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**Numerical Solution of Eigenvalue
Problems for Alternating Matrix
Polynomials and Their Application in
Control Problems for Descriptor Systems**



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Abstract

Numerical methods for eigenvalue problems associated to alternating matrix pencils and polynomials are discussed. These problems arise in a large number of control applications for differential-algebraic equations ranging from regular and singular linear-quadratic optimal and robust control to dissipativity checking. We present a survey of several of these applications and give a systematic overview over the theory and the numerical solution methods. Our solution concept is based throughout on the computation of eigenvalues and deflating subspaces of even matrix pencils. The unified approach allows to generalize and improve several techniques that are currently in use in systems and control.

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Notation

\mathbb{R}, \mathbb{C}	the fields of real and complex numbers, resp.;
\mathbb{C}^+	the set of complex numbers with positive real part;
i	the imaginary unit;
\mathbf{u}	the machine precision;
$\mathbb{R}(\lambda)$	the field of real-rational functions in the indeterminate λ ;
$\mathbb{R}^{m,n}, \mathbb{C}^{m,n}$	the sets of $m \times n$ matrices with entries in \mathbb{R} and \mathbb{C} , resp.;
A^T, A^H, A^{-1}	transpose, conjugate transpose, and inverse of the matrix A ;
range A , ker A	range and kernel of the matrix A , resp.;
$\text{diag}(A_1, \dots, A_k)$	$:= \begin{bmatrix} A_1 & & \\ & \ddots & \\ & & A_k \end{bmatrix};$
$A \otimes B$	$:= \begin{bmatrix} a_{11}B & \dots & a_{1n}B \\ \vdots & & \vdots \\ a_{m1}B & \dots & a_{mn}B \end{bmatrix}$ (Kronecker product).

1 Introduction

In this paper we discuss the numerical solution of structured linear or polynomial eigenvalue problems and their use in control problems for descriptor systems, i.e., systems where the dynamics are described by a system of differential-algebraic equations (DAEs). The numerical solution of polynomial eigenvalue problems is an important task in the vibration analysis of buildings, machines, and vehicles [23, 50, 71, 86, 95]. In many of the applications, several of which are summarized in [76], the coefficient matrices have further structure which reflects the properties of the underlying physical model, and it is important that numerical methods respect this structure. In this paper we discuss the special class of eigenvalue problems $P(\lambda)x = 0$ associated to *alternating (even/odd) matrix polynomials* of the form

$$P(\lambda) = \sum_{i=0}^d \lambda^i A_i, \quad A_0, \dots, A_d \in \mathbb{R}^{n,n}, \quad A_d \neq 0, \quad (1)$$

satisfying $P(-\lambda)^T = \pm P(\lambda)$, i.e., the coefficients are alternating between real symmetric and real skew-symmetric matrices, see e.g., [38, 76, 78, 84], and [66, 83] for the more general variable coefficient case. We focus on the case of even matrix polynomials, where the last coefficient A_0 is real symmetric, the odd case can be treated in a similar fashion. Even matrix polynomials of order two arise in the study of corner singularities in anisotropic elastic materials [3, 4, 72, 87] and gyroscopic systems [95]. Even polynomial problems of any order arise in optimal control of systems that are constrained by higher order differential equations, see [76] and the references therein or [23]. In the first order case, i.e., in the case of *even matrix pencils*, one has $P(\lambda) = \lambda N - M$, where $N = -N^T \in \mathbb{R}^{n,n}$ and $M = M^T \in \mathbb{R}^{n,n}$ and we will discuss in Section 4 when and how the eigenvalue problems for even matrix polynomials can be reduced to that of even pencils.

If the dimension of an even matrix polynomial is even, i.e., $n = 2m$, then it is closely related to so called *skew-Hamiltonian/Hamiltonian* matrix polynomials [11, 81, 84, 87]. A real matrix pencil $P(\lambda) = \lambda S - H$ is called *skew-Hamiltonian/Hamiltonian* if $\mathcal{J}P(\lambda)$ is even, where

$$\mathcal{J} = \begin{bmatrix} 0 & I_m \\ -I_m & 0 \end{bmatrix},$$

- (iii) each $\mathcal{I}_{2\epsilon_j+1}$ is a $(2\epsilon_j + 1) \times (2\epsilon_j + 1)$ block that contains a single block corresponding to the eigenvalue $\lambda = \infty$ of size $2\epsilon_j + 1$. It has the form

$$\lambda \left[\begin{array}{c|ccc} & & & 1 & 0 \\ & & & \ddots & \ddots \\ & & & \ddots & \ddots \\ & & & 1 & 0 \\ \hline & & -1 & 0 & \\ & & 0 & & \\ \hline -1 & & & & \\ \ddots & & & & \\ 0 & & & & \end{array} \right] - \left[\begin{array}{c|ccc} & & & 0 & 1 \\ & & & \ddots & \ddots \\ & & & 0 & 1 \\ \hline & & 0 & 1 & \\ & & 1 & & \\ \hline 0 & & & & \\ \ddots & & & & \\ 1 & & & & \end{array} \right];$$

where $s \in \{1, -1\}$ is the sign-characteristic of the block;

- (iv) each $\mathcal{I}_{2\delta_j}$ is a $4\delta_j \times 4\delta_j$ block that combines two $2\delta_j \times 2\delta_j$ blocks associated to $\lambda = \infty$ of size δ_j . It has the form

$$\lambda \left[\begin{array}{c|ccc} & & & 1 & 0 \\ & & & \ddots & \ddots \\ & & & \ddots & \ddots \\ & & & 1 & 0 \\ \hline & & -1 & 0 & \\ & & 0 & & \\ \hline -1 & & & & \\ \ddots & & & & \\ 0 & & & & \end{array} \right] - \left[\begin{array}{c|ccc} & & & & 1 \\ & & & & \ddots \\ & & & & \ddots \\ & & & & 1 \\ \hline & & 1 & & \\ & & & & \\ \hline 1 & & & & \end{array} \right];$$

- (v) each $\mathcal{Z}_{2\sigma_j+1}$ is a $(4\sigma_j + 2) \times (4\sigma_j + 2)$ block that combines two $(2\sigma_j + 1) \times (2\sigma_j + 1)$ Jordan blocks corresponding to the eigenvalue $\lambda = 0$. It has the form

$$\lambda \left[\begin{array}{c|ccc} & & & & 1 \\ & & & & \ddots \\ & & & & \ddots \\ & & & & 1 \\ \hline & & -1 & & \\ & & 0 & & \\ \hline -1 & & & & \\ \ddots & & & & \\ 0 & & & & \end{array} \right] - \left[\begin{array}{c|ccc} & & & 1 & 0 \\ & & & \ddots & \ddots \\ & & & 1 & 0 \\ \hline & & 1 & 0 & \\ & & 0 & & \\ \hline 1 & & & & \\ \ddots & & & & \\ 0 & & & & \end{array} \right];$$

- (vi) each $\mathcal{Z}_{2\rho_j}$ is a $2\rho_j \times 2\rho_j$ block that contains a single Jordan block corresponding to the eigenvalue $\lambda = 0$. It has the form

$$\lambda \left[\begin{array}{c|ccc} & & & & 1 \\ & & & & \ddots \\ & & & & \ddots \\ & & & & 1 \\ \hline & & -1 & & \\ & & 0 & & \\ \hline -1 & & & & \\ \ddots & & & & \\ 0 & & & & \end{array} \right] - \left[\begin{array}{c|ccc} & & & 1 & 0 \\ & & & \ddots & \ddots \\ & & & 1 & 0 \\ \hline & & s & 0 & \\ & & 0 & & \\ \hline 1 & & & & \\ \ddots & & & & \\ 0 & & & & \end{array} \right];$$

where $s \in \{1, -1\}$ is the sign-characteristic of this block;

- (vii) each \mathcal{R}_{ϕ_j} is a $2\phi_j \times 2\phi_j$ block that combines two $\phi_j \times \phi_j$ Jordan blocks corresponding to nonzero real eigenvalues a_j and $-a_j$. It has the form

$$\lambda \left[\begin{array}{c|ccc} & & & & 1 \\ & & & & \ddots \\ & & & & \ddots \\ & & & & 1 \\ \hline & & -1 & & \\ & & 0 & & \\ \hline -1 & & & & \\ \ddots & & & & \\ 0 & & & & \end{array} \right] - \left[\begin{array}{c|ccc} & & & 1 & a_j \\ & & & \ddots & \ddots \\ & & & 1 & a_j \\ \hline & & 1 & a_j & \\ & & a_j & & \\ \hline 1 & & & & \\ \ddots & & & & \\ a_j & & & & \end{array} \right].$$

(viii) The entries \mathcal{C}_{ψ_j} take two slightly different forms.

(a) One possibility is that \mathcal{C}_{ψ_j} is a $2\psi_j \times 2\psi_j$ block combining two $\psi_j \times \psi_j$ Jordan blocks with purely imaginary eigenvalues $ib_j, -ib_j$ ($b_j > 0$). In this case it has the form

$$\lambda \left[\begin{array}{c|c} & \begin{matrix} & & & 1 \\ & & \ddots & \\ & & & \\ -1 & & & \end{matrix} \\ \hline & \begin{matrix} 1 & & & \\ & \ddots & & \\ & & & \\ & & & \end{matrix} \\ \hline -1 & \begin{matrix} \ddots & & & \\ & & & \\ & & & \\ & & & \end{matrix} \end{array} \right] - s \left[\begin{array}{c|c} \begin{matrix} & & & 1 & b_j \\ & & \ddots & & \\ & & & & \\ & & & & \\ 1 & & & & \\ b_j & & & & \end{matrix} \\ \hline & \begin{matrix} & & & & \\ & & \ddots & & \\ & & & & \\ & & & & \\ & & & & \\ & & & & \\ 1 & & & & \\ b_j & & & & \end{matrix} \end{array} \right],$$

where $s \in \{1, -1\}$ is the sign-characteristic.

(b) The other possibility is that \mathcal{C}_{ψ_j} is a $4\psi_j \times 4\psi_j$ block combining $\psi_j \times \psi_j$ Jordan blocks for each of the complex eigenvalues $a_j + ib_j, a_j - ib_j, -a_j + ib_j, -a_j - ib_j$ (with $a_j \neq 0$ and $b_j \neq 0$). In this case it has form

$$\lambda \left[\begin{array}{c|c} & \begin{matrix} & & & \Omega \\ & & \ddots & \\ & & & \\ -\Omega & & & \end{matrix} \\ \hline & \begin{matrix} \Omega & & & \\ & \ddots & & \\ & & & \\ & & & \end{matrix} \\ \hline -\Omega & \begin{matrix} \ddots & & & \\ & & & \\ & & & \\ & & & \end{matrix} \end{array} \right] - \left[\begin{array}{c|c} & \begin{matrix} & & & \Omega & \Lambda_j \\ & & \ddots & & \\ & & & & \\ & & & & \\ \Omega & & & & \\ \Lambda_j & & & & \end{matrix} \\ \hline & \begin{matrix} & & & & \\ & & \ddots & & \\ & & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & & \\ \Omega & & & & \\ \Lambda_j & & & & \end{matrix} \end{array} \right]$$

$$\text{with } \Omega = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \text{ and } \Lambda_j = \begin{bmatrix} -b_j & a_j \\ a_j & b_j \end{bmatrix}.$$

This structured Kronecker canonical form is unique up to permutation of the blocks, i.e., the kind, size, and number of the blocks as well as the sign-characteristics are invariants of the pencil $\lambda N - M$ under congruence transformations.

An even pencil is called *regular* if and only if no blocks of type (i) and (ii) occur in the even Kronecker form. The (*Kronecker*) *index* of the pencil is the size of the largest block of type (iii) and (iv) in the even Kronecker form, thus a regular pencil is of *index at most one* if and only if there are no blocks of type (iv) and the blocks of type (iii) are of size at most one. In some of the applications discussed below, it will be necessary to detect whether an even matrix polynomial is regular and of index at most one and whether there exist *finite eigenvalues* with real part 0. In other applications the computation of the deflating subspace, i.e., the subspace spanned by the eigenvectors and generalized eigenvectors, associated to all eigenvalues in the open left half plane is the goal. The structured Kronecker form reveals this information but usually it cannot be computed numerically, because arbitrary small perturbations may change the structural information and since the transformation matrices may be unbounded.

A computationally attractive alternative is the staircase form under orthogonal transformations. It allows to check regularity and to determine the index within the usual limitations of rank computations in finite precision arithmetic, see [38] for a detailed discussion of the difficulties. This is an essential preparation for the computation of the eigenvalues and deflating subspaces.

