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Balanced Truncation of Linear Time-Invariant Systems at a Single Frequency

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Abstract

This work considers the model reduction of linear time-invariant systems with focus on good approximation at a particular frequency. A family of frequency-dependent extended systems preserving some special frequency-dependent properties is constructed first, then the corresponding controllability and observability Lyapunov equations are defined. Finally, those results are used to develop the desired frequency-dependent balanced truncation, which generates the desired reduced-order model with an explicit frequency-dependent approximation error bound. Both continuous-time and discrete-time systems are considered. Several benchmark examples are tested to illustrate the advantage of the proposed frequency-dependent balanced truncation method.

1 Introduction

Model reduction is of fundamental importance in many modeling and control applications and has attracted considerable attention in the past three decades [1] [2] [3]. A general idea of model reduction is to approximate a large-scale system by a much smaller model that captures the input-output behavior of the original system to a required accuracy and also preserves essential physical properties such as stability and passivity.

One model reduction scheme that is well grounded in theory and commonly used in the control community is balanced truncation, first introduced in the systems and control literature by Moore [5]. The advantages of balanced truncation is preservation of stability, as well as the existence of *a priori* known computable entire-frequency approximation error bounds. Unfortunately, the standard balanced truncation (SBT) [5] is frequency-independent, and many practical model reduction problems are inherently frequency dependent (see [12]-[36]) (i.e., the requirement on the approximation accuracy at some frequency ranges are more important than others). The behavior of the reduced-order model near resonances or in an *a priori* known operating frequency interval should often be as close as possible to that of the high-order model, even at the expense of larger errors at other frequencies.

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During the past several decades, there have been many attempts at modifying the SBT to deal with the frequency-dependent model reduction problems. One way is to introduce frequency-sensitive weighting transfer functions and reformulate the problem into a frequency-weighted one (eg, see [12]-[29] and the references therein). The other way to take the frequency range into consideration is to replace the frequency-independent *controllability and observability* Gramian matrices involved in SBT by the so-called frequency-dependent Gramians (See [30])-[36] and the references therein). However, in contrast to the frequency-independent error bound for the standard balanced truncation, no frequency-dependent approximation error bound can be established by those methods . Therefore, those generalizations are all not fully successful [37].

On the one hand, it has been revealed that the Kalman-Yakubovich-Popov (KYP) lemma plays an important role in the development of the standard balanced truncation [7]. On the other hand, the Generalized KYP (GKYP) lemma, which can be used to treat frequency-dependent performance analysis and viewed as a comparable fundamental machinery like the KYP Lemma has emerged in recent years [9]. Therefore, it is a natural idea to further generalize the standard balanced truncation from a frequency-independent one to a frequency-dependent one with the aid of the GKYP lemma.

In this paper, we will revisit the frequency-limited model reduction problem inspired by the GKYP lemma, with focus on a single frequency. First, a group of *frequency-dependent extended systems* which preserve many interesting frequency domain properties is introduced. Based on the *frequency-dependent extended systems*, corresponding concepts such as *frequency-dependent Gramians*, *frequency-dependent Lyapunov equations*, and *frequency-dependent balanced realization* are defined subsequently. Finally, a *frequency-dependent balanced truncation* method which generates a reduced-order model with an explicit frequency-dependent approximation error bound is proposed. Several applications of frequency-dependent balanced truncation for typical examples are included to illustrate its effectiveness and advantages.

2 Problem Statements and Preliminaries

Let a finite dimensional linear dynamical system be described by the following linear constant coefficient differential equations:

$$\delta[x(t)] = Ax(t) + Bu(t), \quad y(t) = Cx(t) + Du(t) \tag{1}$$

where the symbol $\delta[.]$ represents the differential operator for continuous-time and the forward-shift operator for discrete-time system. Here, $x(t) \in \mathbb{R}^n$ is the state vector, $u(t) \in \mathbb{R}^m$ is the input signal, $y(t) \in \mathbb{R}^p$ is the output signal. The corresponding transfer function of (1) describing the mapping $u \to y$ in the frequency domain for $x(0) = 0$ can be represented as:

$$G(\jmath\omega) = C(\jmath\omega I - A)^{-1}B + D \tag{2}$$

in the continuous-time case and as

$$G(e^{\jmath\theta}) = C(e^{\jmath\theta}I - A)^{-1}B + D \tag{3}$$

in the discrete-time case, respectively. Generally, model reduction schemes aim at finding a proper low-order system model:

$$\delta[x_r(t)] = A_r x_r(t) + B_r u(t), \quad y_r(t) = C_r x_r(t) + D_r u(t) \tag{4}$$

which efficiently approximates the original system (1). In other words, the low-order model should generate output signals closely similar to the original system under the same input signal, where $A_r \in \mathbb{R}^{r\times r}, B_r \in \mathbb{R}^{r\times m}, C_r \in \mathbb{R}^{p\times r}, D_r \in \mathbb{R}^{p\times m}$ with $r \ll n$. In this paper, we focus on the case that the frequency of the input signal is pre-known and belongs to a singleton set (i.e. $\omega = \varpi$ in the continuous-time case and $\theta = \vartheta$ in the discrete-time case). With the single frequency assumption, the desired specifications for the model reduction problem under consideration can be naturally concluded as follows, where $\|M\|_2 := \sigma_{\max}(M)$ denotes the spectral norm, i.e. the largest singular value of the matrix M.

(1) Preserving the finite gain input-output stability at the pre-specified frequency, i.e.
$\|G_r(\jmath\varpi)\|_2 < \infty \; if \; \|G(\jmath\varpi)\|_2 < \infty$, $\left\|G_r(e^{\jmath\vartheta})\right\|_2 < \infty \; if \; \left\|G(e^{\jmath\vartheta})\right\|_2 < \infty$.

(2) Small approximation error at the pre-specified frequency, i.e.
$\|G(\jmath\varpi) - G_r(\jmath\varpi)\|_2 < e_\varpi$, $\left\|G(e^{\jmath\vartheta}) - G_r(e^{\jmath\vartheta})\right\|_2 < e_\vartheta$.

