_i^1 = (1-s)p_i^b + s\left(p^0 + \left(p_0^1 - p^0\right)\frac{\sin{(k-i)\theta}}{\sin k\theta} + \left(p_k^1 - p^0\right)\frac{\sin i\theta}{\sin k\theta}\right) \tag{5.7}$$

In the matrix form, this transformation can be written as

$$T = \left(\begin{array}{c|c} M & 0 \\ \hline 0 & I \end{array}\right)$$

Multiplying this matrix by the subdivision matrix on the left, we see that the eigenvalues of the product ST are eigenvalues of the blocks $B_{00}M$ and B_{11}. By construction. eigenvalues of $B_{00}M$ are 1, 1/2, $(1-s)\,(2\delta\cos j\theta_k + \gamma)$, $j = 1\ldots k-1$. As we have seen before, the eigenvalues of B_{11} are 1/8 and 1/16.

By choosing the value of s so that $(1-s)\,(2\delta\cos\theta_k + \gamma) < 1/2$, we can ensure that the $\beta = 1/2$ is the subdominant eigenvalue. The parameter s can be viewed as a tension parameter for the corner, which determines how flat the surface is near the corner.

We can therefore consider the case of convex and concave corners together:

Proposition 5.2. *Let $\beta = 1/2, \alpha = 0, \delta = 1/8$ and $\gamma = 1/2 - 1/4\cos(\theta)$ where $0 < \theta < \pi$ for convex corners and $\pi < \theta < 2\pi$ for concave corners. Then Φ, the characteristic map is defined by the eigenvectors corresponding to $\beta = 1/2$. Then*

1. *the preimage $\Phi^{-1}(0)$ contains only one element, 0;*
2. *the characteristic map has a Jacobian of constant sign at all points of the domain besides 0;*
3. *the image of the boundary of the characteristic map has no self-intersections;*
4. *the image of the characteristic map is not the whole plane.*

Proof. The proof is done exactly the same way as in the non-corner case. The characteristic map we need to check has a parameter θ. In the case of the concave corner the convex hull of the control points (see Figure 7) no longer lies in the upper half plane. However we can look at the sectors individually and see that the limit function does not span the whole complex plane. □

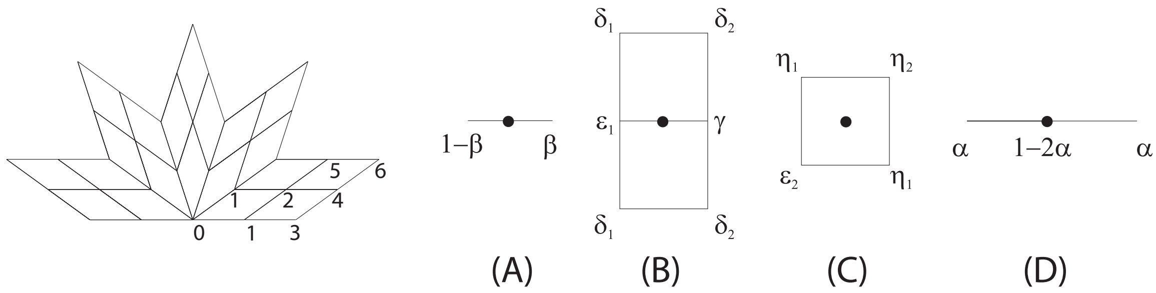


Figure 8: Control mesh for a boundary patch of a Catmull-Clark subdivision surface and masks of the subdivision rules. (A) The rule for the boundary vertices adjacent to the central vertex (type 1). (B) The rule for the interior edge vertices adjacent to the central vertex (type 1). (C) The rule for the face vertices adjacent to the central vertex (type 2). (D) The rule for the central vertex (type 0). The rules for vertices of type 4, 5 and 6 are the standard Catmull-Clark rules; the rule for the vertex of type 3 is the standard one-dimensional cubic spline rule.

With this we have established that the characteristic map is injective and regular. Now we need to check condition 4 in Theorem 4.2. Since the boundary of the control mesh away form 0 is a straight line for $k > 1$ the limit curve which is a B-spline is also a straight line. This means it satisfies the condition.

5.2 Catmull-Clark scheme

The analysis of the eigenstructure of the boundary subdivision matrices becomes more complex in the case of the Catmull-Clark scheme. Using the Catmull-Clark scheme as an example, we describe a technique that can be used to analyze schemes with larger support.

The control mesh for a boundary patch surrounding an extraordinary vertex is shown in Figure 8.

There are 6 different types of vertices in the control mesh, shown in the same figure. For two types (1 and 3) there are two different masks that are used for boundary and interior vertices respectively. As we did in the case of the Loop scheme, we introduce a number of undefined coefficients into the masks and find eigenvalues and eigenvectors of the subdivision matrix as functions of coefficients. The choice of the parameters is guided by the same considerations as for the Loop scheme.

Various types of boundary behavior (smooth convex corner, smooth boundary) can be obtained by choosing appropriate values of the parameters. Again, we can show that no scheme from this class can generate surfaces with smooth concave corners.

Subdivision matrix. The subdivision matrix has somewhat more complex structure for the Catmull-Clark scheme.