Theorem 2.2. [38] For every even pencil $\lambda N - M$, with $N = -N^T, M = M^T \in \mathbb{R}^{n,n}$, there

exists a real orthogonal matrix $U \in \mathbb{R}^{n,n}$, such that

$$U^T N U = \begin{array}{c} s_1 \\ \vdots \\ \vdots \\ s_w \\ \hline l \\ q_w \\ \vdots \\ q_2 \\ q_1 \end{array} \left[\begin{array}{cccc|ccc|c} N_{11} & \cdots & \cdots & N_{1,w} & N_{1,w+1} & N_{1,w+2} & \cdots & N_{1,2w} & 0 \\ \vdots & \ddots & & \vdots & \vdots & \vdots & \ddots & \ddots & \\ \vdots & \vdots & \ddots & \vdots & \vdots & N_{w-1,w+2} & \ddots & & \\ \hline -N_{1,w}^T & \cdots & \cdots & N_{w,w} & N_{w,w+1} & 0 & & & \\ \hline -N_{1,w+1}^T & \cdots & \cdots & -N_{w,w+1}^T & N_{w+1,w+1} & & & & \\ \hline -N_{1,w+2}^T & \cdots & -N_{w-1,w+2}^T & 0 & & & & & \\ \vdots & \vdots & \ddots & & & & & & \\ \vdots & \vdots & \ddots & & & & & & \\ \hline -N_{1,2w}^T & \ddots & & & & & & & \\ \hline 0 & & & & & & & & \end{array} \right] \quad (3)$$

$$U^T M U = \begin{array}{c} s_1 \\ \vdots \\ \vdots \\ s_w \\ \hline l \\ q_w \\ \vdots \\ \vdots \\ q_1 \end{array} \left[\begin{array}{cccc|ccc|c} M_{11} & \cdots & \cdots & M_{1,w} & M_{1,w+1} & M_{1,w+2} & \cdots & \cdots & M_{1,2w+1} \\ \vdots & \ddots & & \vdots & \vdots & \vdots & \ddots & \ddots & \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & & \\ \hline M_{1,w}^T & \cdots & \cdots & M_{w,w} & M_{w,w+1} & M_{w,w+2} & & & \\ \hline M_{1,w+1}^T & \cdots & \cdots & M_{w,w+1}^T & M_{w+1,w+1} & & & & \\ \hline M_{1,w+2}^T & \cdots & \cdots & M_{w,w+2}^T & & & & & \\ \vdots & \vdots & \ddots & & & & & & \\ \vdots & \vdots & \ddots & & & & & & \\ \hline M_{1,2w+1}^T & & & & & & & & \end{array} \right],$$

where $q_1 \geq s_1 \geq q_2 \geq s_2 \geq \dots \geq q_w \geq s_w$, $l = r_{w+1} + a_{w+1}$, and for $i = 1, \dots, w$ we have $N_{ii} = -N_{ii}^T$, $M_{ii} = M_{ii}^T$. Furthermore,

$$\begin{aligned} N_{j,2w+1-j} &\in \mathbb{R}^{s_j, q_{j+1}}, \quad 1 \leq j \leq w-1, \\ N_{w+1,w+1} &= \begin{bmatrix} \Delta & 0 \\ 0 & 0 \end{bmatrix}, \quad \Delta = -\Delta^T \in \mathbb{R}^{r_{w+1}, r_{w+1}}, \\ M_{j,2w+2-j} &= [\Gamma_j \ 0] \in \mathbb{R}^{s_j, q_j}, \quad \Gamma_j \in \mathbb{R}^{s_j, s_j}, \quad 1 \leq j \leq w, \\ M_{w+1,w+1} &= \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix}, \quad \Sigma_{11} \in \mathbb{R}^{r_{w+1}, r_{w+1}}, \quad \Sigma_{22} \in \mathbb{R}^{a_{w+1}, a_{w+1}}, \\ M_{w+1,w+1} &= M_{w+1,w+1}^T, \end{aligned}$$

and the blocks Σ_{22} and Δ and Γ_j , $j = 1, \dots, w$ (if they occur) are nonsingular.

Production code implementations for the computation of these and other related structured staircase forms via a sequence of singular value decompositions have been presented in [32]. Since the staircase form uses congruence transformations, all the invariants of the even Kronecker canonical form are preserved, as discussed in the following corollary.

Corollary 2.3. [38] Consider an even pencil and its staircase form (3).

1. The pencil is regular if and only if $s_i = q_i$ for $i = 1, \dots, w$.
2. The pencil is regular and of index at most one if and only if $w = 0$.
3. The block $(N_{w+1,w+1}, M_{w+1,w+1})$ contains the regular part associated to finite eigenvalues and blocks associated to the infinite eigenvalues of index at most one.

4. The finite eigenvalues of the pencil are the eigenvalues of

$$\lambda\Delta - (\Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}).$$

5. For every purely imaginary eigenvalue $\lambda_0 = i\alpha_0$ with $\alpha_0 \in \mathbb{R}$, satisfying

$$(i\alpha_0\Delta - (\Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}))x_0 = 0.$$

the sign-characteristic of $i\alpha_0$ is given by the sign of the real number $ix_0^*\Delta x_0$.

Thus, once the staircase form has been computed, for the computation of eigenvalues and invariant subspaces one can restrict the methods to the middle regular index one block of the staircase form. We recall the appropriate methods in the next section.

3 Computing Eigenvalues and Deflating Subspaces of Regular Index One Even Pencils

For the computation of eigenvalues, eigenvectors, and deflating subspaces associated to finite eigenvalues of even pencils, we need eigenvalue methods for regular even pencils of index at most one that can be applied to the middle block in the staircase form (3)

$$\lambda N_{w+1,w+1} - M_{w+1,w+1} = \lambda \begin{bmatrix} \Delta & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix}, \quad (4)$$

In the special case that this even pencil has no infinite eigenvalues, i.e., if the second block row and column are not occurring, and hence $a_{w+1,w+1} = 0$, then we have a pencil $\lambda\Delta - \Sigma_{11}$, where Δ is nonsingular (and thus of even dimension). In this case one can perform a Cholesky-like decomposition, see [8, 34] of the form $\Delta = U^T \mathcal{J} U$ with an upper-triangular matrix U . If the factorization is well-conditioned and if U is well-conditioned with respect to inversion, then one can turn this even eigenvalue problem into an eigenvalue problem for the Hamiltonian matrix $\mathcal{H} = \mathcal{J}^T U^{-T} \Sigma_{11} U^{-1}$ and apply the structure-preserving methods for Hamiltonian eigenvalue problems [42, 85]. If, however, the computation and inversion of U is ill-conditioned or if the pencil $\lambda N_{w+1,w+1} - M_{w+1,w+1}$ has infinite eigenvalues, then it is better to stay with the pencil formulation.

Since for skew-Hamiltonian/Hamiltonian pencils eigenvalue methods are well established and have been professionally implemented [11, 13, 19, 20, 48, 74, 84, 87], we just adapt these for the even case. However, we suggest that in the long run these methods should be implemented to directly work for the even case, since it may happen that the middle block $\lambda N_{w+1,w+1} - M_{w+1,w+1}$ (i.e., the regular index one part) is of odd dimension. To apply the methods for skew-Hamiltonian/Hamiltonian pencils to this middle block in the odd-dimensional case, we consider an embedded $2k \times 2k$ pencil

$$\lambda S - H = \mathcal{J} \left(\lambda \begin{bmatrix} N_{w+1,w+1} & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} M_{w+1,w+1} & 0 \\ 0 & 1 \end{bmatrix} \right)$$

which has an additional eigenvalue ∞ , right eigenvector e_{2k} (the $2k$ -th unit vector) and left eigenvector $\mathcal{J}^T e_{2k}$, which are orthogonal to all the other eigenvectors. So in the following, whenever an eigenvalue method for regular even pencils of index at most one is needed, then we can perform this embedding and employ a solver for the skew-Hamiltonian/Hamiltonian pencil $\lambda S - H$, with $S, H \in \mathbb{R}^{2k,2k}$.

For the computation of the eigenvalues and deflating subspaces of skew-Hamiltonian/Hamiltonian pencils we make use of \mathcal{J} -congruence transformations of the form

$$\lambda \tilde{S} - \tilde{H} := \mathcal{J} \mathcal{Q}^T \mathcal{J}^T (\lambda S - H) \mathcal{Q}$$

with nonsingular matrices \mathcal{Q} , which preserve the skew-Hamiltonian/Hamiltonian structure. In general we would hope that we can compute an *orthogonal* matrix \mathcal{Q} such that

$$\mathcal{J}\mathcal{Q}^T\mathcal{J}^T(\lambda S - H)\mathcal{Q} = \lambda \begin{bmatrix} S_{11} & S_{12} \\ 0 & S_{11}^T \end{bmatrix} - \begin{bmatrix} H_{11} & H_{12} \\ 0 & -H_{11}^T \end{bmatrix}$$

is in skew-Hamiltonian/Hamiltonian Schur form, i.e., the subpencil $\lambda S_{11} - H_{11}$ is in generalized Schur form [51]. Unfortunately, not every skew-Hamiltonian/Hamiltonian pencil has this structured Schur form, since certain simple purely imaginary eigenvalues, or multiple purely imaginary eigenvalues with even algebraic multiplicity, but uniform sign-characteristic, cannot be represented in this structure. An embedding into a pencil of the double size solves this issue as follows.

We introduce the orthogonal matrices

$$\mathcal{Y} = \frac{\sqrt{2}}{2} \begin{bmatrix} I_{2k} & I_{2k} \\ -I_{2k} & I_{2k} \end{bmatrix}, \quad \mathcal{P} = \begin{bmatrix} I_k & 0 & 0 & 0 \\ 0 & 0 & I_k & 0 \\ 0 & I_k & 0 & 0 \\ 0 & 0 & 0 & I_k \end{bmatrix}, \quad \mathcal{X} = \mathcal{Y}\mathcal{P}$$

and define the $4k \times 4k$ pencil

$$\lambda\mathcal{B}_S - \mathcal{B}_H := \mathcal{X}^T \left(\lambda \begin{bmatrix} S & 0 \\ 0 & S \end{bmatrix} - \begin{bmatrix} H & 0 \\ 0 & -H \end{bmatrix} \right) \mathcal{X},$$

which is still regular and of index at most one.

It can be easily observed, that $\lambda\mathcal{B}_S - \mathcal{B}_H$ is again real skew-Hamiltonian/Hamiltonian with the same eigenvalues (now with double algebraic, geometric and partial multiplicities, but with appropriate mixed sign-characteristic) as the pencil $\lambda S - H$. To compute the eigenvalues of $\lambda\mathcal{B}_S - \mathcal{B}_H$ one uses the generalized symplectic URV decomposition of $\lambda S - H$, see [15, 16], i.e., there exist $4k \times 4k$ real orthogonal matrices Q_1, Q_2 , such that

$$\begin{aligned} Q_1^T S J Q_1 J^T &= \begin{bmatrix} S_{11} & S_{12} \\ 0 & S_{11}^T \end{bmatrix}, \\ J Q_2^T J^T S Q_2 &= \begin{bmatrix} T_{11} & T_{12} \\ 0 & T_{11}^T \end{bmatrix}, \\ Q_1^T H Q_2 &= \begin{bmatrix} H_{11} & H_{12} \\ 0 & H_{22} \end{bmatrix}, \end{aligned} \tag{5}$$

where S_{12} and T_{12} are skew-symmetric and the generalized matrix product $S_{11}^{-1} H_{11} T_{11}^{-1} H_{22}^T$ is in periodic Schur form [26, 56, 61].

Applying this result to the specially structured pencil $\lambda\mathcal{B}_S - \mathcal{B}_H$, we can compute an orthogonal matrix \mathcal{Q} such that

$$\mathcal{J}\mathcal{Q}^T\mathcal{J}^T(\lambda\mathcal{B}_S - \mathcal{B}_H)\mathcal{Q} = \lambda \left[\begin{array}{cc|cc} S_{11} & 0 & S_{12} & 0 \\ 0 & T_{11} & 0 & T_{12} \\ \hline 0 & 0 & S_{11}^T & 0 \\ 0 & 0 & 0 & T_{11}^T \end{array} \right] - \left[\begin{array}{cc|cc} 0 & H_{11} & 0 & H_{12} \\ -H_{22}^T & 0 & H_{12}^T & 0 \\ \hline 0 & 0 & 0 & H_{22} \\ 0 & 0 & -H_{11}^T & 0 \end{array} \right] \tag{6}$$

with $\mathcal{Q} = \mathcal{P}^T \begin{bmatrix} J Q_1 J^T & 0 \\ 0 & Q_2 \end{bmatrix} \mathcal{P}$.

Note, that for these computations we never explicitly construct the embedded pencils. It is sufficient to compute the necessary parts of the matrices in (5).

The eigenvalues of $\lambda S - H$ can then be computed as $\pm i\sqrt{\lambda_j}$ where the $\lambda_j, j = 1, \dots, k$ are the eigenvalues of $S_{11}^{-1} H_{11} T_{11}^{-1} H_{22}^T$ which can be determined by evaluating the entries on the 1×1 and 2×2 diagonal blocks of the matrices only. In particular, the finite, purely imaginary eigenvalues correspond to the 1×1 diagonal blocks of this matrix product. Provided that the pairwise distance

of the simple, finite, purely imaginary eigenvalues with mixed sign-characteristics is sufficiently large, they can be computed in a robust way without any error in the real part. This property of the algorithm plays an essential role for many of the applications that we will consider in subsequent sections.

If also the deflating subspaces associated to certain eigenvalues are desired, then one computes the real skew-Hamiltonian/Hamiltonian Schur form of the embedded pencil where the eigenvalues are reordered in such a way such that the desired eigenvalues appear in the leading principal sub-pencil. By determining also the sign-characteristic of the purely imaginary eigenvalues, one can (at least in exact arithmetic) check whether a Hamiltonian Schur form exists. It should be noted that if the problem has computed eigenvalues very close to the imaginary axis (within a strip of width $\sqrt{\mathbf{u}}$), then these may have resulted from a perturbation of size \mathbf{u} of double purely imaginary eigenvalue with mixed sign-characteristic. This does not prevent the existence of a Hamiltonian Schur form, however, in the neighborhood of this problem there is then a problem with two simple purely imaginary eigenvalues of mixed sign-characteristic, where the problem has no Hamiltonian Schur form, see [1].

The structure-preserving Algorithm 1 was introduced in [10] and has been updated and improved in [74]. It is available as the SLICOT¹ subroutine MB04BD. While the classic unstructured QZ algorithm applied to the $2k \times 2k$ pencil would require $528k^3$ flops or $240k^3$ flops for the eigenvalues [51], this algorithm needs roughly 60% of that [10]. Note that there are many more structure-exploiting algorithms for Hamiltonian and even eigenvalues problems in the dense but also sparse setting, see, e.g., [13, 14, 42, 62, 73, 84, 85, 92].

In later sections, when discussing applications for even pencils, we will always use the algorithm presented here, since the preservation of the spectral symmetry is essential for the robustness of the methods. For illustration, Figure 1 from [20] plots the computed eigenvalues of a skew-Hamiltonian/Hamiltonian pencil that results from the stability analysis of a linearized gyroscopic system. A necessary condition for stability is that all eigenvalues are on the imaginary axis. The figure shows that the structure-preserving algorithm captures this behavior whereas the standard QZ algorithm fails to do so and therefore, does not allow to make any statement about stability.