Obviously, the above two specifications capture the intrinsic demands for approximating the original system at a single frequency, and they can also be viewed as a generalization of the counterparts adopted for the frequency-unlimited model reduction problems. In accordance, the aim of this paper is to develop a frequency-dependent balanced truncation model reduction method to meet the above two frequency-dependent specifications.

Proposition 1. ([7]) *Given a linear system (1) and the following statements*
(1) (A, B) is controllable.
(2) Let λ and x be any eigenvalue and corresponding left eigenvector of A, i.e. $x^*A = x^*\lambda$, then $x^*B \neq 0$.
(3) (C, A) is observable.
(4) Let λ and y be any eigenvalue and corresponding right eigenvector of A, i.e. $Ay = \lambda y$, then $Cy \neq 0$.
the above statements (1) and (2) are equivalent, and the statements (3) and (4) are equivalent. Besides, a state space realization (A, B, C, D) of it is minimal if and only if (A, B) is controllable and (C, A) is observable.

Lemma 1. ([8], *KYP Lemma (continuous-time case))*
Consider a continuous-time transfer function matrix $G(\jmath\omega) = C(\jmath\omega I - A)^{-1}B + D$, and let a symmetric matrix Π be given. Then the following statements are equivalent:
(1) The entire frequency inequality

$$\begin{bmatrix} G(\jmath\omega) \\ I \end{bmatrix}^* \Pi \begin{bmatrix} G(\jmath\omega) \\ I \end{bmatrix} \leq 0 \; holds \; for \; all \; \omega \in [-\infty, +\infty]. \tag{5}$$

(2) There exists a symmetric matrix $P > 0$ of appropriate dimension satisfying

$$\begin{bmatrix} A & I \\ C & 0 \end{bmatrix} \begin{bmatrix} 0 & P \\ P & 0 \end{bmatrix} \begin{bmatrix} A & I \\ C & 0 \end{bmatrix}^T + \begin{bmatrix} B & 0 \\ D & I \end{bmatrix} \Pi \begin{bmatrix} B & 0 \\ D & I \end{bmatrix}^T \leq 0 \; . \tag{6}$$

For our purposes, the following frequency-dependent version of the KYP lemma will play an essential role.

Lemma 2. ([9], Generalized KYP (GKYP) Lemma (continuous-time case))
Consider a continuous-time transfer function matrix $G(\jmath\omega) = C(\jmath\omega I - A)^{-1}B + D$, and let a symmetric matrix Π be given. Then the following statements are equivalent:
(1) The finite frequency inequality

$$\begin{bmatrix} G(\jmath\omega) \\ I \end{bmatrix}^* \Pi \begin{bmatrix} G(\jmath\omega) \\ I \end{bmatrix} \leq 0 \; holds \; for \; \omega = \varpi. \tag{7}$$

(2) There exist symmetric matrices P and Q of appropriate dimensions, satisfying $Q > 0$ and

$$\begin{bmatrix} A & I \\ C & 0 \end{bmatrix} \begin{bmatrix} -Q & P + \jmath\varpi Q \\ P - \jmath\varpi Q & -\jmath\varpi^2 Q \end{bmatrix} \begin{bmatrix} A & I \\ C & 0 \end{bmatrix}^T + \begin{bmatrix} B & 0 \\ D & I \end{bmatrix} \Pi \begin{bmatrix} B & 0 \\ D & I \end{bmatrix}^T \leq 0. \tag{8}$$

3 Frequency-dependent Extended Systems

In this section, we first construct a group of frequency-dependent extended systems (FDES) and present some interesting and important frequency-related frequency domain properties of the FDES, and then the corresponding frequency-dependent Lyapunov equations, frequency-dependent Gramians, frequency-dependent balanced realization are defined, which establish a foundation of our development for frequency-dependent balanced truncation.

Definition 1: (Frequency-dependent Extended Systems)
(a) Given a linear continuous-time system (1) and a pre-specified operating frequency ϖ, then one can construct a group of related ϖ-dependent extended systems as follows:

$$\begin{aligned}\dot{x}(t) &= A_\varpi x(t) + B_\varpi u(t),\\ y(t) &= C_\varpi x(t) + D_\varpi u(t),\end{aligned} \tag{9}$$

where

$$\left[\begin{array}{c|c}A_\varpi & B_\varpi\\ \hline C_\varpi & D_\varpi\end{array}\right] = \left[\begin{array}{c|c}\jmath\varpi I - \epsilon(\epsilon I + \jmath\varpi I - A)^{-1}(\jmath\varpi I - A) & \epsilon(\epsilon I + \jmath\varpi I - A)^{-1}B\\ \hline \epsilon C(\epsilon I + \jmath\varpi I - A)^{-1} & D + C(\epsilon I + \jmath\varpi I - A)^{-1}B\end{array}\right], \tag{10}$$

Alternatively, the ϖ-dependent extended systems (9) may be represented by a rational transfer-function matrix

$$G_\varpi(\jmath\omega) = C_\varpi(\jmath\omega I - A_\varpi)^{-1}B_\varpi + D_\varpi \tag{11}$$