In the block form, the matrix can be written as

$$\left(\begin{array}{c|c|c|c} A_{00} & & & \\ \hline A_{10} & \frac{1}{8}I_2 & & \\ \hline A_{20} & A_{21} & A_{22} & \\ \hline A_{30} & A_{31} & A_{32} & \frac{1}{64}I_k \end{array}\right)$$

where the diagonal blocks are

$$
A_{00} = \left(\begin{array}{c|cc|cccc|cccc}
1-2\alpha & \alpha & \alpha & & & & & & & & \\
1-\beta & \beta & & & & & & & & & \\
1-\beta & & \beta & & & & & & & & \\
\hline
\epsilon_1 & \delta_1 & & \gamma & \delta_1 & & & \delta_2 & \delta_2 & & \\
\epsilon_1 & & & \delta_1 & \gamma & \delta_1 & & & \delta_2 & \delta_2 & \\
 & & & & \ddots & \ddots & \ddots & & & \ddots & \ddots \\
\epsilon_1 & & \delta_1 & & & \delta_1 & \gamma & & & \delta_2 & \delta_2 \\
\hline
\epsilon_2 & \eta_2 & & \eta_2 & & & & \eta_1 & & & \\
\epsilon_2 & & & \eta_2 & \eta_2 & & & & \eta_1 & & \\
\vdots & & & & \ddots & \ddots & & & & \ddots & \\
\epsilon_2 & & \eta_2 & & & & \eta_2 & & & & \eta_1
\end{array}\right)
\qquad
A_{22} = \left(\begin{array}{cccc|cccc|ccccc}
\frac{3}{32} & & & & 0 & \frac{1}{64} & & & \frac{1}{64} & & & & \\
 & \frac{3}{32} & & & & & \frac{1}{64} & & & \frac{1}{64} & & & \\
 & & \ddots & & & & & \ddots & & & \ddots & & \\
 & & & \frac{3}{32} & & & & \frac{1}{64} & & & & \frac{1}{64} & 0 \\
\hline
0 & & & & \frac{1}{16} & & & & & & & & \\
\frac{1}{16} & & & & & \frac{1}{16} & & & & & & & \\
 & \frac{1}{16} & & & & & \ddots & & & & & & \\
 & & \ddots & & & & & \ddots & & & & & \\
 & & & \frac{1}{16} & & & & \frac{1}{16} & & & & & \\
\hline
\frac{1}{16} & & & & & & & & \frac{1}{16} & & & & \\
 & \frac{1}{16} & & & & & & & & \frac{1}{16} & & & \\
 & & \ddots & & & & & & & & \ddots & & \\
 & & & \frac{1}{16} & & & & & & & & \ddots & \\
 & & & 0 & & & & & & & & & \frac{1}{16}
\end{array}\right)
\tag{5.8}
$$

Note that all eigenvalues of A_{22} are guaranteed to be less than 1/8 (the sum of the magnitudes of the entries on any line does not exceed 1/8). Thus, only the eigenvalues of A_{00} are of interest to us. Next, we observe that the matrix A_{00} itself has two blocks on the diagonal; the first 3×3 block is identical to the block that we have considered for the Loop scheme; it has eigenvalues 1, β and $\beta - 2\alpha$. The remaining block denoted by $\bar{A}_{00}$ is the one we need to consider.

The matrix $\bar{A}_{00}$ acts on control points of types 1 and 2, excluding boundary control points of type 1.

Transformation of the subdivision matrix. Assume $k > 1$ (we will consider the case $k = 1$ separately). The eigenvalues and eigenvectors of $\bar{A}_{00}$ can be found directly from the recurrences derived from the subdivision rules. We take a somewhat different approach, similar to the DFT analysis used for interior extraordinary vertices. This approach has somewhat greater generality and can potentially be applied to analyze subdivision schemes with larger supports. To find the eigenvalues of $\bar{A}_{00}$, we introduce a new set of control points. We replace control points of type 2, p_i^2, $i = 0 \dots k-1$, with $k+1$ control points $\tilde{p}_i^2$ satisfying

$$
p_i^2 = \frac{1}{2}\left(\tilde{p}_i^2 + \tilde{p}_{i+1}^2\right) \tag{5.9}
$$

for $i = 0 \dots k-1$. Also, let $\tilde{p}_i^1 = p_i^1$ for control points of type 1. Note that we increase the number of control points. These equations clearly do not define the new control points uniquely. However, it is not relevant for our purposes. In matrix form, the relation between the original vector of control points of types 1 and 2 and the transformed vector $\tilde{p}$ can be written as $p = T\tilde{p}$, where T is a $2k+1 \times 2k+2$ matrix.

In addition, we define the subdivision rules for the new control points. We choose the rules for $\tilde{p}$ in such a way that the relations (5.9) also hold after the subdivision rules are applied to p and $\tilde{p}$. Let $\tilde{S}$ be the subdivision matrix for $\tilde{p}$. Then our choice of rules means that

$$
ST\tilde{p} = T\tilde{S}\tilde{p}.
$$

If λ is an eigenvalue of $\tilde{S}$, then $\tilde{S}\tilde{p}^\lambda = \lambda\tilde{p}^\lambda$ where $\tilde{p}^\lambda$ is the corresponding eigenvector, and

$$
ST\tilde{p}^\lambda = T\tilde{S}\tilde{p}^\lambda = \lambda T\tilde{p}^\lambda.
$$

Therefore, λ is also an eigenvalue of S, unless $T\tilde{p}^\lambda = 0$. Note that the nullspace of T has dimension 1 and contains the vector $p_i^1 = 0$, $\tilde{p}_i^2 = (-1)^i$. Hence a complete set of eigenvalues and eigenvectors of S can be obtained from eigenvalues and eigenvectors of $\tilde{S}$ once we exclude the eigenvalue corresponding to this vector, if it happens to be an eigenvector.

We choose the subdivision rule for $\tilde{p}_i^2$ as follows:

$$[\tilde{S}\tilde{p}]_i^2 = \epsilon_2 p^0 + 2\eta_2 p_i^1 + \eta_1 \tilde{p}_i^2 \tag{5.10}$$

In terms of new control points, the rule for control points of type 1 becomes

$$[Sp]_i^1 = \epsilon_1 p^0 + \delta_1 \left(p_{i-1}^1 + p_{i+1}^1\right) + \gamma p_i^1 + \frac{\delta_2}{2}\left(\tilde{p}_{i-1}^2 + 2\tilde{p}_i^2 + \tilde{p}_{i+1}^2\right)$$