4 Structured Linearizations

The numerical solution of eigenvalue problems for matrix polynomials of the form (1) is usually achieved by first carrying out a *linearization*, in which the given polynomial (1) is transformed into a $dn \times dn$ matrix pencil $L(\lambda) = \lambda X + Y$ that satisfies

$$E(\lambda)L(\lambda)F(\lambda) = \begin{bmatrix} P(\lambda) & 0 \\ 0 & I_{(d-1)n} \end{bmatrix}, \quad (7)$$

where $E(\lambda)$ and $F(\lambda)$ are *unimodular matrix polynomials* [50]. A matrix polynomial is *unimodular* if its determinant is a nonzero constant, independent of λ . Standard methods for linear eigenvalue problems can then be applied to $L(\lambda)$. The companion forms [50] provide the standard examples of linearizations for a matrix polynomial. Unfortunately, since the companion linearizations do not reflect the structure present in even matrix polynomials, the corresponding linearized pencil can usually only be treated with methods for general matrix pencils. Then, in a finite precision environment, a numerical method that ignores the structure may destroy symmetries in the spectrum and hence produce physically meaningless results [95]. Furthermore, properties such as the sign-characteristic are not preserved when considering the standard companion forms.

Therefore, it is important to construct linearizations that reflect the structure of the given matrix polynomial. This topic has been extensively studied in recent years, see [58, 59, 76, 77]. To achieve this, in [77], new vector spaces of pencils were introduced that generalize the classical

¹<http://slicot.org/>

Algorithm 1 Computation of stable eigenvalues and associated stable deflating subspaces of a real skew-Hamiltonian/Hamiltonian pencil

Input: A regular real skew-Hamiltonian/Hamiltonian pencil $\lambda S - H$ of index at most one, with $S, H \in \mathbb{R}^{2k, 2k}$.

Output: The eigenvalues of $\lambda S - H$ and a matrix P_V^- whose columns form an orthogonal basis of the r -dimensional deflating subspace associated to the eigenvalues in the open left half plane.

- 1: Compute the generalized symplectic URV decomposition [74, Algorithm 2] of the pencil $\lambda S - H$ and determine orthogonal matrices Q_1, Q_2 such that

$$\begin{aligned} Q_1^T S J Q_1 J^T &= \begin{bmatrix} S_{11} & S_{12} \\ 0 & S_{11}^T \end{bmatrix}, \\ J Q_2^T J^T S Q_2 &= \begin{bmatrix} T_{11} & T_{12} \\ 0 & T_{11}^T \end{bmatrix}, \\ Q_1^T H Q_2 &= \begin{bmatrix} H_{11} & H_{12} \\ 0 & H_{22} \end{bmatrix}, \end{aligned}$$

where the generalized matrix product $S_{11}^{-1} H_{11} T_{11}^{-1} H_{22}^T$ is in periodic Schur form.

- 2: Apply [74, Algorithm 3] to determine orthogonal matrices Q_3 and Q_4 such that

$$\lambda \mathcal{S}_{11} - \mathcal{H}_{11} := Q_4^T \left(\lambda \begin{bmatrix} S_{11} & 0 \\ 0 & T_{11} \end{bmatrix} - \begin{bmatrix} 0 & H_{11} \\ -H_{22}^T & 0 \end{bmatrix} \right) Q_3$$

is in generalized Schur form. Update

$$\mathcal{S}_{12} := Q_4^T \begin{bmatrix} S_{12} & 0 \\ 0 & T_{12} \end{bmatrix} Q_4, \quad \mathcal{H}_{12} := Q_4^T \begin{bmatrix} 0 & H_{12} \\ H_{12}^T & 0 \end{bmatrix} Q_4$$

and set

$$\lambda \mathcal{B}_S - \mathcal{B}_H := \lambda \begin{bmatrix} \mathcal{S}_{11} & \mathcal{S}_{12} \\ 0 & \mathcal{S}_{11}^T \end{bmatrix} - \begin{bmatrix} \mathcal{H}_{11} & \mathcal{H}_{12} \\ 0 & -\mathcal{H}_{11}^T \end{bmatrix}.$$

- 3: Apply the eigenvalue reordering method [74, Algorithm 4] to the pencil $\lambda \mathcal{B}_S - \mathcal{B}_H$ to determine an orthogonal matrix \hat{Q} such that

$$\lambda \tilde{\mathcal{B}}_S - \tilde{\mathcal{B}}_H := J \hat{Q}^T J^T (\lambda \mathcal{B}_S - \mathcal{B}_H) \hat{Q}$$

is still in structured Schur form but the eigenvalues with negative real part of $\lambda \tilde{\mathcal{B}}_S - \tilde{\mathcal{B}}_H$ are contained in the leading $2r \times 2r$ principal subpencil of $\lambda \mathcal{S}_{11} - \mathcal{H}_{11}$.

- 4: Set

$$V = \begin{bmatrix} I_{2k} & 0 \end{bmatrix} \left(\mathcal{Y} \begin{bmatrix} J Q_1 J^T & 0 \\ 0 & Q_2 \end{bmatrix} \mathcal{P} \begin{bmatrix} Q_3 & 0 \\ 0 & Q_4 \end{bmatrix} \hat{Q} \right) \begin{bmatrix} I_{2r} \\ 0 \end{bmatrix}$$

and compute P_V^- , an orthogonal basis of range V , using any numerically stable orthogonalization scheme.

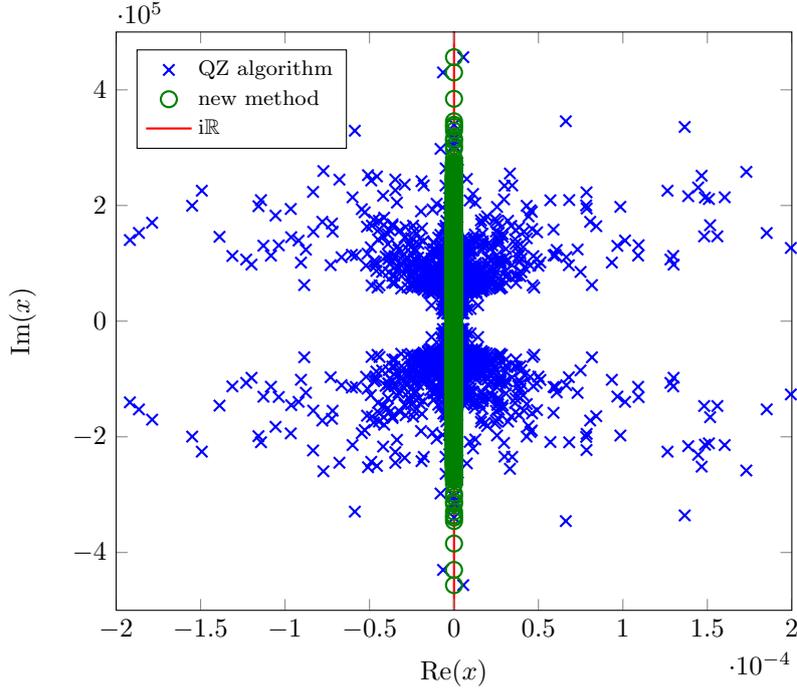


Figure 1: Computed eigenvalues from a skew-Hamiltonian/Hamiltonian pencil with only purely imaginary eigenvalues resulting from a linearized gyroscopic system

companion forms, namely

$$\begin{aligned} \mathbb{L}_1(P) &:= \{L(\lambda) = \lambda X + Y : L(\lambda) \cdot (\Lambda \otimes I_n) = v \otimes P(\lambda), v \in \mathbb{R}^d\}, \\ \mathbb{L}_2(P) &:= \{L(\lambda) = \lambda X + Y : (\Lambda^T \otimes I_n) \cdot L(\lambda) = w^T \otimes P(\lambda), w \in \mathbb{R}^d\}, \\ \mathbb{DL}(P) &:= \mathbb{L}_1(P) \cap \mathbb{L}_2(P), \end{aligned}$$

where $\Lambda = [\lambda^{d-1} \ \lambda^{d-2} \ \dots \ \lambda \ 1]^T$. In [78, 79] a characterization via structured Smith forms has been given when even (and other structure preserving) linearizations exist in these classes. In the following we briefly summarize these results.

When $P(\lambda)$ is regular, then the mild hypothesis of a pencil in $\mathbb{L}_j(P)$ being regular is sufficient to guarantee that it is indeed a linearization for P . Regularity makes these pencils strong linearizations for $P(\lambda)$, i.e., the reverse polynomial $\text{rev } L(\lambda)$ with coefficients in the opposite order is also a linearization for $\text{rev } P(\lambda)$. This ensures that the Jordan structures associated to both the finite and infinite eigenvalues of P are always faithfully reflected in L , just as is in the companion forms.

Eigenvectors of $P(\lambda)$ are easily recoverable from those of $L(\lambda)$. Indeed, the definition of $\mathbb{L}_1(P)$ implies that $L(\lambda) \cdot (\Lambda \otimes x) = v \otimes (P(\lambda)x)$ for all $x \in \mathbb{R}^n$. Thus, whenever x is a right eigenvector of $P(\lambda)$ associated to an eigenvalue λ then $\Lambda \otimes x$ is a right eigenvector of $L(\lambda)$ associated to λ . Similar observations hold for $L(\lambda) \in \mathbb{L}_2(P)$ and left eigenvectors and then clearly also in $\mathbb{DL}(P)$. In [76] a simple procedure to construct pencils in $\mathbb{L}_1(P)$ and $\mathbb{L}_2(P)$ has been devised. It has been shown that to obtain linearizations in $\mathbb{DL}(P)$, the left and right ansatz vectors v and w have to be equal. Furthermore, for even linearizations one needs $\Sigma v = v$, where $\Sigma = \text{diag}((-1)^{d-1}, \dots, (-1)^1, (-1)^0)$, and for any such v the resulting pencil in $\mathbb{DL}(P)$ is unique. These results lead to a simple procedure for the construction of even linearizations:

- (1) Choose a right ansatz vector $v \in \mathbb{R}^d$ that satisfies $\Sigma v = v$.

(2) Construct the unique $\tilde{L}(\lambda) \in \mathbb{DL}(P)$ with ansatz vector $w = v$.

For step (2) one uses a canonical basis for $\mathbb{DL}(P)$ derived in [57]. This basis is given by the block diagonal matrices $X_j = \text{diag}(\mathcal{L}_j, -\mathcal{U}_{d-j})$, $j = 0, \dots, d$, whose diagonal blocks are the block $j \times j$ block-Hankel matrices

$$\mathcal{L}_j = \begin{bmatrix} & & & A_k \\ & & \ddots & A_{d-1} \\ & & \ddots & \vdots \\ A_d & A_{d-1} & \dots & A_{d-j+1} \end{bmatrix} \quad \text{and} \quad \mathcal{U}_j = \begin{bmatrix} A_{j-1} & \dots & A_1 & A_0 \\ \vdots & \ddots & \ddots & \\ A_1 & \ddots & & \\ A_0 & & & \end{bmatrix},$$

see also [49, 70, 71, 87].

After constructing these pencils it remains to check whether this really is a linearization. This is the case if for regular $P(\lambda)$ the structured linear pencil is also regular and in [77], for pencils in $\mathbb{DL}(P)$ a criterion has been given that characterizes regularity directly in terms of the ansatz vector. Let $v = [v_1 \ v_2 \ \dots \ v_d]^T$ be the ansatz vector of $L(\lambda) \in \mathbb{DL}(P)$, and define the associated *v-polynomial*

$$p(x, v) := v_1 x^{d-1} + v_2 x^{d-2} + \dots + v_{d-1} x + v_d.$$

By convention, we say that ∞ is a root of $p(x, v)$ if $v_1 = 0$. Then regularity of $L(\lambda) \in \mathbb{DL}(P)$ can be expressed in terms of the roots of this *v-polynomial* and the eigenvalues of $P(\lambda)$, as follows.

Theorem 4.1. [76] *Suppose the regular matrix polynomial $P(\lambda)$ and the nonzero pencil $L(\lambda) \in \mathbb{L}_1(P)$ with nonzero right ansatz vector v , are even. Then $L(\lambda)$ is a (strong) linearization for $P(\lambda)$ if and only if no root of the *v-polynomial* $p(x, v)$ is an eigenvalue of $P(\lambda)$.*

It has been shown in [58], that the condition number of an eigenvalue of $L(\lambda)$ is inverse proportional to the value of the *v-polynomial* evaluated at λ . This result allows to choose ansatz vectors with *v-polynomials* that lead to well conditioned eigenvalues.

Example 4.2. Even linearizations for the quadratic even polynomial $P(\lambda) = \lambda^2 A_2 + \lambda A_1 + A_0$ have $v = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and take the form

$$\lambda \begin{bmatrix} 0 & -A_2 \\ A_2 & A_1 \end{bmatrix} + \begin{bmatrix} A_2 & 0 \\ 0 & A_0 \end{bmatrix}.$$

The *v-polynomial* $p(x, v)$ has the root ∞ . So if $P(\lambda)$ has an infinite eigenvalue then there do not exist even linearizations in $\mathbb{L}_1(P)$. One could instead consider the reverse polynomial $\text{rev } P(\lambda)$, which is also even and this would make the eigenvalue 0 the exceptional case. But if both 0 and ∞ are eigenvalues of $P(\lambda)$ then there simply are no even linearizations in $\mathbb{L}_1(P)$, $L_2(P)$, or $\mathbb{DL}(P)$. For more details see [76, 78].

In the case that even linearizations do not exist, it is necessary to first deflate the critical eigenvalues that prevent these linearizations, see [39]. If, however, a structured linearization is possible, then the simplest method is to construct the linearization and then use the structure-preserving methods for matrix pencils.

After having reviewed some of the basic results on even pencils and their linearizations, we recall some properties of descriptor systems in the next section.

5 Descriptor Systems and Their Properties

In this section we give a short introduction to continuous-time linear time-invariant descriptor systems, pointing out some general concepts and properties of this system class that will play important roles in the forthcoming sections. Furthermore, we summarize numerical methods for checking these properties.

A continuous-time linear time-invariant descriptor system is a dynamical system of the form

$$E\dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0, \quad (8)$$

$$y(t) = Cx(t) + Du(t), \quad (9)$$

with matrices $E, A \in \mathbb{R}^{k,n}$, $B \in \mathbb{R}^{k,m}$, $C \in \mathbb{R}^{p,n}$ and $D \in \mathbb{R}^{p,m}$. Furthermore, $x : [0, \infty) \rightarrow \mathbb{R}^n$ represents the state, $u : [0, \infty) \rightarrow \mathbb{R}^m$ denotes a control input signal, and $y : [0, \infty) \rightarrow \mathbb{R}^p$ is the output signal. The general solution theory for descriptor systems is much more involved than that for standard ODE systems [63, 96], however, this will not play an important role in this paper. Here we focus on properties like controllability, stabilizability and the related dual concepts observability and detectability. For brevity we only introduce these for the case of square systems, i.e., for $k = n$. The general case is covered by [22, 41]. Also, instead of defining them in system theoretic terms we directly introduce equivalent algebraic characterizations. These are very useful for numerically checking these properties. Note that there are several different concepts of controllability at infinity introduced in [91, 99] and compared in [21, 22, 35, 43].