(b) Given a linear discrete-time system (1) and a pre-specified operating frequency ϑ, then one can construct a group of ϑ-dependent extended systems $G_\vartheta(e^{\jmath\vartheta})$ as follows:

$$\begin{aligned}x(k+1) &= A_\vartheta x(k) + B_\vartheta u(k),\\ y(k) &= C_\vartheta x(k) + D_\vartheta u(k),\end{aligned} \tag{12}$$

where

$$\left[\begin{array}{c|c}A_\vartheta & B_\vartheta\\ \hline C_\vartheta & D_\vartheta\end{array}\right] = \left[\begin{array}{c|c}e^{\jmath\vartheta}I - \epsilon(\epsilon I + e^{\jmath\vartheta}I - A)^{-1}(e^{\jmath\vartheta}I - A) & \epsilon(\epsilon I + e^{\jmath\vartheta}I - A)^{-1}B\\ \hline \epsilon C(\epsilon I + e^{\jmath\vartheta}I - A)^{-1} & D + C(\epsilon I + e^{\jmath\vartheta}I - A)^{-1}B\end{array}\right], \tag{13}$$

if $\vartheta \in [-\pi/2, \pi/2]$, and

$$\left[\begin{array}{c|c}A_\vartheta & B_\vartheta\\ \hline C_\vartheta & D_\vartheta\end{array}\right] = \left[\begin{array}{c|c}-e^{\jmath\vartheta}I - \epsilon(\epsilon I - e^{\jmath\vartheta}I - A)^{-1}(-e^{\jmath\vartheta}I - A) & \epsilon(\epsilon I - e^{\jmath\vartheta}I - A)^{-1}B\\ \hline \epsilon C(\epsilon I - e^{\jmath\vartheta}I - A)^{-1} & D + C(\epsilon I - e^{\jmath\vartheta}I - A)^{-1}B\end{array}\right], \tag{14}$$

if $\vartheta \in [-\pi, -\pi/2]$ or $\vartheta \in [\pi/2, \pi]$, and ϵ is a positive scalar. Alternatively, the ϑ-dependent extended systems (12) may be represented by a rational transfer-function matrix

$$G_\vartheta(\jmath\omega) = C_\vartheta(\jmath\omega I - A_\vartheta)^{-1}B_\varpi + D_\vartheta \tag{15}$$

Theorem 1. (Relationships between the original system and the FDES)
(a) Given a continuous-time system (1) and one of its corresponding ϖ-dependent extended systems (9), then the following statements are true:
(a.1) If $G(\jmath\omega)$ is stable then $G_\varpi(\jmath\omega)$ is stable for any $\epsilon > 0$.
(a.2) If $G(\jmath\omega)$ is unstable then $G_\varpi(\jmath\omega)$ is stable for $0 < \epsilon < \hat{\epsilon}_\varpi$, where $\hat{\epsilon}_\varpi = \min\{\hat{\epsilon}(\lambda_1), \hat{\epsilon}(\lambda_2), \ldots, \hat{\epsilon}(\lambda_i), \ldots, \hat{\epsilon}(\lambda_{n_u})\}$, $\hat{\epsilon}(\lambda_i) = (\varpi - \lambda_i^i)^2/\lambda_i^r + \lambda_i^r, i = 1, \ldots, n_u$, and λ_i^r, λ_i^i denote the real and imaginary parts of the unstable eigenvalue $\lambda_i = \lambda_i^r + \lambda_i^i, i = 1, \ldots, n_u$ of A, respectively.
(a.3) (A_ϖ, B_ϖ) is controllable if (A, B) is controllable.
(a.4) (A_ϖ, C_ϖ) is observable if (A, C) is observable.
(a.5) $(A_\varpi, B_\varpi, C_\varpi, D_\varpi)$ is a minimal realization of $G_\varpi(\jmath\omega)$ if (A, B, C, D) is a minimal realization of $G(\jmath\omega)$.
(a.6) $G_\varpi(\jmath\varpi) = G(\jmath\varpi)$.
(a.7) If $\|G(\jmath\omega)\|_\infty \le \gamma$ for all $\omega \in [-\infty, +\infty]$, then $\|G_\varpi(\jmath\omega)\|_\infty \le \gamma$ for all $\omega \in [-\infty, +\infty]$.
(a.8) If $\|G_\varpi(\jmath\omega)\|_\infty \le \gamma_\varpi$ for all $\omega \in [-\infty, +\infty]$, then $\|G(\jmath\omega)\|_2 \le \gamma_\varpi$ for $\omega = \varpi$.

(b).Given a discrete-time system (1) and its corresponding ϑ-dependent extended systems (12), then the following statements are true:

(b.1) If $G(e^{\jmath\theta})$ is stable then $G_\vartheta(e^{\jmath\theta})$ is stable for $\epsilon > 0$.
(b.2) If $G(e^{\jmath\theta})$ is unstable then $G_\vartheta(e^{\jmath\theta})$ is stable for $0 < \epsilon < \hat{\epsilon}_\vartheta$, where $\hat{\epsilon}_\vartheta = \min\{\hat{\epsilon}(\lambda_1), \hat{\epsilon}(\lambda_2), \ldots, \hat{\epsilon}(\lambda_i), \ldots, \hat{\epsilon}(\lambda_{n_u})\}$ and $\hat{\epsilon}(\lambda_i)$ is the minimal positive solution of the following equation

$$\left|e^{\jmath\vartheta} - \varepsilon(e^{\jmath\vartheta} - \lambda)/(\varepsilon + e^{\jmath\vartheta} - \lambda)\right| = 1$$

and $\lambda_i, i = 1, \ldots, n_u$ denote the unstable eigenvalues of A.
(b.3) $(A_\vartheta, B_\vartheta)$ is controllable if (A, B) is controllable.
(b.4) $(A_\vartheta, C_\vartheta)$ is observable if (A, C) is observable.
(b.5) $(A_\vartheta, B_\vartheta, C_\vartheta, D_\vartheta)$ is a minimal realization of $G_\vartheta(e^{\jmath\theta})$ if (A, B, C, D) is a minimal realization of $G(e^{\jmath\theta})$.
(b.6) $G_{e\vartheta}(e^{\jmath\vartheta}) = G(e^{\jmath\vartheta})$.
(b.7) If $\left\|G(e^{\jmath\theta})\right\|_\infty \leq \gamma\ \forall\theta \in [-\pi, +\pi]$, then $\left\|G_\vartheta(e^{\jmath\theta})\right\|_\infty \leq \gamma\ \forall\theta \in [-\pi, +\pi]$.
(b.8) If $\left\|G_\vartheta(e^{\jmath\theta})\right\|_\infty \leq \gamma_\vartheta\ \forall\theta \in [-\pi, +\pi]$, then $\left\|G(e^{\jmath\theta})\right\|_2 \leq \gamma_\vartheta\ \theta = \vartheta$.