The matrix $\bar{A}_{00}$ is transformed into

$$\tilde{A}_{00} = \left(\begin{array}{cccc|cccccc}
\gamma & \delta_1 & & & \frac{\delta_2}{2} & \delta_2 & \frac{\delta_2}{2} & & & \\
\delta_1 & \gamma & \delta_1 & & & \frac{\delta_2}{2} & \delta_2 & \frac{\delta_2}{2} & & \\
 & \cdot & \cdot & \cdot & & & \cdot & \cdot & \cdot & \\
 & & \delta_1 & \gamma & & & & \frac{\delta_2}{2} & \delta_2 & \frac{\delta_2}{2} \\
\hline
 & & & & \eta_1 & & & & & \\
2\eta_2 & & & & & \eta_1 & & & & \\
 & 2\eta_2 & & & & & \eta_1 & & & \\
 & & \cdot & & & & & \eta_1 & & \\
 & & & 2\eta_2 & & & & & \eta_1 & \\
 & & & & & & & & & \eta_1
\end{array}\right) \tag{5.11}$$

Note that $\tilde{p}_0^2$ and $\tilde{p}_k^1$ depend on p_0^1 and p_k^1 which are outside this matrix.

This matrix has a double eigenvalue η_1. The rest of the eigenvalues are eigenvalues of the matrix A consisting only of the 4 tridiagonal subblocks. We have already observed that three diagonal matrices have eigenvectors independent from the entries of the matrix. Denote H the matrix with entries $\sin ij\theta_k$, with $\theta_k = \pi/k$ as before, $i, j = 1 \dots k-1$. This matrix to some extent has the same role in the analysis of subdivision matrices of boundary vertices as the DFT matrix has in the analysis of subdivision matrices of interior vertices. The transform $\mathcal{H}$ is defined as $\operatorname{diag}(H, H)$. The inverse of this matrix is $\mathcal{H}^{-1} = \operatorname{diag}((2/k)H, (2/k)H)$.

$$\mathcal{H}A\mathcal{H}^{-1} = \frac{2}{k}\begin{pmatrix} H & 0 \\ 0 & H \end{pmatrix}\begin{pmatrix} B_{00} & B_{01} \\ B_{10} & B_{11} \end{pmatrix}\begin{pmatrix} H & 0 \\ 0 & H \end{pmatrix} = \frac{2}{k}\begin{pmatrix} HB_{00}H & HB_{01}H \\ HB_{10}H & HB_{11}H \end{pmatrix} \tag{5.12}$$

each block $HB_{ij}H$ is a diagonal matrix. Finally, we apply the following permutation to the components of the vector: $\left[p_1^1, p_2^1, \dots p_{k-1}^1, \tilde{p}_1^2, \tilde{p}_2^2, \dots \tilde{p}_{k-1}^2\right] \rightarrow \left[p_1^1, \tilde{p}_1^2, p_2^1, \tilde{p}_2^2 \dots p_{k-1}^1, \tilde{p}_{k-1}^2\right]$. Let P be the corresponding permutation matrix. The matrix A is reduced to the block diagonal form

$$P\mathcal{H}A\mathcal{H}^{-1}P^{-1} = \begin{pmatrix} B(1) & & & \\ & B(2) & & \\ & & \cdot & \\ & & & B(k-1) \end{pmatrix} \tag{5.13}$$

where the blocks $B(i)$, $i = 1 \dots k-1$, are 2×2 matrices

$$B(i) = \begin{pmatrix} \gamma + 2\delta_1 \cos \frac{i\pi}{k} & \delta_2\left(1 + \cos\frac{i\pi}{k}\right) \\ 2\eta_2 & \eta_1 \end{pmatrix} \tag{5.14}$$

The explicit expressions for the eigenvalues are not particularly enlightening in the general case and we omit them here.

Case $k = 1$. In this case, the eigenvalues and eigenvectors can be computed directly. The eigenvalues are β, $\beta - 2\alpha$, η_2, 1/8, 1/16 and 1/64.

Eigenvectors. We start with eigenvectors of the matrix A_{00}. We assume that $\eta_1 \neq 0$ and $\delta_1 \neq 0$ and none of the eigenvalues of the blocks $B(i)$ coincide with η_1. In this case, the eigenvectors corresponding to each block $B(i)$ can be taken to be $[0, \dots 0, 1, r, 0, \dots 0]$, where the only two nonzero entries are in positions $2i-1$ and $2i$, $r = -2\eta_2/(\eta_1 - \lambda)$, and λ is the eigenvalue. Applying the inverse permutation and transform H, we get eigenvectors of the form $[v^i, rv^i]$, with v^i being a vector of length $k-1$ with entries $\sin(j\theta_k)$,

$j = 1 \ldots k-1$. The entries of the eigenvector of A_{00} corresponding to $\tilde{p}_0^2$ and $\tilde{p}_k^2$ are zero. The remaining possible eigenvalues of A_{00} are 1, β, $\beta - 2\alpha$ and η_1. Once the eigenvalue is known, the expressions for the eigenvectors can be found directly from the subdivision rules. Keeping in mind that for all eigenvectors except the eigenvector of the eigenvalue 1 and $\beta - 2\alpha$ for $\alpha \neq 0$ we have $p^0 = 0$. An interior control point of type 1, p_i^1 and a control point of type 2 p_i^2 from an eigenvector p with eigenvalue λ should satisfy

$$
\begin{aligned}
&\lambda p_i^1 = \delta_1 \left(p_{i-1}^1 + p_{i+1}^1\right) + \delta_2 \left(p_i^2 + p_{i-1}^2\right) + \gamma p_i^1 \quad i = 1 \ldots k-1 \\
&\lambda p_i^2 = \eta_2 \left(p_i^1 + p_{i+1}^1\right) + \eta_1 p_i^2 \quad i = 0 \ldots k-1 \\
&\lambda p_0^1 = \beta p_0^1 \\
&\lambda p_k^1 = \beta p_k^1
\end{aligned}
\tag{5.15}
$$

For $\lambda \neq \eta_1$, this leads to the following system of equations for p_i^1, $i = 1 \ldots k-1$,

$$
0 = \left(\delta_1 + \delta_2 \frac{\eta_2}{\lambda - \eta_1}\right)\left(p_{i-1}^1 + p_{i+1}^1\right) + \left(\gamma - \lambda + \frac{2\delta_2\eta_2}{\lambda - \eta_1}\right) p_i^1 \tag{5.16}
$$

Denote $\tilde{\eta}_1 = \delta_2\eta_2/\delta_1$. Then, if $\lambda = \eta_1 - \tilde{\eta}_1$, the equation is reduced to $p_i^1(\gamma - \eta_1 + \tilde{\eta}_1 - 2\delta_2) = 0$, which has nontrivial solutions only if $(\gamma - \eta_1 + \tilde{\eta}_1 - 2\delta_2) = 0$.