Definition 5.1. Let $E, A \in \mathbb{R}^{n,n}$, $B \in \mathbb{R}^{n,m}$, $C \in \mathbb{R}^{p,n}$. Furthermore, let T_∞, S_∞ be matrices with $\text{range } T_\infty = \ker E^T$ and $\text{range } S_\infty = \ker E$. Then, the triple (E, A, B) is called

- (i) *finite dynamics stabilizable* if $\text{rank} \begin{bmatrix} \lambda E - A & B \end{bmatrix} = n$ for all $\lambda \in \mathbb{C}^+$;
- (ii) *finite dynamics controllable* if $\text{rank} \begin{bmatrix} \lambda E - A & B \end{bmatrix} = n$ for all $\lambda \in \mathbb{C}$;
- (iii) *impulse controllable* if $\text{rank} \begin{bmatrix} E & AS_\infty & B \end{bmatrix} = n$;
- (iv) *strongly stabilizable* if it is both finite dynamics stabilizable and impulse controllable;
- (v) *strongly controllable* if it is both finite dynamics controllable and impulse controllable;
- (vi) *completely controllable* if it is both finite dynamics controllable and $\text{rank} \begin{bmatrix} E & B \end{bmatrix} = n$.

In a dual fashion, the triple (E, A, C) is called

- (vii) *finite dynamics detectable* if $\text{rank} \begin{bmatrix} \lambda E^T - A^T & C^T \end{bmatrix} = n$ for all $\lambda \in \mathbb{C}^+$;
- (viii) *finite dynamics observable* if $\text{rank} \begin{bmatrix} \lambda E^T - A^T & C^T \end{bmatrix} = n$ for all $\lambda \in \mathbb{C}$;
- (ix) *impulse observable* if $\text{rank} \begin{bmatrix} E^T & A^T T_\infty & C^T \end{bmatrix} = n$;
- (x) *strongly detectable* if it is both finite dynamics detectable and impulse observable;
- (xi) *strongly observable* if it is both finite dynamics observable and impulse observable;
- (xii) *completely observable* if it is both finite dynamics observable and $\text{rank} \begin{bmatrix} E^T & C^T \end{bmatrix} = n$.

To check these conditions one can use the condensed form of [35, 36].

Theorem 5.2. [36] *If $E, A \in \mathbb{R}^{n,n}$, $B \in \mathbb{R}^{n,m}$, $C \in \mathbb{R}^{p,n}$, then there exist orthogonal matrices*

$U, V \in \mathbb{R}^{n,n}$, $W \in \mathbb{R}^{m,m}$, $Y \in \mathbb{R}^{p,p}$ such that

$$\begin{aligned}
U^T E V &= \begin{matrix} & t_1 & n-t_1 \\ t_1 & \left[\begin{array}{cc} \Sigma_E & 0 \\ 0 & 0 \end{array} \right] \\ n-t_1 & \end{matrix}, \\
U^T A V &= \begin{matrix} & t_1 & s_2 & t_5 & t_4 & t_3 & s_6 \\ t_1 & \left[\begin{array}{cccccc} A_{11} & A_{12} & A_{13} & A_{14} & A_{15} & A_{16} \\ A_{21} & A_{22} & A_{23} & A_{24} & 0 & 0 \\ A_{31} & A_{32} & A_{33} & A_{34} & \Sigma_{35} & 0 \\ A_{41} & A_{42} & A_{43} & \Sigma_{44} & 0 & 0 \\ A_{51} & 0 & \Sigma_{53} & 0 & 0 & 0 \\ A_{61} & 0 & 0 & 0 & 0 & 0 \end{array} \right] \\ t_2 & \\ t_3 & \\ t_4 & \\ t_5 & \\ t_6 & \end{matrix}, \\
U^T B W &= \begin{matrix} & k_1 & k_2 & k_3 \\ t_1 & \left[\begin{array}{ccc} B_{11} & B_{12} & 0 \\ B_{21} & 0 & 0 \\ B_{31} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array} \right] \\ t_2 & \\ t_3 & \\ t_4 & \\ t_5 & \\ t_6 & \end{matrix}, \\
Y^T C V &= \begin{matrix} & t_1 & s_2 & t_5 & t_4 & t_3 & s_6 \\ l_1 & \left[\begin{array}{cccccc} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ C_{21} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right] \\ l_2 & \\ l_3 & \end{matrix}.
\end{aligned} \tag{10}$$

The matrices $\Sigma_E, \Sigma_{35}, \Sigma_{44}, \Sigma_{53}$ are nonsingular diagonal matrices, B_{12} has full column rank, C_{21} has full row rank and the matrices

$$\begin{bmatrix} B_{21} \\ B_{31} \end{bmatrix} \in \mathbb{R}^{k_1, k_1}, \quad [C_{12} \ C_{13}] \in \mathbb{R}^{l_1, l_1},$$

with $k_1 = t_2 + t_3$ and $l_1 = s_2 + t_5$ are nonsingular.

Impulse controllability and observability and some of the other properties can be checked via the following corollary.

Corollary 5.3. [36] Let E, A, B, C be given as in the condensed form (10).

- (i) The triple (E, A, B) is impulse controllable if and only if $t_6 = 0$, i.e., the last block row of A is void.
- (ii) The triple (E, A, C) is impulse observable if and only if $s_6 = 0$, i.e., the last block column of A is void.
- (iii) The condition $\text{rank} [E \ B] = n$ is satisfied if and only if $t_4 = t_5 = t_6 = 0$.
- (iv) The condition $\text{rank} [E^T \ C^T] = n$ is satisfied if and only if $t_4 = t_3 = s_6 = 0$.
- (v) The triple (E, A, B) is completely controllable if and only if $t_4 = t_5 = t_6 = 0$ and the system is finite dynamics controllable.
- (vi) The triple (E, A, C) is completely observable if and only if $t_4 = t_3 = s_6 = 0$ and the system is finite dynamics observable.

When the system is not impulse controllable, then it is possible to perform a regularization of the system, see [37]. If one partitions the transformed state vector $V^T x = [x_1^T \ \dots \ x_6^T]^T$ conformal to the condensed form (10) then one sees that $x_1 = 0$ and $x_3 = 0$ and the last two block rows can be removed. The remaining subsystem of the first four block rows is then impulse controllable. If the system is not impulse observable then this is critical because this cannot be achieved by removing equations and variables. In this case, impulses in the solution cannot be seen and this is an indication of a modeling error, see [37]. In some of the applications that we discuss below, the solvability depends on these properties and a regularization process or an alternative modeling process should ensure that these properties are satisfied.

If the system is impulse controllable and impulse observable, then the other properties, i.e., that the system is finite dynamics controllable or stabilizable and finite dynamics observable or detectable can be checked via the following *controllability/observability decompositions*, see [44, 97, 98]. Let Q_c, Z_c be real orthogonal matrices, such that

$$\begin{aligned} Q_c^T E Z_c &= \begin{bmatrix} E_c & * \\ 0 & E_{nc} \end{bmatrix}, & Q_c^T A Z_c &= \begin{bmatrix} A_c & * \\ 0 & A_{nc} \end{bmatrix}, \\ Q_c^T B &= \begin{bmatrix} B_c \\ 0 \end{bmatrix}, & C Z_c &= [C_c \ C_{nc}], \end{aligned} \quad (11)$$

where the subsystem given by the matrices E_c, A_c, B_c, C_c contains the *controllable subsystem* of the original system, i.e., the triple (E_c, A_c, B_c) is finite dynamics controllable. If the subpencil $\lambda E_{nc} - A_{nc}$ corresponding to the uncontrollable part of the system has no finite eigenvalues with positive real part, then the system is finite dynamics stabilizable, otherwise it is not.

Similarly, one can determine an *observability decomposition*

$$\begin{aligned} Q_o^T E Z_o &= \begin{bmatrix} E_o & 0 \\ * & E_{no} \end{bmatrix}, & Q_o^T A Z_o &= \begin{bmatrix} A_o & 0 \\ * & A_{no} \end{bmatrix}, \\ Q_o^T B &= \begin{bmatrix} B_o \\ B_{no} \end{bmatrix}, & C Z_o &= [C_o \ 0], \end{aligned} \quad (12)$$

where Q_o, Z_o are orthogonal matrices and the subsystem given by the matrices E_o, A_o, B_o, C_o contains the *observable subsystem* of the original system, i.e., (E_o, A_o, C_o) is finite dynamics observable. If the subpencil $\lambda E_{no} - A_{no}$ corresponding to the unobservable part of the system has no finite eigenvalues with positive real part, then the system is finite dynamics detectable, otherwise it is not. Methods for the computation of these decompositions are described in [98] and implemented as TG01HD, TG01ID in the SLICOT library.

For many applications, in particular those where the influence of the inputs to the outputs is crucial, it is not suitable to analyze the descriptor system in the time domain, i.e., in the form (8)–(9). Instead, one turns to the frequency domain. For this, assume that the system is square and that the pencil $\lambda E - A$ is regular. Then, under the assumption that $Ex(0) = 0$ we can apply the Laplace transformation to the functions $x(\cdot), u(\cdot)$, and $y(\cdot)$ and obtain the transfer function

$$G(s) := C(sE - A)^{-1}B + D, \quad (13)$$

that directly maps the Laplace transformed inputs to the Laplace transformed outputs [44]. These transfer functions are typically associated to certain Banach spaces. Consider the Banach spaces $\mathcal{H}_\infty^{p,m}$ and $\mathcal{L}_\infty^{p,m}$ of all $\mathbb{C}^{p,m}$ -valued functions that are analytic and bounded in the open right half-plane \mathbb{C}^+ ; and bounded on the imaginary axis $i\mathbb{R}$, respectively. Obviously, the inclusion $\mathcal{H}_\infty^{p,m} \subset \mathcal{L}_\infty^{p,m}$ holds. For $G \in \mathcal{L}_\infty^{p,m}$, the \mathcal{L}_∞ -norm is given by

$$\|G\|_{\mathcal{L}_\infty} = \sup_{\omega \in \mathbb{R}} \sigma_{\max}(G(i\omega)),$$

where $\sigma_{\max}(\cdot)$ denotes the maximum singular value. For $G \in \mathcal{H}_\infty^{p,m}$, the \mathcal{L}_∞ -norm is equal to the \mathcal{H}_∞ -norm. These norms play an important role in many applications, in particular as robustness measures in robust control. Details on this will be pointed out in Sections 7 and 8.

After briefly introducing the basic concepts, some of the system theoretic properties and numerical methods to check these properties, we now turn to several important applications in control theory.

6 Linear-Quadratic Optimal Control

In this section we consider the linear quadratic optimal control problem of minimizing

$$\mathcal{J}(x(\cdot), u(\cdot)) = \frac{1}{2} \int_0^\infty (x(t)^T Q x(t) + 2x(t)^T S u(t) + u(t)^T R u(t)) dt \quad (14)$$

subject to the *square* linear descriptor system of the form (8) with the stabilization condition $\lim_{t \rightarrow \infty} E x(t) = 0$, and with $Q = Q^T \in \mathbb{R}^{n,n}$, $S \in \mathbb{R}^{n,m}$, and $R = R^T \in \mathbb{R}^{m,m}$. If an output equation (9) is also given, then the cost functional is usually given as $\mathcal{J}(y(\cdot), u(\cdot))$ which can then easily be transformed to the form given in (14) by inserting the output equation. This yields

$$\tilde{\mathcal{J}}(x(\cdot), u(\cdot)) = \frac{1}{2} \int_0^\infty (x(t)^T \tilde{Q} x(t) + 2x(t)^T \tilde{S} u(t) + u(t)^T \tilde{R} u(t)) dt$$

with

$$\tilde{Q} := C^T Q C, \quad \tilde{S} := C^T Q D + C^T S, \quad \tilde{R} := D^T Q D + D^T S + S^T D + R. \quad (15)$$

Optimal control problems for equations of this form arise in mechanical multibody systems [53, 54, 93], electrical circuits [52] and many other applications like the linearization of general nonlinear systems along stationary trajectories [40].

To solve this problem in the most general situation, it has been shown in [41, 64] how to replace the DAE constraint by a so called *strangeness-free formulation*

$$\hat{E} \dot{x}(t) = \hat{A} x(t) + \hat{B} u(t), \quad (16)$$

where

$$\hat{E} = \begin{bmatrix} \hat{E}_1 \\ 0 \end{bmatrix}, \quad \hat{A} = \begin{bmatrix} \hat{A}_1 \\ \hat{A}_2 \end{bmatrix}, \quad \hat{B} = \begin{bmatrix} \hat{B}_1 \\ \hat{B}_2 \end{bmatrix},$$

with the additional property that the matrix

$$\begin{bmatrix} \hat{E}_1 & 0 \\ \hat{A}_2 & \hat{B}_2 \end{bmatrix}$$

has full row rank. The necessary optimality system is then given by

$$\begin{bmatrix} 0 & \hat{E} & 0 \\ -\hat{E}^T & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \frac{d}{dt} \begin{bmatrix} \lambda(t) \\ x(t) \\ u(t) \end{bmatrix} = \begin{bmatrix} 0 & \hat{A} & \hat{B} \\ \hat{A}^T & -Q & -S \\ \hat{B}^T & -S^T & -R \end{bmatrix} \begin{bmatrix} \lambda(t) \\ x(t) \\ u(t) \end{bmatrix}, \quad (17)$$

with boundary conditions $\hat{E} x(0) = \hat{E} x_0$, and $\lim_{t \rightarrow \infty} \hat{E}^T \lambda(t) = 0$. Solving this system will give the optimal input $u(\cdot)$, state $x(\cdot)$, and the Lagrange multiplier $\lambda(\cdot)$.