Proof.
(a.1) Let us denote $\lambda_i, i = 1, 2\ldots, n$, and $\lambda_{\varpi,i}, i = 1, 2\ldots, n$ as the eigenvalues of the matrices A and A_ϖ, respectively. According to the mapping between A and A_ϖ (10), we know that $\lambda_{\varpi,i} = \jmath\varpi - \epsilon(\jmath\varpi - \lambda_i)(\epsilon + \jmath\varpi - \lambda_i)^{-1}$. Noticing that $\lambda_i^r = Re(\lambda_i) < 0$ if the system $G(\jmath\omega)$ is stable and

$$\lambda_{\varpi i}^r = Re(\lambda_{\varpi i}) = -(\epsilon\lambda_i^r(\epsilon - \lambda_i^r)) + \epsilon(\varpi - \lambda_i^i)^2)/((\epsilon - \lambda_i^r)^2 + (\varpi - \lambda_i^i)^2) < 0, \forall\epsilon > 0, \varpi_i^r < 0,$$

thus the proof can be completed.

(a.2) Noticing that the following inequalities

$$\lambda_{\varpi,i}^r = Re(\lambda_{\varpi,i}) = -(\epsilon\lambda_i^r(\epsilon - \lambda_i^r)) + \epsilon(\varpi - \lambda_i^i)^2)/((\epsilon - \lambda_i^r)^2 + (\varpi - \lambda_i^i)^2) < 0, i = 1, \ldots, n_u$$

hold if $0 < \epsilon < \hat{\epsilon}_\varpi$ and $\varpi_i^r > 0$, which means for any unstable eigenvalue λ_i, the corresponding mapped eigenvalue $\lambda_{\varpi i}$ is stable.

(a.3) Let us denote λ and x^* as any eigenvalue and the corresponding eigenvector of A, i.e. $x^*A = x^*\lambda$. According to (41) we have $x^*A_\varpi = x^*\lambda_\varpi$, where λ_ϖ is the corresponding mapped eigenvalue of A_ϖ. On the other side,

$$x^*B_\varpi = x^*\epsilon(\epsilon I + \jmath\varpi I - A)^{-1}B = x^*\epsilon(\epsilon + \jmath\varpi - \lambda)^{-1}B = \epsilon(\epsilon + \jmath\varpi - \lambda)^{-1}x^*B.$$

From Proposition 1, we know that $x^*B \neq 0$ as (A, B) is assumed to be controllable here. This leads us to $x^*B_\varpi \neq 0$ since $\epsilon(\epsilon + \jmath\varpi - \lambda)^{-1}$ for any $\epsilon > 0$. Thus, one can conclude that (A_ϖ, B_ϖ) is controllable.

(a.4) The proof can be completed similarly to the proof of (a.3) by utilizing Proposition 1, the details are omitted here for brevity.

(a.5) According to Proposition 1, we know that (A, B) is controllable and (C, A) is observable if (A, B, D, C) is a minimal realization, as proved above, we have (A_ϖ, B_ϖ) is controllable and (C_ϖ, A_ϖ) is observable, which further implies $(A_\varpi, B_\varpi, C_\varpi, D_\varpi)$ is a minimal realization.

(a.6) The equality $G_\varpi(\jmath\varpi) = G(\jmath\varpi)$ can be easily validated by direct matrix manipulation:

$$\begin{aligned}
G_\varpi(\jmath\varpi) &= C_\varpi(\jmath\varpi I - A_\varpi)^{-1}B_\varpi + D_\varpi \\
&= C(\varepsilon I + \jmath\varpi I - A)^{-1}(\jmath\varpi I - A)^{-1}(\varepsilon I + \jmath\varpi I - A)\varepsilon(\varepsilon I + \jmath\varpi I - A)^{-1}B \\
&\quad + D + C(\varepsilon I + \jmath\varpi I - A)^{-1}B \\
&= C\varepsilon(\varepsilon I + \jmath\varpi I - A)^{-1}(\jmath\varpi I - A)^{-1}B + D + C(\varepsilon I + \jmath\varpi I - A)^{-1}B \\
&= C(\jmath\varpi I - A)^{-1}B + D \\
&= G(\jmath\varpi)
\end{aligned} \tag{16}$$

(a.7) According to the KYP lemma, if $\|G(\jmath\omega)\|_\infty \leq \gamma\ \forall\omega \in [-\infty, +\infty]$, then there exists a positive symmetric matrix P such that the following equation holds:

$$\Psi = \begin{bmatrix} AP + PA^* + BB^* & PC^* + BD^* \\ * & DD^* - \gamma^2 I \end{bmatrix} = 0,$$

defining the Lyapunov variable P_ϖ as: $P_\varpi = P$ and denote $N = \varepsilon(\varepsilon I + \jmath\varpi I - A)^{-1}$ for simplicity, then the following equations can be validated:

$$\begin{aligned}
\Psi_{\varpi 11} &= A_\varpi P_\varpi + P_\varpi A_\varpi{}^* + B_\varpi B_\varpi{}^* \\
&= (\jmath\varpi I - \varepsilon(\varepsilon I + \jmath\varpi I - A)^{-1}(\jmath\varpi I - A))P \\
&+ P(\jmath\varpi I - j\varepsilon(\varepsilon I + \jmath\varpi I - A)^{-1}(\jmath\varpi I - A))^* \\
&\quad + \varepsilon(\varepsilon I + \jmath\varpi I - A)^{-1}BB^*\varepsilon(\varepsilon I + \jmath\varpi I - A)^{-*} \\
&= -\varepsilon^{-1}N((\jmath\varpi I - A)P(\varepsilon I + \jmath\varpi I - A)^* \\
&- (\varepsilon I + \jmath\varpi I - A)P(\jmath\varpi I - A)^*)N^* + NBB^*N^* \\
&= -N((\jmath\varpi I - A)P + P(\jmath\varpi I - A)^*)N^* + NBB^*N^* \\
&- 2\varepsilon^{-1}N(\jmath\varpi I - A)P(\jmath\varpi I - A)^*N^* \\
&= N(AP + PA^* + BB^*)N^* - 2\varepsilon^{-1}N(\jmath\varpi I - A)P(\jmath\varpi I - A)^*N^* \\
&= N\Psi_{11}N^* - \varepsilon^{-1}NAPA^*N^*,
\end{aligned} \tag{17}$$