Now we can find expressions for the eigenvectors. We start with the eigenvector of the eigenvalue η_1. Two cases are possible:

1. $\beta = \eta_1$. Then there are two eigenvectors which both have $p_i^1 = (-1)^i$, and for the first one $p_i^2 = (\lambda - \gamma + 2\delta_2)(-1)^i i/\delta_1$, and for the second one $p_i^2 = (\lambda - \gamma + 2\delta_2)(-1)^i(i+1)/\delta_1$.

2. $\beta \neq \eta_1$. In this case, $p_i^1 = 0$, and $p_i^2 = (-1)^i$.

If one of the eigenvalues β or $\beta - 2\alpha$ coincides with η_1, its eigenvectors are described by the same formulas. Suppose $\beta \neq \eta_1$. Then three cases are possible for the eigenvector of β.

1. $\beta = \eta_1 - \tilde{\eta}_1$, $\gamma + \beta - 2\delta_2 = 0$. In this case, the eigenvalue β has multiplicity $k+1$, and the components p_i^1, $i = 0 \ldots k$ can be chosen arbitrarily.

2. $\beta = \eta_1 - \tilde{\eta}_1$, $\gamma + \beta - 2\delta_2 \neq 0$. In this case, the eigenvalue β has multiplicity 2, the components p_i^1, $i = 1 \ldots k-1$ are zero, and p_0^1, p_k^1 can be chosen arbitrarily.

3. $\beta \neq \eta_1 - \tilde{\eta}_1$, $\gamma + \beta - 2\delta_2 \neq 0$. This is the most useful case. Let

   $$
   r(\lambda) = \frac{\gamma - \lambda + \frac{2\delta_2\eta_2}{\lambda - \eta_1}}{\delta_1 + \delta_2 \frac{\eta_2}{\lambda - \eta_1}} \tag{5.17}
   $$

   then (5.16) reduces to $p_{i-1}^1 + p_{i+1}^1 + r(\beta)p_i^1 = 0$. We have already explored the possible solutions of these equations in Section 5.1. The most useful range of $r(\beta)$ is $(-2, 0)$, in which case the eigenvector can be chosen to be $\sin((i - k/2)\theta)$, with θ such that $r(\beta) = -2\cos\theta$.

Finally, for $\beta - 2\alpha$ there are two possibilities.

1. $\beta - 2\alpha = \eta_1 - \tilde{\eta}_1$, $\gamma + \beta - 2\alpha - 2\delta_2 \neq 0$. In this case, the eigenvalue β has multiplicity $k-1$, the components p_i^1, $i = 1 \ldots k-1$ can be chosen arbitrarily, $p_0^1 = p_k^1 = 0$.

2. $\beta - 2\alpha \neq \eta_1 - \tilde{\eta}_1$, $\gamma + \beta - 2\alpha - 2\delta_2 \neq 0$. This case is similar to the third case for the eigenvalue β, with $r(\beta)$ replaced with $r(\beta - 2\alpha)$.

If $\alpha = 0$, then in the case $\beta \neq \eta_1 - \tilde{\eta}_1$, $\gamma + \beta - 2\delta_2 \neq 0$, the eigenvalue β has two eigenvectors that can be chosen to be $\sin i\theta$ and $\sin(i-k)\theta$

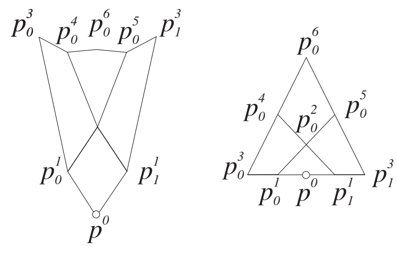


Figure 9: The control mesh for the parametric and characteristic maps in the case $k = 1$ for smooth boundary.

Coefficients for smooth boundary vertices. As it was discussed in Section 5.1, it is desirable to use $\beta = 1/2$ and $\alpha = 1/8$ for smooth boundary vertices. This choice of coefficients leads to a cubic spline boundary curve. It is easy to see that we need only a single parameter in this case to ensure C^1-continuity. We choose the parameter γ, using the standard values for all other parameters: $\eta_1 = \eta_2 = 1/4$, $\delta_1 = \delta_2 = 1/16$. In this case, the expression for the eigenvalues λ_j, λ'_j simplifies to

$$\lambda_j, \lambda'_j = \frac{1}{2}\tilde{\eta} + \frac{1}{8} \pm \frac{1}{8}\sqrt{16\tilde{\eta}^2 - 8\tilde{\eta} + 1 + 2\,(1 + \cos j\theta_k)} \quad j = 1 \ldots k-1$$