Instead of first computing a strangeness-free formulation and forming the optimality system (17), we can instead directly form and solve the *formal optimality system* [7, 41, 64, 65, 69] given by

$$\begin{bmatrix} 0 & E & 0 \\ -E^T & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \frac{d}{dt} \begin{bmatrix} \tilde{\lambda}(t) \\ x(t) \\ u(t) \end{bmatrix} = \begin{bmatrix} 0 & A & B \\ A^T & -Q & -S \\ B^T & -S^T & -R \end{bmatrix} \begin{bmatrix} \tilde{\lambda}(t) \\ x(t) \\ u(t) \end{bmatrix}, \quad (18)$$

with boundary conditions $E x(0) = E x_0$, and $\lim_{t \rightarrow \infty} E^T \tilde{\lambda}(t) = 0$. One has the following relation between the true and the formal optimality system which we cite here for constant coefficient systems, for the general case of variable coefficient systems see [65].

Theorem 6.1. *Suppose that the formal necessary optimality system (18) has a solution $\begin{bmatrix} \tilde{\lambda}(\cdot)^T & x(\cdot)^T & u(\cdot)^T \end{bmatrix}^T$. Then, there exists a function $\lambda(\cdot)$ replacing $\tilde{\lambda}(\cdot)$ such that the function $\begin{bmatrix} \lambda(\cdot)^T & x(\cdot)^T & u(\cdot)^T \end{bmatrix}^T$ solves the necessary optimality conditions (17).*

Theorem 6.1 shows that it is enough to solve the boundary value problem (18) in the original data, provided it is solvable. Since this is a homogeneous differential-algebraic system, the solvability of the boundary value problem depends on the consistency of the boundary conditions and the solvability of the linear system that relates initial and terminal conditions, see [5, 67, 68]. Since the boundary value problem is of the form

$$N\dot{z}(t) = Mz(t), \quad P_1z(0) = P_1z_0, \quad \lim_{t \rightarrow \infty} P_2z(t) = 0,$$

with $z(\cdot) = [\lambda(\cdot)^T \quad x(\cdot)^T \quad u(\cdot)^T]^T$, and some matrices P_1 , and P_2 , the simplest way to perform these computations is to apply the congruence transformation to even staircase form

$$U^T N U \dot{\tilde{z}}(t) = U^T M U \tilde{z}(t), \quad P_1 U \tilde{z}(0) = P_1 U \tilde{z}_0, \quad \lim_{t \rightarrow \infty} P_2 U \tilde{z}(t) = 0,$$

with $\tilde{z}(\cdot) = U^T z(\cdot)$, and $\tilde{z}_0 = U^T z_0$.

This allows to check the unique solvability by checking the regularity as in Corollary 2.3 and the consistency of the boundary conditions, see [38] for details. By partitioning $\tilde{z}(\cdot) = [\tilde{z}_1(\cdot)^T, \dots, \tilde{z}_{2w+1}(\cdot)^T]^T$ analogous to (3), the last w blocks yield the consistency conditions $\tilde{z}_1(\cdot) \equiv 0, \dots, \tilde{z}_w(\cdot) \equiv 0$. The middle block system can be expressed as

$$N_{w+1, w+1} \dot{\tilde{z}}_{w+1}(t) = M_{w+1, w+1} \tilde{z}_{w+1}(t), \quad (19)$$

with appropriately transformed boundary conditions. This system is regular and has index at most one. If we make use of the semi-explicit form (4) and split

$$\tilde{z}_{w+1}(\cdot) = \begin{bmatrix} \xi(\cdot) \\ \zeta(\cdot) \end{bmatrix},$$

then we obtain

$$\begin{bmatrix} \Delta & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{\xi}(t) \\ \dot{\zeta}(t) \end{bmatrix} = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix} \begin{bmatrix} \xi(t) \\ \zeta(t) \end{bmatrix}.$$

It follows that $\zeta(\cdot) = -\Sigma_{22}^{-1} \Sigma_{21} \xi(\cdot)$, which gives further consistency conditions on $\tilde{z}_{w+1}(\cdot)$ and

$$\Delta \dot{\xi}(t) = (\Sigma_{11} - \Sigma_{12} \Sigma_{22}^{-1} \Sigma_{21}) \xi(t).$$

Then we can perform a factorization $\Delta = U^T \mathcal{J} U$ with nonsingular upper triangular matrix U [34]. If the factorization is well-conditioned and the factor U is well-conditioned with respect to inversion, then we set $H := \mathcal{J}^T U^{-T} (\Sigma_{11} - \Sigma_{12} \Sigma_{22}^{-1} \Sigma_{21}) U^{-1}$ to obtain the Hamiltonian boundary value problem

$$\dot{\xi}(t) = H \xi(t). \quad (20)$$

with appropriate boundary conditions $\Pi_1 \xi(0) = \Pi_1 \xi_0$, and $\lim_{t \rightarrow \infty} \Pi_2 \xi(t) = 0$. This system has the general solution $\xi(t) = \exp(Ht) \xi_0$ and therefore,

$$\tilde{z}_{w+1}(t) = \begin{bmatrix} \exp(Ht) \xi_0 \\ -\Sigma_{22}^{-1} \Sigma_{21} \exp(Ht) \xi_0 \end{bmatrix}. \quad (21)$$

It is important to note that one does not have to compute the exponential function in (21) but one rather uses a transformation to Hamiltonian Schur form [82] if it exists. Therefore, assume that there exists an orthogonal symplectic matrix $V \in \mathbb{R}^{r_{w+1}, r_{w+1}}$ such that

$$V^T H V = \begin{bmatrix} H_{11} & H_{12} \\ 0 & -H_{11}^T \end{bmatrix},$$

where H_{11} is upper quasi-triangular with all eigenvalues in the open left half-plane and H_{12} is symmetric.

If a Hamiltonian Schur form exists, then the boundary value problem (20) decouples into

$$\begin{bmatrix} \tilde{\xi}_1(t) \\ \tilde{\xi}_2(t) \end{bmatrix} = \begin{bmatrix} H_{11} & H_{12} \\ 0 & -H_{11}^T \end{bmatrix} \begin{bmatrix} \tilde{\xi}_1(t) \\ \tilde{\xi}_2(t) \end{bmatrix}, \quad \text{with} \quad V^T \xi(t) = \begin{bmatrix} \tilde{\xi}_1(t) \\ \tilde{\xi}_2(t) \end{bmatrix} =: \tilde{\xi}(t),$$

with appropriately transformed boundary conditions $\tilde{\xi}_1(0) = \tilde{\xi}_{1,0}$, and $\lim_{t \rightarrow \infty} \tilde{\xi}_2(t) = 0$. Since $-H_{11}^T$ is an unstable matrix, we obtain $\tilde{\xi}_2(\cdot) \equiv 0$ by backwards integration. This results in

$$\dot{\tilde{\xi}}_1(t) = H_{11} \tilde{\xi}_1(t), \quad \tilde{\xi}_1(0) = \tilde{\xi}_{1,0},$$

which can now be efficiently solved due to the quasi triangular structure of H_{11} . From that we can easily reconstruct $\tilde{z}_{w+1}(\cdot)$, given by

$$\tilde{z}_{w+1}(t) = \begin{bmatrix} V \tilde{\xi}(t) \\ -\Sigma_{22}^{-1} \Sigma_{21} V \tilde{\xi}(t) \end{bmatrix}.$$

This can be used to determine $\tilde{z}_{w+2}(\cdot), \dots, \tilde{z}_{2w+1}(\cdot)$ in terms of $\tilde{z}_{w+1}(\cdot)$, and the consistency conditions $\tilde{z}_1(\cdot) \equiv 0, \dots, \tilde{z}_w(\cdot) \equiv 0$ via a backward substitution process applied to the first w block rows of (3). This recursive process leads to

$$\tilde{z}_{w+j+1}(t) = \Gamma_{w-j+1}^{-1} \left(\sum_{i=w+1}^{w+j} N_{w-j+1,i} \dot{\tilde{z}}_i(t) - \sum_{i=w+1}^{w+j} M_{w-j+1,i} \tilde{z}_i(t) \right),$$

which requires w differentiations to be carried out, see [38]. The complete procedure is graphically displayed in 2.

Remark 6.2. A similar decoupling procedure can also be constructed in the finite-time horizon problem by decoupling the forward and backward integration via the solution of a Riccati differential equation or by using other boundary value methods [5].

7 \mathcal{H}_∞ Optimal Control

Our second application is the \mathcal{H}_∞ optimal control problem which is one of the major tasks in robust control. We consider descriptor system of the form

$$\mathbf{P} : \begin{cases} E \dot{x}(t) = Ax(t) + B_1 v(t) + B_2 u(t), & x(0) = x_0, \\ z(t) = C_1 x(t) + D_{11} v(t) + D_{12} u(t), \\ y(t) = C_2 x(t) + D_{21} v(t) + D_{22} u(t), \end{cases} \quad (22)$$

where $E, A \in \mathbb{R}^{n,n}$, $B_i \in \mathbb{R}^{n,m_i}$, $C_i \in \mathbb{R}^{p_i,n}$, and $D_{ij} \in \mathbb{R}^{p_i,m_j}$ for $i, j = 1, 2$. In this system, $x : [0, \infty) \rightarrow \mathbb{R}^n$ is the state, $u : [0, \infty) \rightarrow \mathbb{R}^{m_2}$ is the control input, and $v : [0, \infty) \rightarrow \mathbb{R}^{m_1}$ is an exogenous input that may include noise, linearization errors and unmodeled dynamics. The function $y : [0, \infty) \rightarrow \mathbb{R}^{p_2}$ contains measured outputs, while $z : [0, \infty) \rightarrow \mathbb{R}^{p_1}$ is a regulated output or an estimation error.

The \mathcal{H}_∞ optimal control problem is typically formulated in the frequency domain. Its goal is to stabilize the system, while minimizing the \mathcal{H}_∞ -norm of the closed loop transfer function $T_{zv}(\cdot)$ mapping noise or disturbance to error signals [105], is minimized. The value of $\|T_{zv}\|_{\mathcal{H}_\infty}$ is used as a measure for the worst-case influence of the disturbances v on the output z . A more rigorous formulation is given in the following definition [75].

Definition 7.1 (The optimal \mathcal{H}_∞ control problem). For the descriptor system (22), determine a controller (dynamic compensator)

$$\mathbf{K} : \begin{cases} \hat{E} \dot{\hat{x}}(t) = \hat{A} \hat{x}(t) + \hat{B} y(t), \\ u(t) = \hat{C} \hat{x}(t) + \hat{D} y(t), \end{cases} \quad (23)$$

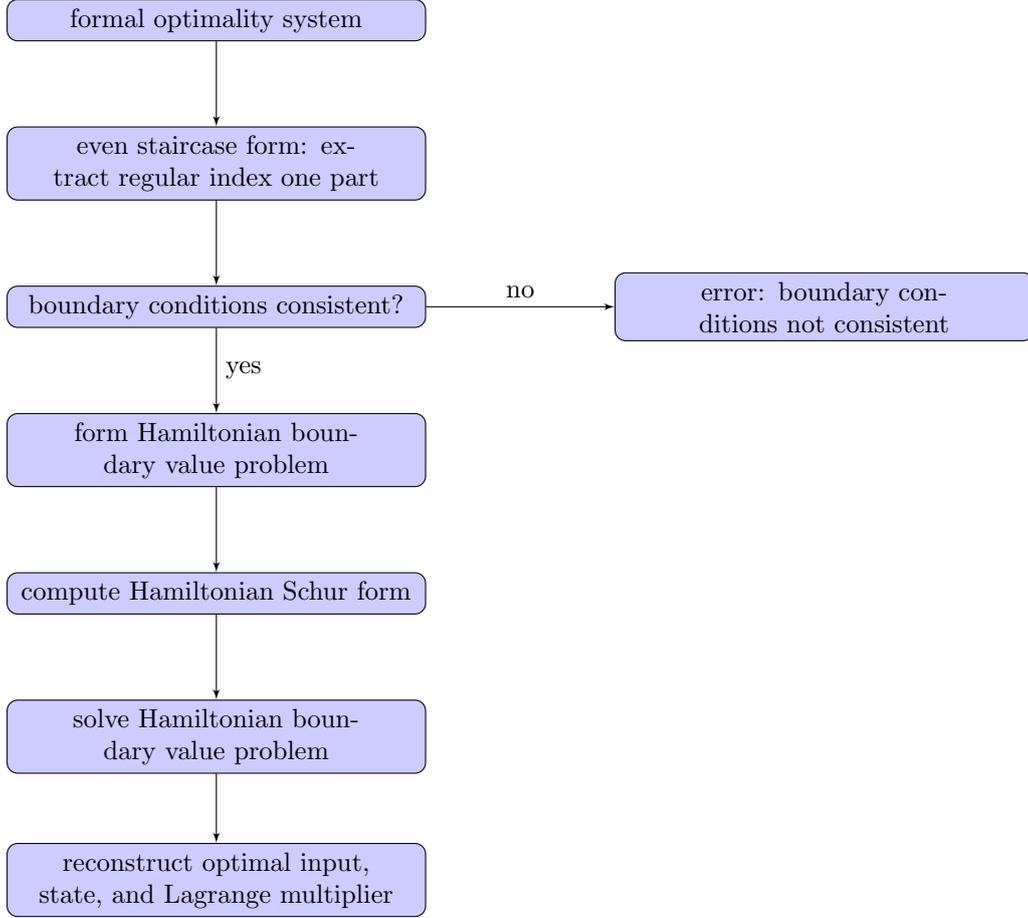


Figure 2: Algorithm flowchart for solving linear-quadratic optimal control problems

with $\hat{E}, \hat{A} \in \mathbb{R}^{N,N}$, $\hat{B} \in \mathbb{R}^{N,p_2}$, $\hat{C} \in \mathbb{R}^{m_2,N}$, $\hat{D} \in \mathbb{R}^{m_2,p_2}$, and transfer function $K(s) = \hat{C}(s\hat{E} - \hat{A})^{-1}\hat{B} + \hat{D}$ such that the closed-loop system resulting from the combination of (22) and (23), that is given by

$$\begin{aligned}
 E\dot{x}(t) &= \left(A + B_2\hat{D}Z_1C_2\right)x(t) + B_2Z_2\hat{C}\hat{x}(t) + \left(B_1 + B_2\hat{D}Z_1D_{21}\right)v(t), \\
 \hat{E}\dot{\hat{x}}(t) &= \hat{B}Z_1C_2x(t) + \left(\hat{A} + \hat{B}Z_1D_{22}\hat{C}\right)\hat{x}(t) + \hat{B}Z_1D_{21}v(t), \\
 z(t) &= \left(C_1 + D_{12}Z_2\hat{D}C_2\right)x(t) + D_{12}Z_2\hat{C}\hat{x}(t) + \left(D_{11} + D_{12}\hat{D}Z_1D_{21}\right)v(t)
 \end{aligned} \tag{24}$$

with $Z_1 = \left(I_{p_2} - D_{22}\hat{D}\right)^{-1}$ and $Z_2 = \left(I_{m_2} - \hat{D}D_{22}\right)^{-1}$, has the following properties:

- 1.) System (24) is *internally stable*, i.e., the solution $\begin{bmatrix} x(\cdot) \\ \hat{x}(\cdot) \end{bmatrix}$ of the system with $v \equiv 0$ is *asymptotically stable*, in other words $\lim_{t \rightarrow \infty} \begin{bmatrix} x(t) \\ \hat{x}(t) \end{bmatrix} = 0$.
- 2.) The closed-loop transfer function $T_{zv}(\cdot)$ from v to z satisfies $T_{zv} \in \mathcal{H}_\infty^{p_1, m_1}$ and is minimized in the \mathcal{H}_∞ -norm.