$$\begin{aligned}
\Psi_{\varpi 12} &= P_\varpi C_\varpi{}^* + B_\varpi D_\varpi{}^* \\
&= P\varepsilon(\varepsilon I + \jmath\varpi I - A)^{-*}C^* + B_\varpi D^* + NBB^*(\varepsilon I + \jmath\varpi I - A)^{-*}C^* \\
&= N\varepsilon^{-1}(\varepsilon I + \jmath\varpi I - A)P\varepsilon(\varepsilon I + \jmath\varpi I - A)^{-*}C^* + NBD^* + NBB^*(\varepsilon I + \jmath\varpi I - A)^{-*}C^* \\
&= NP\varepsilon(\varepsilon I + \jmath\varpi I - A)^{-*}C^* + NBD^* + N\varepsilon^{-1}(\jmath\varpi I - A)P\varepsilon(\varepsilon I + \jmath\varpi I - A)^{-*}C^* \\
&\quad + N(\jmath\varpi I - A)P(\varepsilon I + \jmath\varpi I - A)^{-*}C^* + NP(\jmath\varpi I - A)^*(\varepsilon I + \jmath\varpi I - A)^{-*}C^* \\
&= NPC^* + NBD^* + 2\varepsilon^{-1}N(\jmath\varpi I - A)PC_\varpi{}^* = N\Psi_{12} + 2\varepsilon^{-1}N(\jmath\varpi I - A)PC_\varpi{}^*,
\end{aligned} \tag{18}$$

$$\begin{aligned}
\Psi_{\varpi 22} &= D_\varpi D_\varpi{}^* - \gamma^2 I \\
&= DD^* - \gamma^2 I + C(\varepsilon I + \jmath\varpi I - A)^{-1}BD^* + DB^*(\varepsilon I + \jmath\varpi I - A)^{-*}C^* \\
&\quad + C(\varepsilon I + \jmath\varpi I - A)^{-1}BB^*(\varepsilon I + \jmath\varpi I - A)^{-*}C^* \\
&= DD^* - \gamma^2 I - C(\varepsilon I + \jmath\varpi I - A)^{-1}PC^* - CP(\varepsilon I + \jmath\varpi I - A)^{-*}C^* \\
&\quad + C(\varepsilon I + \jmath\varpi I - A)^{-1}((\varepsilon I + \jmath\varpi I - A)P + P(\varepsilon I + \jmath\varpi I - A)^*)(\varepsilon I + \jmath\varpi I - A)^{-*}C^* \\
&\quad - C(\varepsilon I + \jmath\varpi I - A)^{-1}(2\varepsilon P)(\varepsilon I + \jmath\varpi I - A)^{-*}C^* \\
&= DD^* - \gamma^2 I - 2\varepsilon C(\varepsilon I + \jmath\varpi I - A)^{-1}P(\varepsilon I + \jmath\varpi I - A)^{-*}C^* \\
&= \Psi_{22} - 2\varepsilon^{-1}C_\varpi PC_\varpi^*.
\end{aligned} \tag{19}$$

Therefore, we have

$$\begin{aligned}
\Psi_\varpi &= \begin{bmatrix} A_\varpi P_\varpi + P_\varpi A_\vartheta{}^* + B_\varpi B_\varpi{}^* & P_\varpi C_\varpi^* + B_\varpi D_\varpi^* \\ * & DD^* - \gamma^2 I \end{bmatrix} \\
&= -\begin{bmatrix} N(\jmath\varpi I - A) \\ C_\varpi \end{bmatrix}(2\varepsilon P)\begin{bmatrix} N(\jmath\varpi I - A) \\ C_\varpi \end{bmatrix}^* + \begin{bmatrix} N & 0 \\ 0 & I \end{bmatrix}\Phi\begin{bmatrix} N & 0 \\ 0 & I \end{bmatrix}^* \\
&= -\begin{bmatrix} N(\jmath\varpi I - A) \\ C_\varpi \end{bmatrix}(2\varepsilon P)\begin{bmatrix} N(\jmath\varpi I - A) \\ C_\varpi \end{bmatrix}^* \\
&\leq 0.
\end{aligned}$$

According to the KYP Lemma, we have $\|G_\varpi(\jmath\omega)\|_\infty \leq \gamma\ \forall\omega \in [-\infty, +\infty]$.

(a.8). According to the KYP lemma, if $\|G_\varpi(\jmath\omega)\|_\infty \leq \gamma_\varpi, \forall\omega \in [-\infty, +\infty]$, then there exists a positive symmetric matrix P_ϖ such that the following equation holds:

$$\Psi_\varpi = \begin{bmatrix} A_\varpi P_\varpi + P_\varpi A_\varpi^* + B_\varpi B_\varpi^* & P_\varpi C_\varpi^* + B_\varpi D_\varpi^* \\ * & D_\varpi D_\varpi^* - \gamma_\varpi^2 I \end{bmatrix} = 0.$$