Note that for any k, j and any $0 < \gamma < 1$, $|\lambda_j| < \lambda_1$ and $|\lambda'_j| < \lambda_1$. From the formulas for the eigenvectors we can tell that it is desirable to have subdominant eigenvalues $\beta = 1/2$ and λ_1. For λ_1 to be equal to $1/2$, we can take $\gamma = 3/8 - (1/4)\cos\theta_k$. Note that for the regular case $k = 2$ we get the standard value $\gamma = 3/8$. In general, for $1/2$ to be one of the subdominant eigenvalues, it is necessary that $\gamma < 3/8 - (1/4)\cos 2\theta_k$. If one wishes to use a single value of γ for all valences, then the maximal possible choice of γ is $1/8$.
Case $k = 1$. For the regular choices of parameters, the subdominant eigenvalues are $1/2$ and $1/4$, where $1/4$ has a Jordan block of size 2. The resulting scheme is C^1, although the normals converge to the limit slower than in other cases due to the presence of the Jordan block. In this case the *parametric map* does not coincide with the characteristic map. The parametric map can be informally characterized as the map approximating, up to affine invariance, any subdivision surface generated near the central control point. Typically, it coincides with the characteristic map, but in the case when one of the subdominant eigenvalues has a nontrivial Jordan block, these maps can be different. The tangent vectors are actually determined by the control vectors of the parametric map. The control net of the characteristic and parametric maps for $k = 1$ and the standard choice of coefficients is shown in Figure 9. Assuming the ordering of components $x^1 = [p^0, p^1_0, p^1_1, p^2, p^3_0, p^3_1, p^4_0, p^5_0, p^6_0]$, the eigenvectors defining the maps are $x^1 = [0, 1, -1, 0, 2, -2, 1, -1, 0]$ (eigenvalue 1/2), $x^2 = [0, 0, 0, 1, 0, 0, 2, 2, 4]$ (eigenvalue 1/4 regular eigenvector) and $x'^2 = [-1, 2, 2, 5, 11, 11, 10, 10, 51/5]$ (eigenvalue 1/4, generalized eigenvector). The characteristic map is defined by the pair (x^1, x^2), the parametric map is defined by the pair (x^1, x'^2).

Coefficients for convex corner vertices. For the corner vertices we choose $\alpha = 0$, $\beta = 1/2$. In this case, we have to ensure that the two eigenvectors of the double eigenvalue β are the subdominant eigenvectors. The necessary condition for this is $\lambda_1 < \beta$. In addition, we have to verify that the resulting corner is indeed convex. As it was the case for the Loop scheme, if the characteristic map is regular, for convexity it is sufficient that the control mesh of the characteristic map has a convex corner at the central vertex. As the subdominant eigenvectors for the eigenvalue β can be chosen to have components p^1_i equal to $\sin i\theta$ and $\cos i\theta$, with θ such that $-2\cos\theta = r(\beta)$ and $r(\beta)$ defined by (5.17), the condition for convexity is $r < -2\cos\theta_k$. As it was the case for the Loop scheme, this condition turns out to be exactly equivalent to the condition for the eigenvalue β to be subdominant. We arrive at the same conclusion: *no scheme from the class that we have defined can produce smooth concave corners.*

Coefficients for concave corner vertices. To obtain coefficients that would allow us to generate surfaces with smooth concave corners, we use the same approach that we used for the Loop scheme: we modify the coefficients in such a way that all eigenvalues of the matrix A_{00} except 1 and $\beta = 1/2$ are scaled by the constant $s < 1$. Recall that the idea is to use subdivision rules with γ chosen in such a way that the eigenvectors of the eigenvalue $\beta = 1/2$ produce a concave configuration, and use additional modification of control points to ensure that β is subdominant. The additional rules were derived from the expression

$$\tilde{p} = (1-s)p + s\left((l^0, p)x^0 + (l, p)x^\beta + (l', p)x'^\beta\right)$$

where x_0 is the eigenvector $[1, \dots, 1]$ of the eigenvalue 1. The vectors x^β and x'^β are eigenvectors of the eigenvalue β, and l^0, l and l' are corresponding left eigenvectors. The left eigenvectors l^0, l and l' are exactly the same as for the Loop scheme: $l = [-1, 0, 1, 0, \dots]$, $l' = [-1, 1, 0, 0, \dots 0]$ and $l^0 = [1, 0, \dots 0]$. The eigenvectors x^β and x'^β coincide with the eigenvectors for the Loop scheme when restricted to the vertices of type 1. To obtain the desired scaling of eigenvalues we also need to modify vertices of type 2. The components of the eigenvectors corresponding to the vertices of type 2 are easily computed using subdivision rules (cf. (5.15)):

$$p_i^2 = \frac{\eta_2}{\lambda - \eta_1}\left(p_i^1 + p_{i+1}^1\right) = \left(p_i^1 + p_{i+1}^1\right)$$

Therefore, the analog of rules (5.7) for the Catmull-Clark subdivision is

$$\begin{aligned} [Sp]_i^1 &= (1-s)p_i^1 + s\left(p^0 + \left(p_0^1 - p^0\right)\frac{\sin{(k-i)\theta}}{\sin k\theta} + \left(p_k^1 - p^0\right)\frac{\sin i\theta}{\sin k\theta}\right) \\ [Sp]_i^2 &= (1-s)p_i^2 + s\left(p^0 + \left(p_0^1 - p^0\right)\frac{\sin{(k-i)\theta} + \sin{(k-i+1)\theta}}{\sin k\theta} + \left(p_k^1 - p^0\right)\frac{\sin i\theta + \sin{(i+1)\theta}}{\sin k\theta}\right) \end{aligned} \tag{5.18}$$

Proposition 5.3. *Let Φ be the characteristic map defined by the eigenvectors described as above for each of the different cases. Then*

1. *the preimage $\Phi^{-1}(0)$ contains only one element, 0;*
2. *the characteristic map has a Jacobian of constant sign at all points of the domain besides 0.*
3. *The image of the boundary of the characteristic map has no self-intersections;*
4. *the image of the characteristic map is not the whole plane.*

Proof. We will consider the smooth boundary case and the corner case separately. For the smooth boundary case we use the coefficients $\beta = 1/2$, $\alpha = 1/8$, $\eta_1 = \eta_2 = 1/4$, $\delta_1 = \delta_2 = 1/16$ and $\gamma = 3/8 - 1/4\cos\theta_k$. The characteristic map is then formed by the eigenvectors to the eigenvalue $\beta = 1/2$ and $\lambda_1 = 1/2$. In the corner case we use the same coefficients for $\beta, \eta_1, \eta_2, \delta_1$ and δ_2 and $\alpha = 0$ and $\gamma = 3/8 - 1/4\cos\theta$ where θ is such that $r(\beta) = -2\cos\theta$. We construct the 2 ring control mesh (see Figure 10(a) for k=4) given by these eigenvalues and subdivide them twice. The quadrilaterals in the 3rd and 4th ring are then surrounded by one ring of regular quadrilaterals. In each sector we have 12 quadrilaterals. We compute the surrounding 16 control values (Figure 10 (c)) for the different cases for the smooth boundary as well as for the corner case. We distinguish the same cases as in the Loop scheme.