Such an interconnection of a system with a controller is depicted in Figure 3. Solving the optimal

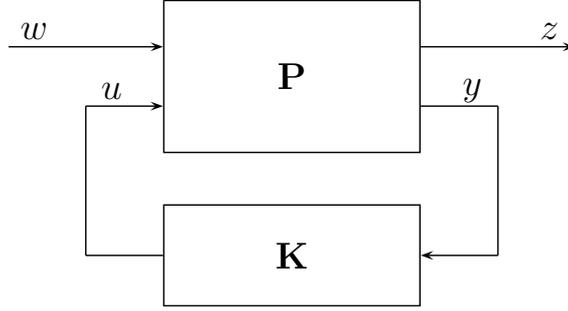


Figure 3: Interconnection of a system \mathbf{P} with a controller \mathbf{K}

\mathcal{H}_∞ control problem by trying to directly minimize the \mathcal{H}_∞ -norm of $T_{zv}(\cdot)$ over the complicated set of internally stabilizing controllers proves difficult or impossible by conventional optimization methods, since it is often unclear if a minimizing controller exists [105] and if one exists, it is typically not unique, there even exist infinitely many. So usually one studies two closely related optimization problems, the *modified optimal \mathcal{H}_∞ control problem* and the *suboptimal \mathcal{H}_∞ control problem* [12, 105].

Definition 7.2 (The modified optimal \mathcal{H}_∞ control problem). For the descriptor system (22), let Γ be the set of positive real numbers γ for which there exists an internally stabilizing dynamic controller of the form (23) so that the transfer function $T_{zv}(\cdot)$ of the closed loop system (24) satisfies $T_{zv} \in \mathcal{H}_\infty^{p_1, m_1}$ with $\|T_{zv}\|_{\mathcal{H}_\infty} < \gamma$. Determine $\gamma_{mo} = \inf \Gamma$. If no internally stabilizing dynamic controller exists, we set $\Gamma = \emptyset$ and $\gamma_{mo} = \infty$.

This problem is usually solved by an iterative process, which is often called the γ -iteration.

Definition 7.3 (The suboptimal \mathcal{H}_∞ control problem). For the descriptor system (22) and $\gamma \in \Gamma$ with $\gamma > \gamma_{mo}$, determine an internally stabilizing dynamic controller of the form (23) such that the closed loop transfer function satisfies $T_{zv} \in \mathcal{H}_\infty^{p_1, m_1}$ with $\|T_{zv}\|_{\mathcal{H}_\infty} < \gamma$. We call such a controller γ -suboptimal controller or simply *suboptimal controller*.

To obtain an existence and uniqueness result we make the following assumptions:

A1) The triple (E, A, B_2) is strongly stabilizable and the triple (E, A, C_2) is strongly detectable.

A2) $\text{rank} \begin{bmatrix} A - i\omega E & B_2 \\ C_1 & D_{12} \end{bmatrix} = n + m_2$ for all $\omega \in \mathbb{R}$.

A3) $\text{rank} \begin{bmatrix} A - i\omega E & B_1 \\ C_2 & D_{21} \end{bmatrix} = n + p_2$ for all $\omega \in \mathbb{R}$.

A4) With matrices $T_\infty, S_\infty \in \mathbb{R}^{n, n-r}$ satisfying $\text{range } S_\infty = \ker E$ and $\text{range } T_\infty = \ker E^T$ and $r := \text{rank } E$ we have

$$\begin{aligned} \text{rank} \begin{bmatrix} T_\infty^T A S_\infty & T_\infty^T B_2 \\ C_1 S_\infty & D_{12} \end{bmatrix} &= n + m_2 - r, \\ \text{rank} \begin{bmatrix} T_\infty^T A S_\infty & T_\infty^T B_1 \\ C_2 S_\infty & D_{21} \end{bmatrix} &= n + p_1 - r. \end{aligned}$$

In Assumption **A1)** the condition of impulse controllability and observability is necessary to avoid impulsive solutions which cannot be controlled or observed. To check these conditions one can use the condensed forms of Theorem 5.2 with the characterization of Corollary 5.3. The property that the system is finite dynamics stabilizable and detectable is necessary for the existence of an

internally stabilizing controller. To verify that these conditions we use the decompositions (11) and (12) which can be computed via the codes TG01HD, TG01ID in the SLICOT library. These routines can also be used to check **A2)** and **A3)**.

To verify that assumption **A4)** is satisfied, we check that the ranks of the extended matrices fulfill

$$\text{rank} \begin{bmatrix} 0 & E & 0 \\ E^T & A & B_2 \\ 0 & C_1 & D_{12} \end{bmatrix} = n + m_2 + r,$$

and

$$\text{rank} \begin{bmatrix} 0 & E & 0 \\ E^T & A & B_1 \\ 0 & C_2 & D_{21} \end{bmatrix} = n + p_1 + r.$$

This check is performed by applying a rank-revealing QR (RRQR) decomposition [51]. The corresponding routine DGEQP3 is available in LAPACK¹. For details on the implementation we refer to [24, 25, 47].

Once we have assured that the assumptions **A1) – A4)** hold, we can form the two even matrix pencils

$$\lambda N_H - M_H(\gamma) = \left[\begin{array}{cc|ccc} 0 & -\lambda E^T - A^T & 0 & 0 & -C_1^T \\ \lambda E - A & 0 & -B_1 & -B_2 & 0 \\ \hline 0 & -B_1^T & -\gamma^2 I_{m_1} & 0 & -D_{11}^T \\ 0 & -B_2^T & 0 & 0 & -D_{12}^T \\ -C_1 & 0 & -D_{11} & -D_{12} & -I_{p_1} \end{array} \right], \quad (25)$$

and

$$\lambda N_J - M_J(\gamma) = \left[\begin{array}{cc|ccc} 0 & -\lambda E - A & 0 & 0 & -B_1 \\ \lambda E^T - A^T & 0 & -C_1^T & -C_2^T & 0 \\ \hline 0 & -C_1 & -\gamma^2 I_{p_1} & 0 & -D_{11} \\ 0 & -C_2 & 0 & 0 & -D_{21} \\ -B_1^T & 0 & -D_{11}^T & -D_{21}^T & -I_{m_1} \end{array} \right]. \quad (26)$$

We determine the deflating subspaces of both pencils associated to the eigenvalues in the closed left half complex plane and check whether the dimension of both subspaces is $r = \text{rank } E$. Suppose that these subspaces are spanned by the columns of the matrices

$$X_H(\gamma) = \begin{bmatrix} X_{H,1}(\gamma) \\ X_{H,2}(\gamma) \\ X_{H,3}(\gamma) \\ X_{H,4}(\gamma) \\ X_{H,5}(\gamma) \end{bmatrix}, \quad X_J(\gamma) = \begin{bmatrix} X_{J,1}(\gamma) \\ X_{J,2}(\gamma) \\ X_{J,3}(\gamma) \\ X_{J,4}(\gamma) \\ X_{J,5}(\gamma) \end{bmatrix},$$

which are partitioned according to the block structure of the pencils $\lambda N_H - M_H$ and $\lambda N_J - M_J$.

We use the following result to solve the modified optimal \mathcal{H}_∞ control problem.

Theorem 7.4. [75] *Consider system (22) and the even pencils $\lambda N_H - M_H(\gamma)$ and $\lambda N_J - M_J(\gamma)$ as in (25) and (26), respectively. Suppose that assumptions **A1) – A4)** hold.*

*Then there exists an internally stabilizing controller such that the transfer function from v to z satisfies $T_{zv} \in \mathcal{H}_\infty^{p_1, m_1}$ with $\|T_{zv}\|_{\mathcal{H}_\infty} < \gamma$ if and only if γ is such that the following conditions **C1) – C4)** hold.*

C1) *The index of both pencils (25) and (26) is at most one.*

C2) *There exists a matrix $X_H(\gamma)$ such that*

C2.a) *the space $\text{range } X_H(\gamma)$ is a semi-stable deflating subspace of $\lambda N_H - M_H(\gamma)$ and*

$\text{range} \begin{bmatrix} E X_{H,1}(\gamma) \\ X_{H,2}(\gamma) \end{bmatrix}$ is an r -dimensional isotropic subspace of \mathbb{R}^{2n} ;

¹<http://www.netlib.org/lapack/>

C2.b) $\text{rank } EX_{H,1}(\gamma) = r$.

C3) *There exists a matrix $X_J(\gamma)$ such that*

C3.a) *the space $\text{range } X_J(\gamma)$ is a semi-stable deflating subspace of $\lambda N_J - M_J(\gamma)$ and $\text{range } \begin{bmatrix} E^T X_{J,1}(\gamma) \\ X_{J,2}(\gamma) \end{bmatrix}$ is an r -dimensional isotropic subspace of \mathbb{R}^{2n} ;*

C3.b) $\text{rank } E^T X_{J,1}(\gamma) = r$.

C4) *The matrix*

$$\mathcal{Y}(\gamma) = \begin{bmatrix} -\gamma X_{H,2}^T(\gamma) EX_{H,1}(\gamma) & X_{H,2}^T(\gamma) EX_{J,2}(\gamma) \\ X_{J,2}^T(\gamma) E^T X_{H,2}(\gamma) & -\gamma X_{J,2}^T(\gamma) E^T X_{J,1}(\gamma) \end{bmatrix} \quad (27)$$

is symmetric, positive semi-definite and satisfies $\text{rank } \mathcal{Y}(\gamma) = k_H + k_J$, where k_H and k_J are such that for all sufficiently large $\gamma_{H,1}, \gamma_{H,2}$, and $\gamma_{J,1}, \gamma_{J,2}$ the conditions

$$\begin{aligned} \text{rank } E^T X_{H,2}(\gamma_{H,1}) &= \text{rank } E^T X_{H,2}(\gamma_{H,2}) = k_H, \\ \text{rank } EX_{J,2}(\gamma_{J,1}) &= \text{rank } EX_{J,2}(\gamma_{J,2}) = k_J \end{aligned}$$

hold.

*Furthermore, the set of γ values satisfying the conditions **C1)** – **C4)** is nonempty.*

To check condition **C4)**, we make use of the LDL^T decomposition, described in [6] and implemented in LAPACK by DSPTRF which decomposes a real symmetric matrix A as $A = LDL^T$, where L is a product of permutation and lower triangular matrices, and D is symmetric and block diagonal with 1×1 and 2×2 diagonal blocks.

Using Theorem 7.4, we can use a bisection type algorithm to determine the optimal value γ_{mo} , see [74].

After completing the bisection process, one has the option to either use the result directly, or to perform a strong validation, by dividing the interval $(0, \gamma_{\text{mo}})$ at a desired number of points and checking the four conditions **C1)** – **C4)** again at these points. If the conditions **C1)** – **C4)** are fulfilled for another $\gamma \in (0, \gamma_{\text{mo}})$, we have obviously found a better value for γ_{mo} . We can either use this new value or continue with the γ -iteration to find an even better value. Once a satisfactory γ is found, it remains to compute the controller. The trick that we use to determine the controller is to compute an index-reducing static output feedback $u(t) = Fy(t) + \bar{u}(t)$, whose application leads to a new descriptor system of the form (22) with an index of at most one. It can be shown that the application of the feedback does not change the solution of the modified \mathcal{H}_∞ optimal control problem [74, 75]. The feedback is computed using the condensed form (10) and the techniques presented in [36], which yield $s_2 = t_2$ and

$$F = \begin{bmatrix} F_{11} & 0 \\ 0 & 0 \end{bmatrix} \in \mathbb{R}^{m,p}, \quad F_{11} = \begin{bmatrix} B_{21} \\ B_{31} \end{bmatrix}^{-1} (I_{s_2} - A_{22}) [C_{12} \quad C_{13}]^{-1}. \quad (28)$$

Note that due to the construction of the condensed form (10), the matrices

$$\begin{bmatrix} B_{21} \\ B_{31} \end{bmatrix}, \quad [C_{12} \quad C_{13}]$$

can be kept in factored form as a product of a unitary and a diagonal matrix. So the computation of F can be carried out by the inversion of two diagonal matrices.

We can use this new descriptor system to compute the controller. The controller formulas themselves and their derivation are rather involved. Therefore, we only refer to the robust controller formulas for the standard system case in [9], and based on that, the controller formulas for the descriptor system case in [74].