Defining two Lyapunov variables Q, P as

$$Q = 2\varepsilon^{-1} P_\varpi, P = P_\varpi,$$

the following equations can be validated:

$$\begin{aligned}\Psi_{11} &= -(\jmath\varpi I - A)Q(\jmath\varpi I - A)^* - (\jmath\varpi I - A)P - P(\jmath\varpi I - A)^* + BB^* \\ &= -(\jmath\varpi I - A)2\varepsilon^{-1}P_\varpi(\jmath\varpi I - A)^* - (\jmath\varpi I - A)P_\varpi - P_\varpi(\jmath\varpi I - A)^* + BB^* \\ &= -\varepsilon^{-1}(\jmath\varpi I - A)P_\varpi(\varepsilon I + \jmath\varpi I - A)^* - (\varepsilon I + \jmath\varpi I - A)P_\varpi(\jmath\varpi I - A)^* + BB^* \\ &= \varepsilon^{-1}(\varepsilon I + \jmath\varpi I - A)\Psi_{\varpi 11}(\varepsilon I + \jmath\varpi I - A)^*\varepsilon^{-*},\end{aligned} \tag{20}$$

$$\begin{aligned}\Psi_{12} &= (\jmath\varpi I - A)(\varepsilon I + \jmath\varpi I - A)^{-1}BB^*(\varepsilon I + \jmath\varpi I - A)^{-*}C^* \\ &\quad + \varepsilon(\varepsilon I + \jmath\varpi I - A)^{-1}BB^*(\varepsilon I + \jmath\varpi I - A)^{-*}C^* + BD^* \\ &\quad + (\jmath\varpi I - A)(\varepsilon I + \jmath\varpi I - A)^{-1}P_\varpi C^* + \varepsilon(\varepsilon I + \jmath\varpi I - A)^{-1}P_\varpi C^* - P_\varpi C^* \\ &\quad + (\varepsilon I + \jmath\varpi I - A)P_\varpi(\varepsilon I + \jmath\varpi I - A)^{-*}C^* \\ &= (\varepsilon I + \jmath\varpi I - A)P_\varpi(\varepsilon I + \jmath\varpi I - A)^{-*}C^* + BB^*(\varepsilon I + \jmath\varpi I - A)^{-*}C^* + BD^* \\ &= (\varepsilon I + \jmath\varpi I - A)\Psi_{\varpi 12},\end{aligned} \tag{21}$$

$$\begin{aligned}\Psi_{22} &= -CQC^* + DD^* - \gamma_\varpi{}^2 I \\ &= -C(\varepsilon I + \jmath\varpi I - A)^{-1}BB^*(\varepsilon I + \jmath\varpi I - A)^{-*}C^* - C(\varepsilon I + \jmath\varpi I - A)^{-1}P_\varpi C^* \\ &\quad - CP_\varpi(\varepsilon I + \jmath\varpi I - A)^{-*}C^* + DD^* - \gamma_\varpi{}^2 I \\ &= -C(\varepsilon I + \jmath\varpi I - A)^{-1}BB^*(\varepsilon I + \jmath\varpi I - A)^{-*}C^* \\ &\quad + C(\varepsilon I + \jmath\varpi I - A)^{-1}BB^*(\varepsilon I + \jmath\varpi I - A)^{-*}C^* + C(\varepsilon I + \jmath\varpi I - A)^{-1}BD^* \\ &\quad + C(\varepsilon I + \jmath\varpi I - A)^{-1}BB^*(\varepsilon I + \jmath\varpi I - A)^{-*}C^* + DB^*(\varepsilon I + \jmath\varpi I - A)^{-*}C^* \\ &\quad + DD^* - \gamma_\varpi{}^2 I \\ &= D_\varpi D_\varpi{}^* - \gamma_\varpi{}^2 I \\ &= \Psi_{\varpi 22}.\end{aligned} \tag{22}$$

Therefore, we have

$$\begin{aligned}\Psi &= \begin{bmatrix} A & I \\ C & 0 \end{bmatrix}\begin{bmatrix} -Q & P + \jmath\varpi Q \\ \mathcal{P} - \jmath\varpi Q & -\varpi^2 Q \end{bmatrix}\begin{bmatrix} A & I \\ C & 0 \end{bmatrix}^* + \begin{bmatrix} B & 0 \\ D & I \end{bmatrix}\begin{bmatrix} I & 0 \\ 0 & -\gamma_\varpi^2 I \end{bmatrix}\begin{bmatrix} B & 0 \\ D & I \end{bmatrix}^* \\ &= \begin{bmatrix} (\varepsilon I + \jmath\varpi I - A) & 0 \\ * & I \end{bmatrix}\begin{bmatrix} \Psi_{\varpi 11} & \Psi_{\varpi 12} \\ * & \Psi_{\varpi 22} \end{bmatrix}\begin{bmatrix} (\varepsilon I + \jmath\varpi I - A) & 0 \\ * & I \end{bmatrix}^* = 0\end{aligned} \tag{23}$$

According to the single frequency GKYP Lemma, we have $\|G(\jmath\omega)\|_2 \leq \gamma_\varpi$ for $\omega = \varpi$.

The discrete-time part can be fulfilled in a similar way and is omitted here.

Remark 1: *From the properties proved above, it is clear that the H_∞-performance of a linear system at a given frequency point can be estimated by the KYP lemma instead of the GKYP lemma. This is advantageous as less Lyapunov variables are involved. Certainly, the H_∞-norm at a single frequency can also be obtained by direct computation without using the KYP lemma or the GKYP lemma. However, this result points out that there may exist some ways to balance the complexity and accuracy for frequency-limited performance analysis. If similar results can be developed for the general interval-type frequency ranges, this will serve as a good bridge between the GKYP lemma and the KYP lemma. Besides, it is probable that the presented concepts, definitions and properties about the frequency-dependent extended system may have other interpretations for the linear system theory.*