For each of those cases we get the polynomial by tensor product B-spline which is given by

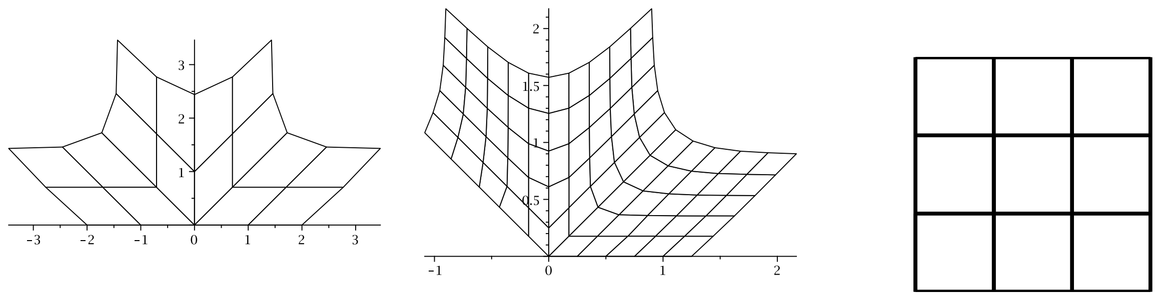
$$p(u, v) = \chi \cdot B \cdot P$$

where

$$\chi = (1, u, u^2, u^3, v, vu, vu^2, cu^3, v^2, v^2u, v^2u^2, v^2u^3, v^3, v^3u, v^3u^2, v^3u^3)$$

and

$$B = \frac{1}{6}\begin{pmatrix} B_1 & 4B_1 & B_1 & 0 \\ -3B_1 & 0 & 3B_1 & 0 \\ 3B_1 & -6B_1 & 3B_1 & 0 \\ -B_1 & 3B_1 & -3B_1 & B_1 \end{pmatrix} \text{ with } B_1 = \frac{1}{6}\begin{pmatrix} 1 & 4 & 1 & 0 \\ -3 & 0 & 3 & 0 \\ 3 & -6 & 3 & 0 \\ -1 & 3 & -3 & 1 \end{pmatrix} \tag{5.19}$$



(a) (b) (c)

Figure 10: Fig (a) show the 2 ring control data for a mesh with 4 sectors which we subdivide twice. We get 5 rings show in Fig (b). Fig (d) shows a control net for a given quadrilateral.

and $P \in \mathbf{R}^{16\times 2}$ given by the eigenvector control data. In order to do our analysis we have to transform the polynomial into Bernstein-Bezier coordinates first. We do that by replacing the vector χ with its Bernstein-Bezier equivalent. This is done by making every expression in χ a homogenous polynomial in $u, v, u' = 1-u, v' = 1-v$ by substituting 1s. Therefore we now have 168 polynomials $p(u, u', v, v') = (f_1, f_2)$ representing the characteristic map in Berstein-Bezier coordinates.

1. For each of those cases we need to check that the radius $f_1^2 + f_2^2$ is strictly bigger than 0. We do that by checking the coefficients in Berstein-Bezier coordinates are positive.

2. We compute the Jacobian on each quadrilateral and check the sign by checking the sign of the coefficients and find that they are all the same.

3. We restrict f_1 and f_2 to the relevant boundary of the quadrilaterals on the 4th ring. We check the monotonicity of the quotient as in the Loop scheme.

4. This follows from the convex hull as the convex hull of the control mesh is not the whole plane. Again we have to take it sector by sector for the concave corner case.

□

We can then follow similarly as for Loop that the scheme is C^1 by finding the cubic B-spline boundary curve and checking the condition of Theorem 4.2

Furthermore it is not necessary to use those exact values for the parameters you just have to make sure that β and λ_1 are subdominant as well as $r(\beta) \in (-2, 0)$ to get similar eigenvectors and by a similar calculation the same results.

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A Appendix

A.1 Loop

The Loop subdivision matrix is given by the matrix

$$
\left(\begin{array}{c|cc|ccccc|cccccc|cc|ccccc}
1-2\alpha & \alpha & \alpha \\
\hline
1-\beta & \beta & \\
1-\beta & & \beta \\
\hline
\epsilon & \delta & & \gamma & \delta \\
\epsilon & & & \delta & \gamma & \delta \\
\cdot & & & & \cdot & \cdot & \cdot \\
\epsilon & & & & & \delta & \gamma & \delta \\
\epsilon & & \delta & & & & \delta & \gamma \\
\hline
1/8 & 3/8 & & 3/8 & & & & & 1/8 \\
1/8 & & & 3/8 & 3/8 & & & & & 1/8 \\
\cdot & & & & \cdot & \cdot & & & & & \cdot \\
\cdot & & & & & \cdot & \cdot & & & & & \cdot \\
1/8 & & & & & & 3/8 & 3/8 & & & & & 1/8 \\
1/8 & & 3/8 & & & & & 3/8 & & & & & & 1/8 \\
\hline
1/8 & 3/4 & & & & & & & & & & & & & 1/8 \\
1/8 & & 3/4 & & & & & & & & & & & & & 1/8 \\
\hline
1/16 & 1/16 & & 5/8 & 1/16 & & & & 1/16 & 1/16 & & & & & & & 1/16 \\
1/16 & & & 1/16 & 5/8 & 1/16 & & & & 1/16 & 1/16 & & & & & & & 1/16 \\
\cdot & & & & & \cdot & & & & & \cdot & \cdot & & & & & & & \cdot \\
1/16 & & & & & 1/16 & 5/8 & 1/16 & & & & 1/16 & 1/16 & & & & & & & 1/16 \\
1/16 & & 1/16 & & & & 1/16 & 5/8 & & & & & 1/16 & 1/16 & & & & & & & 1/16
\end{array}\right) \tag{A.1}
$$

where $\epsilon = 1 - 2\delta - \gamma$.