Figure 4 presents a flow chart for the solution of the optimal solution. First one checks the four assumptions **A1)** – **A4)**, using the condensed forms from Theorem 5.2, the decompositions

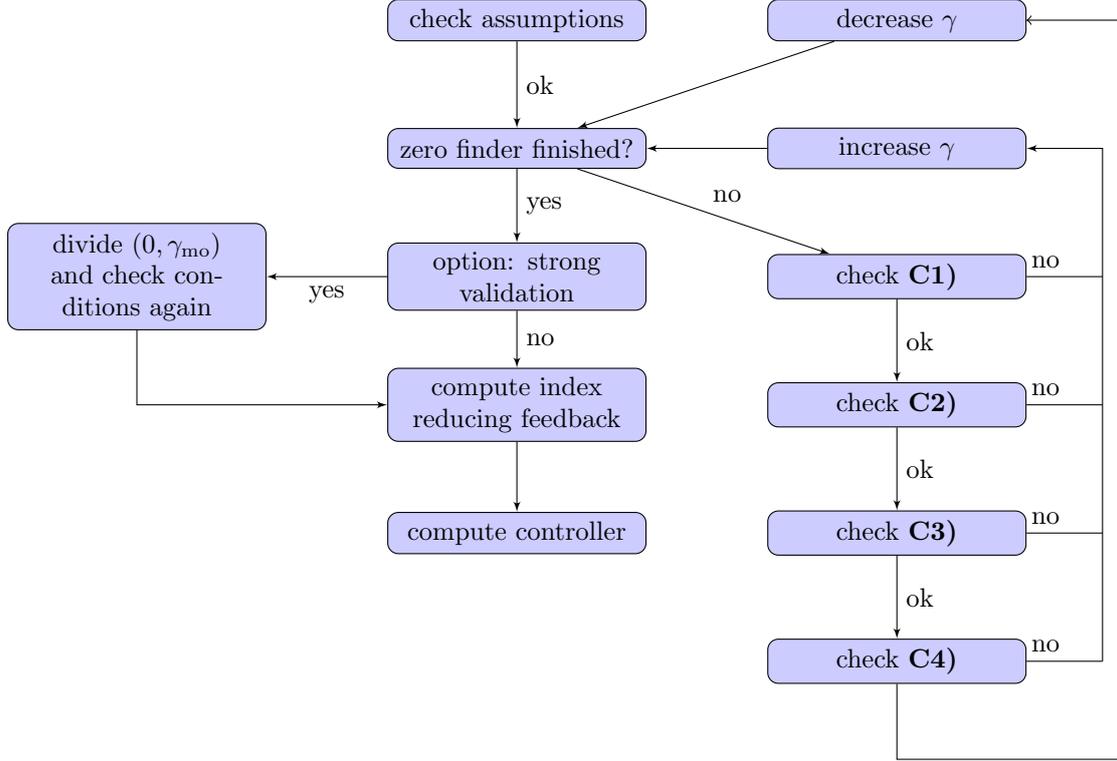


Figure 4: Algorithm flowchart for solving \mathcal{H}_∞ optimal control problems

(11) and performing some rank checks. Then one uses a bisection type algorithm to find the optimal value of γ , by checking the four conditions from Theorem 7.4 in each step by using the staircase form from Theorem 2.2, the computation of the semi-stable deflating subspaces using Algorithm 1, and the LDL^T decomposition from [6]. Here, the structure-preservation aspect of Algorithm 1 is very important since it *cannot* happen, that eigenvalues from the left half-plane move to right half-plane and vice versa due to round-off errors. Therefore, the computed subspaces are guaranteed to have the correct dimension. Once the optimal value is found, one has the option to use a strong validation by checking the aforementioned four conditions again at a desired number of points. Then it remains to compute an index reducing feedback (28) and to compute the controller formulas given in [9, 74].

8 \mathcal{L}_∞ -Norm Computation

In the previous section we have seen that the \mathcal{H}_∞ -norm of a transfer function is an important measure for the robustness of a linear system. This section is devoted to the actual computation of this norm. We will directly present this for the more general case of the \mathcal{L}_∞ -norm. Consider a square descriptor system (8)–(9) with regular pencil $\lambda E - A$ and transfer function $G(\cdot)$ as in (13).

Before we can turn to the actual norm computation, we have to ensure that $G \in \mathcal{L}_\infty^{p,m}$. First, we check whether the transfer function is *proper*, i.e., that $\lim_{\omega \rightarrow \infty} \|G(i\omega)\| < \infty$. For this we make use of the following result of [17, 100] in a modified formulation.

Theorem 8.1. *Consider a descriptor system (8)–(9) given in the condensed form (10). Then, $G(\cdot)$ is proper if and only if the sub-pencil*

$$\lambda \begin{bmatrix} \Sigma_E & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

is regular and of index at most one, i.e., if A_{22} is invertible.

Therefore, to check properness, we first reduce the system to the condensed form (10) and subsequently check A_{22} for invertibility, e.g., by employing condition estimators [51].

When we have checked the transfer function for properness, it remains to check whether $G(\cdot)$ has finite, purely imaginary poles. For this, we first determine the controllability and observability decompositions (11) and (12) to extract the controllable and observable subsystem. The finite eigenvalues of the pencil associated to this subsystem are poles of $G(\cdot)$ and we check whether there are eigenvalues that lie in a thin strip around the imaginary axis. The thickness of this strip depends on the multiplicity of the pole which is generally not known. In finite precision, eigenvalues in this region cannot be distinguished from eigenvalues on the imaginary axis. Generically, a pole will be simple and therefore, in the code we choose the thickness as a small multiple of machine precision. After we have ensured that $G \in \mathcal{L}_\infty^{p,m}$, we can compute the norm value. For this we make use of the even matrix pencils

$$\lambda N - M(\gamma) = \left[\begin{array}{cc|cc} 0 & \lambda E - A & 0 & -B \\ -\lambda E^T - A^T & 0 & -C^T & 0 \\ \hline 0 & -C & \gamma I_p & -D \\ -B^T & 0 & -D^T & \gamma I_m \end{array} \right]. \quad (29)$$

The following theorem connects the singular values of $G(i\omega)$ with the finite, purely imaginary eigenvalues of $\lambda N - M(\gamma)$, see [17, 18, 100] for details.

Theorem 8.2. *Assume that $\lambda E - A$ has no purely imaginary eigenvalues, $G \in \mathcal{L}_\infty^{p,m}$, $\gamma > 0$ and $\omega_0 \in \mathbb{R}$. Then γ is a singular value of $G(i\omega_0)$ if and only if $\lambda N - M(\gamma)$ has the eigenvalue $i\omega_0$.*

A direct consequence of Theorem 8.2 is the following result, see [17, 18].

Theorem 8.3. *Assume that $\lambda E - A$ has no purely imaginary eigenvalues, $G \in \mathcal{L}_\infty^{p,m}$ and let $\gamma > \min_{\omega \in \mathbb{R}} \sigma_{\max}(G(i\omega))$. Then $\|G\|_{\mathcal{L}_\infty} \geq \gamma$ if and only if $\lambda N - M(\gamma)$ in (29) has finite, purely imaginary eigenvalues.*

This directly yields an algorithm for the computation of the \mathcal{L}_∞ -norm, similarly as in [27, 28, 29]. Given an initial value of γ with $\min_{\omega \in \mathbb{R}} \sigma_{\max}(G(i\omega)) < \gamma < \|G\|_{\mathcal{L}_\infty}$, we check if $\lambda N - M(\gamma)$ has purely imaginary eigenvalues. If yes, we denote these eigenvalues with positive imaginary part by $i\omega_1, \dots, i\omega_q$. To obtain the next (larger) value of γ , we determine new test frequencies $m_j = \sqrt{\omega_j \omega_{j+1}}$, $j = 1, \dots, q-1$. Then, the new value of γ is chosen as

$$\gamma = \max_{1 \leq j \leq q-1} \sigma_{\max}(G(im_j)).$$

To check whether a prespecified relative error ε has already been achieved, we would have to check whether the pencil $\lambda N - M(\hat{\gamma})$ with $\hat{\gamma} = \gamma(1 + 2\varepsilon)$ has no purely imaginary eigenvalues. To avoid the additional check in every step, we can directly incorporate this into the algorithm by always working with $\hat{\gamma}$ instead of γ when determining the eigenvalues of the even pencils.

It can be shown that this algorithm converges globally with a quadratic rate and a guaranteed relative error of ε when assuming exact arithmetics. We refer to [17, 18, 100] for details on the implementation and the algorithm properties. Note again that the decision about the existence of purely imaginary eigenvalues is crucial for a robust execution of this algorithm and does require a structured eigensolver as described in Section 3. A graphical interpretation is given in Figure 5.

Note, that when assuming that $G \in \mathcal{L}_\infty^{p,m}$, the algorithm runs on the original data without performing any system reductions beforehand. However, $\lambda E - A$ could still have uncontrollable or unobservable eigenvalues on the imaginary axis. If one does not perform the system reductions to extract the finite dynamics controllable and observable subsystem, then it remains to check if $\lambda E - A$ has no finite, purely imaginary eigenvalues. The complete procedure is summarized in Figure 6.

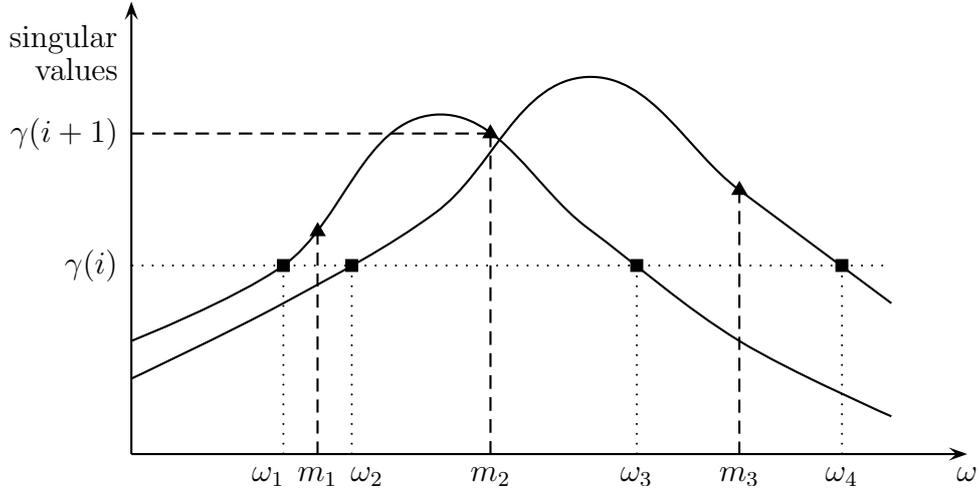


Figure 5: Graphical interpretation of the algorithm for computing the \mathcal{L}_∞ -norm. Here, $\gamma(i)$ and $\gamma(i+1)$ denote the iterates at the i -th and $(i+1)$ -st step, respectively.

9 Dissipativity Check

The notion of dissipativity is one of the most important concepts in systems and control theory. It naturally arises in many physical problems, especially when energy considerations are of importance. Roughly speaking, dissipativity means that the system does not internally generate energy. Equivalently, the system cannot supply more energy to its environment than energy that has been supplied to the system. This means that a fraction of the energy that has been supplied to the system is transformed, e.g., into heat, increase of entropy, or electro-magnetic radiation. When modeling physical processes it is necessary to reflect the dissipative nature of the problem in the model structure. This is important in order to obtain physically meaningful results when performing simulations, see [101, 102, 103]. This section presents a method to check dissipativity for linear time-invariant descriptor systems of the form (8)–(9) based on a spectral characterization for even pencils.

We first introduce a precise mathematical formulation of dissipativity. For this we need the notion of *supply rates* which measure the power supplied to the system at time instance t . In the following we restrict ourselves to quadratic supply functions of the form

$$s(u(t), y(t)) = \begin{bmatrix} y(t) \\ u(t) \end{bmatrix}^T \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \begin{bmatrix} y(t) \\ u(t) \end{bmatrix}, \quad (30)$$

where $Q = Q^T \in \mathbb{R}^{p,p}$, $S \in \mathbb{R}^{p,m}$ and $R = R^T \in \mathbb{R}^{m,m}$. Then the energy supplied to the system in a time interval $[t_0, t_1]$ is measured by

$$\int_{t_0}^{t_1} s(u(t), y(t)) dt.$$

There are many different notions of dissipativity in the literature. In this survey, we stick to the notion of cyclo-dissipativity which has been introduced in [30, 31] in the context of behavior systems.

Definition 9.1. A descriptor system (8)–(9) is called *cyclo-dissipative* with respect to $s(\cdot, \cdot)$, if

$$\int_0^T s(u(t), y(t)) dt \geq 0$$

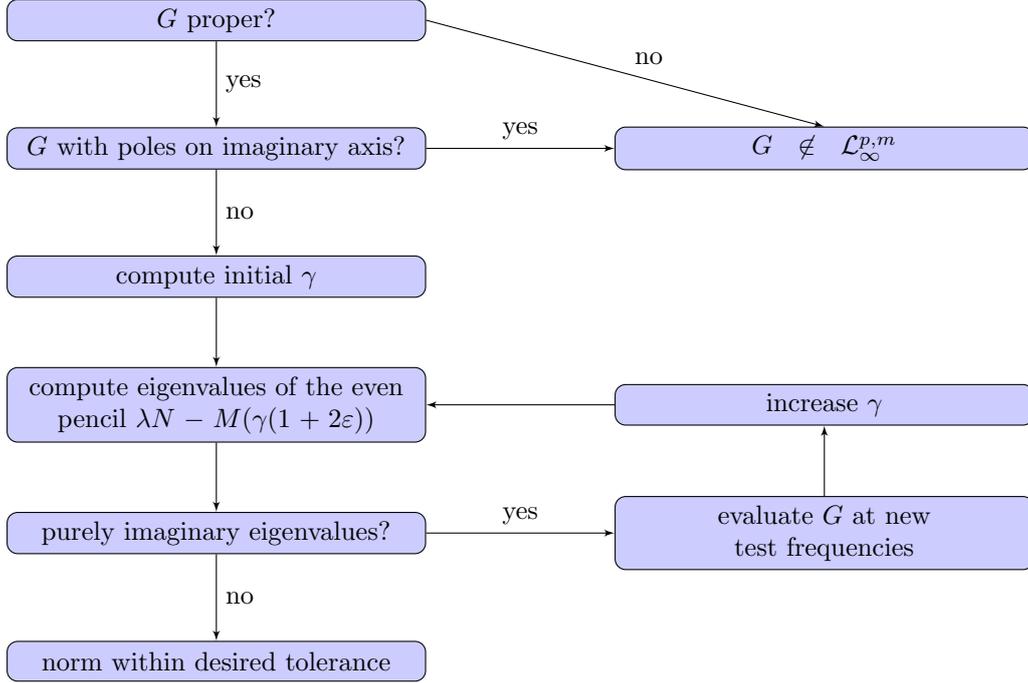


Figure 6: Flowchart for computing the \mathcal{L}_∞ -norm

for all $T \geq 0$ and all smooth trajectories $(u(\cdot), x(\cdot), y(\cdot))$ solving (8)–(9) with the boundary conditions $Ex(0) = Ex(T) = 0$.