Definition 2. (Frequency-dependent Lyapunov Equations)
(a) Given a linear continuous-time system (1) and one of its corresponding Hurwitz stable ϖ-dependent extended systems (9), then the following two Lyapunov equation

$$\begin{aligned}A_\varpi W_{\varpi c} + W_{\varpi c} A_\varpi^* + B_\varpi B_\varpi^* = 0 \\ A_\varpi^* W_{\varpi o} + W_{\varpi o} A_\varpi + C_\varpi^* C_\varpi = 0\end{aligned} \tag{24}$$

are defined as ϖ-dependent controllability and observability Lyapunov equations *of the continuous-time system (1). Furthermore, the solutions* $W_{\varpi c}$ *and* $W_{\varpi o}$ *will be called* ϖ-dependent controllability and observability Gramians *of the continuous-time system (1).*
(b) Given a linear discrete-time system (1) and one of its corresponding Schur stable ϑ*-dependent extended systems (12), then the following two Lyapunov equations*

$$\begin{aligned} A_\vartheta W_{\vartheta c} A_\vartheta^* - W_{\vartheta c} + B_\vartheta B_\vartheta^* = 0 \\ A_\vartheta^* W_{\vartheta o} A_\vartheta - W_{\vartheta o} + C_\vartheta^* C_\vartheta = 0 \end{aligned} \tag{25}$$

are defined as ϑ-dependent controllability and observability Lyapunov equations *of the discrete-time system (1). Furthermore, the solutions* $W_{\vartheta c}$ *and* $W_{\vartheta o}$ *will be called* ϑ-dependent controllability and observability Gramians *of the discrete-time system (1).*

Definition 3. (Frequency-dependent Balanced Realization)
(*a) Given a linear continuous-time system (1) and one of its Hurwitz stable* ϖ*-dependent extended systems (9), the corresponding* ϖ*-dependent controllability and observability Gramians are equal and diagonal, i.e. the following Lyapunov equations*

$$\begin{aligned} A_\varpi \Sigma_\varpi + \Sigma_\varpi A_\varpi^* + B_\varpi B_\varpi^* = 0 \\ A_\varpi^* \Sigma_\varpi + \Sigma_\varpi A_\varpi + C_\varpi^* C_\varpi = 0 \end{aligned} \tag{26}$$

simultaneously hold, then this particular realization will be referred to as a ϖ-dependent balanced realization*.*
(b) Given a linear discrete-time system (1) and one of its Schur stable ϑ*-dependent extended systems (12), the corresponding* ϑ*-dependent controllability and observability Gramians are equal and diagonal, i.e. the following Lyapunov equations*

$$\begin{aligned} A_\vartheta \Sigma_\vartheta A_\vartheta^* + \Sigma_\vartheta + B_\vartheta B_\vartheta^* = 0 \\ A_\vartheta^* \Sigma_\vartheta A_\vartheta + \Sigma_\vartheta + C_\vartheta^* C_\vartheta = 0 \end{aligned} \tag{27}$$

simultaneously hold, then this particular realization will be referred to as a ϑ-dependent balanced realization*.*

Theorem 2.
(a) Suppose the linear continuous-time system (1) is Hurwitz stable, and denote its controllability and observability and balanced Gramian matrices as W_c, W_o, Σ*, then for any* ϖ*-dependent extended system (9), the following statements are true:*
(a.1) $W_c > W_{\varpi c}$, $W_c > W_{\varpi c}$, $\Sigma > \Sigma_\varpi$,
(a.2) $\lim\limits_{\varepsilon \to 0} W_{\varpi c} = 0,\ \lim\limits_{\varepsilon \to 0} W_{\varpi o} = 0,\ \lim\limits_{\varepsilon \to 0} \Sigma_\varpi = 0,$
(a.3) $\lim\limits_{\varepsilon \to \infty} W_{\varpi c} = W_c,\ \lim\limits_{\varepsilon \to \infty} W_{\varpi o} = W_o,\ \lim\limits_{\varepsilon \to \infty} \Sigma_\varpi = \Sigma.$
(b) Suppose the linear discrete-time system (1) is Schur stable, we abuse notation somewhat by denoting its controllability and observability Gramians matrices as W_c, W_o, Σ*, then for any* ϖ*-dependent extended system (12), the following statements are true:*
(b1). $W_c > W_{\vartheta c}$, $W_c > W_{\vartheta c}$, $\Sigma > \Sigma_\vartheta$,
(b.2) $\lim\limits_{\varepsilon \to 0} W_{\vartheta c} = 0,\ \lim\limits_{\varepsilon \to 0} W_{\vartheta o} = 0,\ \lim\limits_{\varepsilon \to 0} \Sigma_\vartheta = 0,$
(b.3) $\lim\limits_{\varepsilon \to \infty} W_{\vartheta c} = W_c,\ \lim\limits_{\varepsilon \to \infty} W_{\vartheta o} = W_o,\ \lim\limits_{\varepsilon \to \infty} \Sigma_\vartheta = \Sigma.$

Proof.
(a.1) It is well known that the controllability and observability Gramian matrices W_c, W_o of system (1) satisfy the following Lyapunov equations:

$$\begin{aligned} AW_c + W_c A^* + BB^* = 0 \\ A^* W_o + W_o A + C^* C = 0. \end{aligned} \tag{28}$$

Post-and-pre multiply the ϖ-dependent Lyapunov Equations (24) by $\epsilon^{-1}(\epsilon I + \jmath\varpi I - A)$, then we have

$$\begin{aligned} AW_{\varpi c} + W_{\varpi c} A^* + 2\epsilon^{-1}(\jmath\varpi I - A) W_{\varpi c} (\jmath\varpi I - A)^* + BB^* = 0 \\ A^* W_{\varpi o} + W_{\varpi o} A + 2\epsilon^{-1}(\jmath\varpi I - A)^* W_{\varpi o} (\jmath\varpi I - A) + BB^* = 0. \end{aligned} \tag{29}$$

Then the following equations can be derived by subtracting the equations (28) from (29)

$$\begin{array}{l} A(W_c - W_{\varpi c}) + (W_c - W_{\varpi c})A^* + 2\epsilon^{-1}(\jmath\varpi I - A)W_{\varpi c}(\jmath\varpi I - A)^* = 0 \\ A^*(W_o - W_{\varpi o}) + (W_o - W_{\varpi o})A + 2\epsilon^{-1}(\jmath\varpi I - A)^* W_{\varpi o}(\jmath\varpi I - A) = 0 \end{array} \tag{30}$$

since the system (1) is supposed to be Hurwitz stable, we can conclude that $(W_c - W_{\varpi c}) > 0$ and $(W_o - W_{\varpi o}) > 0$ since

$$\begin{array}{l} 2\epsilon^{-1}(\jmath\varpi I - A)W_{\varpi c}(\jmath\varpi I - A)^* > 0 \\ 2\epsilon^{-1}(\jmath\varpi I - A)^* W_{\varpi o}(\jmath\varpi I - A) > 0. \end{array} \tag{31}$$