Depending on the range of r Table 1 shows the eigenvector for v^β and Table 2 shows the 2 eigenvectors if β is a double eigenvalue.

The matrix Q needed to evaluate the polynomial is gvien by:

$$
Q = \frac{1}{24}\begin{pmatrix}
2 & 2 & 0 & 2 & 12 & 2 & 0 & 2 & 2 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 12 & 3 & 0 & 3 & 4 & 0 & 0 & 0 \\
1 & 3 & 0 & 0 & 12 & 4 & 0 & 1 & 3 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 8 & 4 & 0 & 4 & 8 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 10 & 6 & 0 & 1 & 6 & 0 & 0 & 0 \\
0 & 4 & 0 & 0 & 8 & 8 & 0 & 0 & 4 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 4 & 3 & 0 & 3 & 12 & 1 & 1 & 0 \\
0 & 1 & 0 & 1 & 6 & 6 & 0 & 1 & 10 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 6 & 10 & 0 & 0 & 6 & 1 & 0 & 0 \\
0 & 3 & 1 & 0 & 4 & 12 & 0 & 0 & 3 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 2 & 2 & 0 & 2 & 12 & 2 & 2 & 2 \\
0 & 0 & 0 & 0 & 3 & 4 & 0 & 1 & 12 & 3 & 0 & 1 \\
0 & 0 & 0 & 0 & 4 & 8 & 0 & 0 & 8 & 4 & 0 & 0 \\
0 & 1 & 0 & 0 & 3 & 12 & 1 & 0 & 4 & 3 & 0 & 0 \\
0 & 2 & 2 & 0 & 2 & 12 & 2 & 0 & 2 & 2 & 0 & 0
\end{pmatrix} \tag{A.2}
$$

The different cases we need distinguish are:

$r > 2,\ k$ odd	$(-1)^i \cosh\left(i - \frac{k}{2}\right)\theta, \quad r = 2\cosh\theta$
$r > 2,\ k$ even	$(-1)^i \sinh\left(i - \frac{k}{2}\right)\theta, \quad r = 2\cosh\theta$
$r = 2,\ k$ odd	$(-1)^i$
$r = 2,\ k$ even	$(-1)^i \left(n - \frac{k}{2}\right),$
$-2 < r < 2$	$\sin\left(i - \frac{k}{2}\right)\theta, \quad r = -2\cos\theta$
$r = -2$	$i - \frac{k}{2}$
$r < -2$	$\sinh\left(i - \frac{k}{2}\right)\theta, \quad r = -2\cosh\theta$

Table 1: Solutions for v^β

$r > 2,$	$(-1)^i \sinh i\theta,\ (-1)^i \sinh(i-k)\theta, \quad r = 2\cosh\theta$
$r = 2,$	$(-1)^i i,\ (-1)^i (i-k)$
$-2 < r < 2$	$\sin i\theta,\ \sin(i-k)\theta, \quad r = -2\cos\theta$
$r = -2$	$i,\ i-k$
$r < -2$	$\sinh i\theta,\ \sinh(i-k)\theta, \quad r = -2\cosh\theta$

Table 2: Solutions for v^β and v'^β

12 for the interior sector with $i = 3, \ldots, k-2$ and arbitrary k
12 for $i = 2$ or $i = k-1$(we only have to check be by symmetry) and arbitrary k
12 for $i = 1$ or $i = k$ and arbitrary k
12 for k=1;
12 for k=2 and $i = 1$ or $i = 2$
12 for k=3 and $i = 1$ or $i = 3$
12 for k=3 and $i = 2$
This means that we have $12 \times 7 = 84$ polynomials.

A.2 Catmull-Clark

The Subdivision matrix for the Catmull-Clark scheme is given by Figure 11.

The matrix we need to find the eigenvalues of is $\bar{A}_{00}$.

$$\bar{A}_{00} = \left(\begin{array}{cccc|ccccc} \gamma & \delta_1 & & & \delta_2 & \delta_2 & & & \\ \delta_1 & \gamma & \delta_1 & & & \delta_2 & \delta_2 & & \\ & \cdot & \cdot & \cdot & & & \cdot & \cdot & \\ & & \delta_1 & \gamma & & & & \delta_2 & \delta_2 \\ \hline \eta_2 & & & & \eta_1 & & & & \\ \eta_2 & \eta_2 & & & & \eta_1 & & & \\ & \cdot & \cdot & & & & \cdot & & \\ & & \cdot & \cdot & & & & \cdot & \\ & & & \eta_2 & & & & & \eta_1 \end{array} \right)$$

After changing variables and rearranging the entries, we get the matrix

$$\left(\begin{array}{cc|cccc|cccc} \eta_1 & & & & & & & & & \\ & \eta_1 & & & & & & & & \\ \hline \frac{\delta_2}{2} & & \gamma & \delta_1 & & & \delta_2 & \frac{\delta_2}{2} & & \\ & & \delta_1 & \gamma & \delta_1 & & \frac{\delta_2}{2} & \delta_2 & \frac{\delta_2}{2} & \\ & & & \cdot & \cdot & \cdot & & \cdot & \cdot & \cdot \\ & \delta_2 & & & \delta_1 & \gamma & & & \frac{\delta_2}{2} & \delta_2 \\ \hline & & 2\eta_2 & & & & \eta_1 & & & \\ & & & 2\eta_2 & & & & \eta_1 & & \\ & & & & \cdot & & & & \eta_1 & \\ & & & & & 2\eta_2 & & & & \eta_1 \end{array} \right)$$

which has 4 diagonal or tridiagonal subblocks of size $k-1 \times k-1$.