Remark 9.2. Cyclo-dissipativity is only a property of the strongly controllable part of the system. A more general definition of dissipativity would require the existence of a *storage function* $\Theta : \text{im } E \rightarrow \mathbb{R}$ with $\Theta(0) = 0$ such that the *dissipation inequality*

$$\Theta(Ex(t_1)) \leq \Theta(Ex(t_0)) + \int_{t_0}^{t_1} s(u(t), y(t)) dt$$

is fulfilled for all $t_0 \leq t_1$ and all smooth solution trajectories $(u(\cdot), x(\cdot), y(\cdot))$ such that the supply rate is locally square-integrable, see [31]. If the system (8)–(9) is strongly controllable, then both definitions coincide. But not every cyclo-dissipative system has to possess a storage function. A counter-example is given in [31].

Remark 9.3. In the definition of cyclo-dissipativity it is only required that trajectories that start in zero and return to zero in some finite time, do not generate energy. A stronger definition, that would require all trajectories that start in zero do not generate energy, exists as well. Closely related to that is positivity of the storage function (if it exists). Unfortunately, its general treatment is much more involved. However, under the condition that the pencil $\lambda E - A$ is regular, stable, and its Kronecker index is at most one, and Q is negative semidefinite, then this stronger definition coincides with Definition 9.1 [33].

In practice, there are two particular cases for the choice of the supply rate. If a descriptor system (8)–(9) is dissipative with respect to the supply rate $s(u(t), y(t)) = u(t)^T y(t)$, i.e., if $k = n$, $p = m$ and

$$\begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 0 & I_m \\ I_m & 0 \end{bmatrix},$$

then the system is called *passive*. This situation typically arises in models for RLC circuits [2, 88, 89, 90].

The other special case is that the supply rate is given by $s(u(t), y(t)) = \|u(t)\|_2^2 - \|y(t)\|_2^2$, i.e., $k = n$ and

$$\begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} = \begin{bmatrix} -I_p & 0 \\ 0 & I_m \end{bmatrix}.$$

In this case, a cyclo-dissipative system is called *contractive*. Usually this structure occurs if (8)–(9) is a realization of scattering parameters [80], but similar structures also appear in \mathcal{H}_∞ control, see Sections 7 and 8.

For square systems (with $k = n$), a well-known relation of dissipativity defined above between the time and frequency domain is given by the so-called *Popov function*

$$H(\mu, \zeta) := \begin{bmatrix} (\mu E - A)^{-1} B \\ I_m \end{bmatrix}^H \begin{bmatrix} \tilde{Q} & \tilde{S} \\ \tilde{S}^T & \tilde{R} \end{bmatrix} \begin{bmatrix} (\zeta E - A)^{-1} B \\ I_m \end{bmatrix},$$

with \tilde{Q} , \tilde{S} , and \tilde{R} as in (15). One has the following theorem of [31, 30].

Theorem 9.4. *The square descriptor system (8)–(9) is cyclo-dissipative with respect to $s(\cdot, \cdot)$ if and only if $H(i\omega, i\omega) \geq 0$ for all $i\omega \notin \Lambda(E, A)$.*

For the cases of passivity and contractivity we get more general relations. These are summarized in the following theorem [2].

Theorem 9.5. *Consider a square descriptor system of the form (8)–(9) with $p = m$.*

1. *The system is passive if and only if $G(\cdot)$ is positive real, i.e.,*
 - (a) *$G(\cdot)$ is analytic in \mathbb{C}^+ ; and*
 - (b) *$H(s, s) = G(s) + G^H(s) \geq 0$ for all $s \in \mathbb{C}^+$.*
2. *The system is contractive if and only if $G(\cdot)$ is bounded real, i.e.,*
 - (a) *$G(\cdot)$ is analytic in \mathbb{C}^+ ; and*
 - (b) *$H(s, s) = I_m - G^H(s)G(s) \geq 0$ for all $s \in \mathbb{C}^+$.*

It is very important to note that the equivalences of Theorem 9.5 does in general *not* hold in the context of general cyclo-dissipativity as in Definition 9.1. A counterexample is given in [104]. There are many algebraic characterizations to check if a given system (8)–(9) is cyclo-dissipative. These are mainly based on solvability of certain linear matrix inequalities or matrix equations, see [55]. Instead we make use of the following spectral characterization of even pencils $\lambda N - M$. For this, we need the *sign-sum function* [30, 31, 33] of a Hermitian matrix T which is defined as

$$\eta(T) = \pi_+ + \pi_0 - \pi_-,$$

where π_+ , π_0 , and π_- are the numbers of positive, zero, and negative eigenvalues of T , respectively. Furthermore, we can define the rank of a polynomial matrix $P(\lambda)$ over the field of real-rational functions (often called normal rank), given by

$$\text{rank}_{\mathbb{R}(\lambda)}(P(\lambda)) := \max_{\lambda_0 \in \mathbb{C}} \text{rank}(P(\lambda_0)). \quad (31)$$

The maximum in (31) is attained for almost all values of $\lambda_0 \in \mathbb{C}$, there is only a finite set of points, where the rank drops.

Theorem 9.6. [31, Theorem 3.11] *Consider the system (8)–(9) with supply rate (30). Assume that*

$$\text{rank}_{\mathbb{R}(\lambda)}([\lambda E - A \quad -B]) = r \quad (32)$$

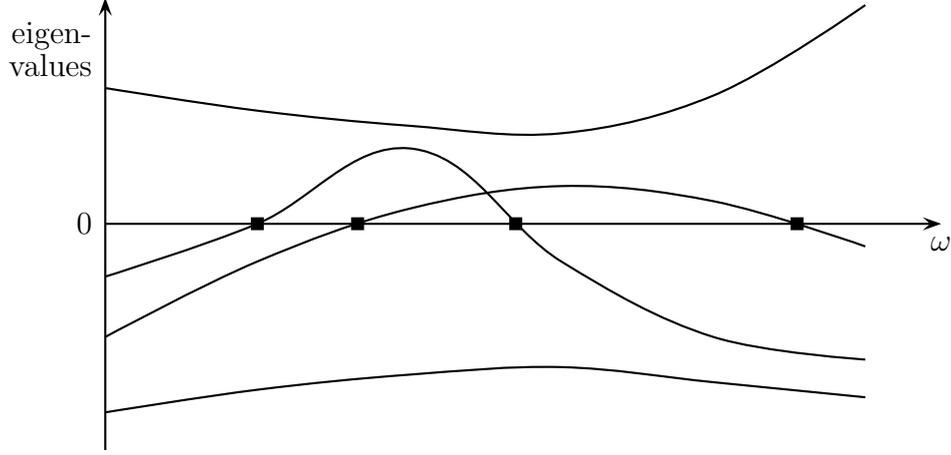


Figure 7: Spectral plot. Here cyclo-dissipativity is violated, since the sign-sum function changes when varying ω .

and let $\ell := k + n + m + 2p$. Consider the $\ell \times \ell$ even pencil

$$\mathcal{N}(\lambda) = \lambda N - M = \begin{bmatrix} 0 & 0 & 0 & \lambda E - A & -B \\ 0 & 0 & I_m & -C & -D \\ 0 & I_m & Q & 0 & S \\ -\lambda E^T - A^T & -C^T & 0 & 0 & 0 \\ -B^T & -D^T & S^T & 0 & R \end{bmatrix}. \quad (33)$$

Then the system given by (8)–(9) is cyclo-dissipative if and only if

$$\eta(\mathcal{N}(i\omega)) = k + n + m - 2r \quad (34)$$

for all $\omega \in \mathbb{R}$ with $\text{rank}([i\omega E - A \quad -B]) = r$.

To better understand this theorem, we present a visualization in terms of the so-called *spectral plot*. This plot is constructed by plotting the ℓ eigenvalues of $\mathcal{N}(i\omega)$ depending on ω , see Figure 7 for an example.

The general framework for checking cyclo-dissipativity then consists of two steps. First, we check if the assumptions of Theorem 9.6 are fulfilled. If the normal rank is unknown, then the GUPTRI form [45, 46, 60] is a suitable tool to compute it. The GUPTRI algorithm¹ delivers two orthogonal matrices $Q \in \mathbb{R}^{k,k}$ and $Z \in \mathbb{R}^{n+m,n+m}$ such that

$$\lambda [E \quad 0] - [A \quad B] = Q \left(\lambda \begin{bmatrix} E_r & * & * & * & * \\ 0 & E_z & * & * & * \\ 0 & 0 & E_f & * & * \\ 0 & 0 & 0 & E_i & * \\ 0 & 0 & 0 & 0 & E_l \end{bmatrix} - \begin{bmatrix} A_r & * & * & * & * \\ 0 & A_z & * & * & * \\ 0 & 0 & A_f & * & * \\ 0 & 0 & 0 & A_i & * \\ 0 & 0 & 0 & 0 & A_l \end{bmatrix} \right) Z, \quad (35)$$

where

- (a) $\lambda E_r - A_r$ contains the right singular structure;
- (b) $\lambda E_z - A_z$ contains the Jordan structure for the zero eigenvalue;
- (c) $\lambda E_f - A_f$ contains all finite eigenvalues;

¹<http://www8.cs.umu.se/~guptri/>

- (d) $\lambda E_i - A_i$ contains the Jordan structure for the infinite eigenvalue;
- (e) $\lambda E_1 - A_1$ contains the left singular structure.

Then one sees that $\text{rank}_{\mathbb{R}(\lambda)}([\lambda E - A \quad -B])$ is nothing but the column dimension k minus the number of left singular blocks which can be obtained by the GUPTRI form.

The next step consists in checking the sign-sum condition in Theorem 9.6. We exploit the fact that $\eta(i\omega N - M)$ can only change at purely imaginary eigenvalues (of the regular index one part) and remains constant between two subsequent purely imaginary eigenvalues. We construct the pencil (33) and apply the even staircase algorithm from Theorem 2.2 to get the regular index one part $\lambda N_{w+1,w+1} - M_{w+1,w+1}$. Then we compute its purely imaginary eigenvalues with positive imaginary part, denoted by $i\omega_1, \dots, i\omega_q$, with $\omega_1 < \omega_2 < \dots < \omega_q$. This is done using Algorithm 1. Next, we set $\omega_0 := 0$ and $\omega_{q+1} := \infty$. For $j = 0, \dots, q$ we choose points $\alpha_j \in (\omega_j, \omega_{j+1})$ with $\text{rank}([i\alpha_j E - A \quad -B]) = r$. Finally, for $j = 0, \dots, q$ we compute the inertia $(\pi_+^j, \pi_0^j, \pi_-^j)$ of the Hermitian matrix $i\alpha_j N - M$ and thus obtain $\eta(i\alpha_j N - M) = \pi_+^j + \pi_0^j - \pi_-^j$. Then the system is dissipative if and only if $\eta(i\alpha_j N - M) = k + n + m - 2r$ for all j . Figure 8 summarizes the complete procedure in a diagram.

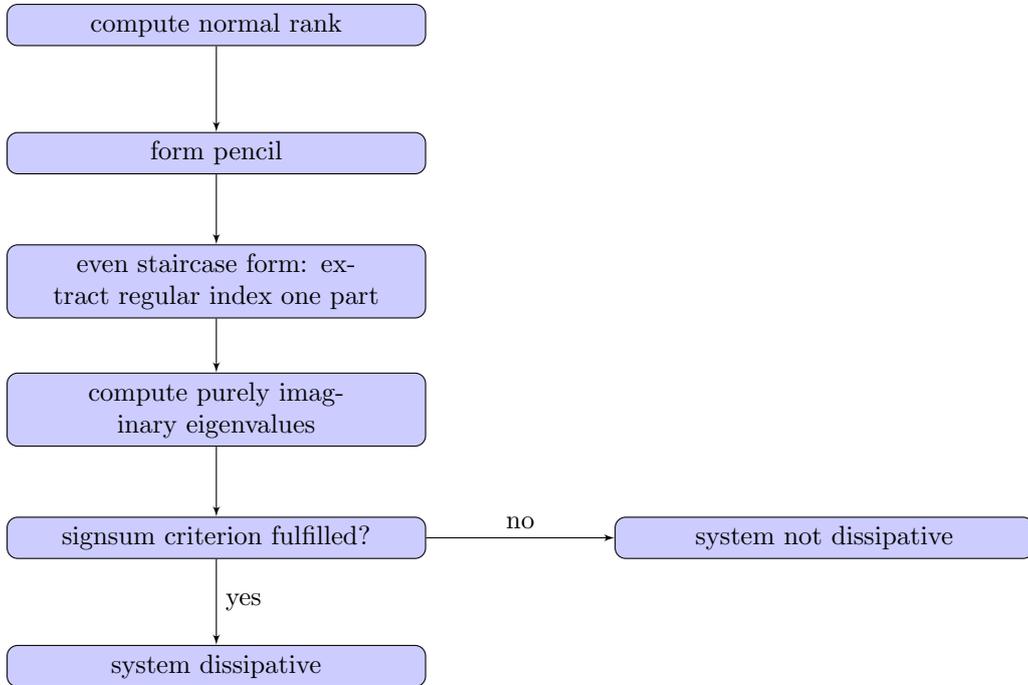


Figure 8: Algorithm flowchart for dissipativity check

10 Conclusions

This paper provides a uniform treatment of eigenvalue problems for even matrix polynomials. We have shown how we can obtain structured linearizations for these polynomials in order to reduce the polynomial problem to a linear even (generalized) eigenvalue problem. Then we have presented several canonical forms of even pencils and discussed their properties. Furthermore, we have presented details on control problems for descriptor systems and we have shown how we can utilize even pencils and their canonical as well as condensed forms to numerically solve these problems. The methods discussed here are usable for small-scale problems, while there are still

many open questions when considering large-scale problems. For instance, then it is not clear how to determine *all* desired eigenvalues of a large-scale even pencil, e.g., the purely imaginary ones or how to approximate the complete subspace associated to all eigenvalues in the left half plane by a sparse representation.

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