Just as in the Loop case we have to compute polynomials for the following different cases.
12 for the interior sector with $i = 3, \ldots, k-2$ and arbitrary k
12 for $i = 2$ or $i = k-1$(we only have to check be by symmetry) and arbitrary k
12 for $i = 1$ or $i = k$ and arbitrary k
12 for k=1;
12 for k=2 and $i = 1$ or $i = 2$
12 for k=3 and $i = 1$ or $i = 3$
12 for k=3 and $i = 2$

$$
\left(\begin{array}{c|cc|cccc|ccccc|cc|cccc|ccccc|ccccc|ccccc}
1-2\alpha & \alpha & \alpha \\ \hline
1-\beta & \beta & \\
1-\beta & & \beta \\ \hline
\epsilon_1 & \delta_1 & & \gamma & \delta_1 & & & \delta_2 & \delta_2 \\
\epsilon_1 & & & \delta_1 & \gamma & \delta_1 & & & \delta_2 & \delta_2 \\
 & & & & \cdot & \cdot & \cdot & & & \cdot & \cdot \\
\epsilon_1 & & \delta_1 & & & \delta_1 & \gamma & & & & \delta_2 & \delta_2 \\ \hline
\epsilon_2 & \eta_2 & & \eta_2 & & & & \eta_1 \\
\epsilon_2 & & & \eta_2 & \eta_2 & & & & \eta_1 \\
\cdot & & & & \cdot & \cdot & & & & \cdot \\
\cdot & & & & & \cdot & \cdot & & & & \cdot \\
\epsilon_2 & & \eta_2 & & & & \eta_2 & & & & & \eta_1 \\ \hline
\frac{1}{8} & \frac{3}{4} & & & & & & & & & & & \frac{1}{8} \\
\frac{1}{8} & & \frac{3}{4} & & & & & & & & & & & \frac{1}{8} \\ \hline
\frac{3}{32} & \frac{1}{64} && \frac{9}{16} & \frac{1}{64} &&& \frac{3}{32} & \frac{3}{32} &&&&&& \frac{3}{32} &&&& 0 & \frac{1}{64} &&&& \frac{1}{64} \\
\frac{3}{32} &&& \frac{1}{64} & \frac{9}{16} & \frac{1}{64} &&& \frac{3}{32} & \frac{3}{32} &&&&&& \frac{3}{32} &&&&& \frac{1}{64} &&&& \frac{1}{64} \\
\cdot &&&& \cdot & \cdot & \cdot &&& \cdot & \cdot &&&&&& \cdot &&&&& \cdot &&&& \cdot \\
\frac{3}{32} && \frac{1}{64} &&& \frac{1}{64} & \frac{9}{16} &&&& \frac{3}{32} & \frac{3}{32} &&&&&& \frac{3}{32} &&&&& \frac{1}{64} &&&& \frac{1}{64} & 0 \\ \hline
\frac{1}{16} & \frac{3}{8} && \frac{1}{16} &&&& \frac{3}{8} &&&&& \frac{1}{16} && 0 &&&& \frac{1}{16} \\
\frac{1}{16} &&& \frac{3}{8} & \frac{1}{16} &&&& \frac{3}{8} &&&&&& \frac{1}{16} &&&&& \frac{1}{16} \\
\cdot &&&& \frac{3}{8} & \cdot &&&& \cdot &&&&&& \frac{1}{16} &&&&& \cdot \\
\cdot &&&&& \cdot & \frac{1}{16} &&&& \cdot &&&&&& \cdot &&&&& \cdot \\
\frac{1}{16} && \frac{1}{16} &&&& \frac{3}{8} &&&&& \frac{3}{8} &&&&&& \frac{1}{16} &&&&& \frac{1}{16} \\ \hline
\frac{1}{16} & \frac{1}{16} && \frac{3}{8} &&&& \frac{3}{8} &&&&&&& \frac{1}{16} &&&&&&&&& \frac{1}{16} \\
\frac{1}{16} &&& \frac{1}{16} & \frac{3}{8} &&&& \frac{3}{8} &&&&&&& \frac{1}{16} &&&&&&&&& \frac{1}{16} \\
\cdot &&&& \frac{1}{16} & \cdot &&&& \cdot &&&&&&& \cdot &&&&&&&&& \cdot \\
\cdot &&&&& \cdot & \frac{3}{8} &&&& \cdot &&&&&&& \frac{1}{16} &&&&&&&&& \cdot \\
\frac{1}{16} && \frac{3}{8} &&&& \frac{1}{16} &&&&& \frac{3}{8} && \frac{1}{16} &&&& 0 &&&&&&&&&& \frac{1}{16} \\ \hline
\frac{1}{64} & \frac{3}{32} && \frac{3}{32} &&&& \frac{9}{16} &&&&& \frac{1}{64} && \frac{1}{64} &&&& \frac{3}{32} &&&&& \frac{3}{32} &&&&& \frac{1}{64} \\
\frac{1}{64} &&& \frac{3}{32} & \frac{3}{32} &&&& \frac{9}{16} &&&&&& \frac{1}{64} & \frac{1}{64} &&&& \frac{3}{32} &&&&& \frac{3}{32} &&&&& \frac{1}{64} \\
\cdot &&&& \frac{3}{32} & \cdot &&&& \cdot &&&&&& \frac{1}{64} & \cdot &&&& \cdot &&&&& \cdot &&&&& \cdot \\
\cdot &&&&& \cdot & \frac{3}{32} &&&& \cdot &&&&&& \cdot & \frac{1}{64} &&&& \cdot &&&&& \cdot &&&&& \frac{1}{64} \\
\frac{1}{64} && \frac{3}{32} &&&& \frac{3}{32} &&&&& \frac{9}{16} && \frac{1}{64} &&&& \frac{1}{64} &&&&& \frac{3}{32} &&&&& \frac{3}{32} &&&&& \frac{1}{64}
\end{array}\right)
$$

Figure 11: The subdivision matrix for the Catmull-Clark scheme.