(a.2) Let us introduce two auxiliary Lyapunov matrices as follow

$$\hat{W}_{\varpi c} = \varepsilon^{-1} W_{\varpi c}, \ \hat{W}_{\varpi o} = \varepsilon^{-1} W_{\varpi o}.$$

This leads to the following Lyapunov equations:

$$\begin{array}{l} (\jmath\varpi I - A)(\varepsilon I + \jmath\varpi I - A)^{-1}\hat{W}_{\varpi c} + \hat{W}_{\varpi c}(\varepsilon I + \jmath\varpi I - A)^{-*}(\jmath\varpi I - A)^* \\ = (\varepsilon I + \jmath\varpi I - A)^{-1}BB^*(\varepsilon I + \jmath\varpi I - A)^{-*} \\ (\jmath\varpi I - A)^*(\varepsilon I + \jmath\varpi I - A)^{-*}\hat{W}_{\varpi o} + \hat{W}_{\varpi o}(\varepsilon I + \jmath\varpi I - A)^{-1}(\jmath\varpi I - A) \\ = (\varepsilon I + \jmath\varpi I - A)^{-*}C^*C(\varepsilon I + \jmath\varpi I - A)^{-1}. \end{array} \tag{32}$$

From (32) one can conclude that:

$$\begin{array}{l} \lim_{\varepsilon\to 0} \hat{W}_{\varpi c} = \frac{1}{2}(\jmath\varpi I - A)^{-1}BB^*(\jmath\varpi I - A)^{-*}, \\ \lim_{\varepsilon\to 0} \hat{W}_{\varpi o} = \frac{1}{2}(\jmath\varpi I - A)^{-*}C^*C(\jmath\varpi I - A)^{-1}. \end{array}$$

Therefore,

$$\lim_{\varepsilon\to 0} W_{\varpi c} = \lim_{\varepsilon\to 0} \varepsilon\hat{W}_{\varpi c} = 0, \quad \lim_{\varepsilon\to 0} W_{\varpi c} = \lim_{\varepsilon\to 0} \varepsilon\hat{W}_{\varpi o} = 0.$$

(a.3) It can be easily observed that the ϖ-dependent matrices $A_\varpi, B_\varpi, C_\varpi$ will recover A, B, C as $\epsilon \to \infty$, i.e.

$$\begin{array}{l} \lim_{\varepsilon\to\infty} A_\varpi = \lim_{\varepsilon\to\infty} (\jmath\varpi I - \epsilon(\epsilon I + \jmath\varpi I - A)^{-1}(\jmath\varpi I - A)) = A, \\ \lim_{\varepsilon\to\infty} B_\varpi = \lim_{\varepsilon\to\infty} \epsilon(\epsilon I + \jmath\varpi I - A)^{-1}B = B, \\ \lim_{\varepsilon\to\infty} C_\varpi = \lim_{\varepsilon\to\infty} \epsilon C(\epsilon I + \jmath\varpi I - A)^{-1} = C. \end{array} \tag{33}$$

Then it is trivial to conclude that

$$\lim_{\epsilon\to\infty} W_{\varpi c} = W_c, \ \lim_{\epsilon\to\infty} W_{\varpi o} = W_o, \ \lim_{\epsilon\to\infty} \Sigma_\varpi = \Sigma.$$

The discrete-time case can be shown in a similar way and is omitted here.

4 Frequency-dependent Balanced Truncation

The following theorem provides the basis for our new model reduction method.

Theorem 3. (Frequency-dependent Balanced Truncation)
(a) Given a linear continuous-time system (1) with a pre-specified operating frequency ϖ, then for any one of its Hurwitz stable ϖ-dependent extended systems (9) given in ϖ-dependent balanced realization with respect to the ϖ-dependent Gramian $\Sigma_\varpi = \text{diag}(\Sigma_{1\varpi}, \Sigma_{2\varpi})$

$$\Sigma_{1\varpi} = \text{diag}(\sigma_{1\varpi}, \sigma_{2\varpi}, \ldots, \sigma_{r\varpi}), \Sigma_{2\varpi} = \text{diag}(\sigma_{(r+1)\varpi}, \sigma_{(r+2)\varpi}, \ldots, \sigma_{n\varpi}),$$

and $\sigma_{1\varpi} \geq \sigma_{2\varpi} \geq \ldots \geq \sigma_{r\varpi} \geq \sigma_{(r+1)\varpi} \geq \sigma_{(r+2)\varpi} \geq \ldots \geq \sigma_{n\varpi}$, *the desired* r^{th}*-order model* $G_r(\jmath\omega) := \left[\begin{array}{c|c} A_r & B_r \\ \hline C_r & D_r \end{array}\right]$ *is given by:*

$$\begin{array}{l} A_r = \jmath\varpi I - \epsilon Z_r(\jmath\varpi I - A_\varpi)Z_r^T(\epsilon I - Z_r(\jmath\varpi I - A_\varpi)Z_r^T)^{-1}, \\ B_r = \epsilon^{-1}(\epsilon I + \jmath\varpi I - A_r)Z_r B_\varpi, \\ C_r = \epsilon^{-1}C_\varpi Z_r^T(\epsilon I + \jmath\varpi I - A_r), \\ D_r = D_\varpi - C_r(\epsilon I + \jmath\varpi I - A_r)^{-1}B_r, \end{array} \tag{34}$$

where $Z_r = [I^{r\times r} \;\; 0^{r\times(n-r)}]$. *Furthermore, the truncated model* $G_r(\jmath\omega)$ *possesses the following properties:*
(a.1) If $\|G(\jmath\varpi)\|_2 < +\infty$, then $\|G_r(\jmath\varpi)\|_2 < +\infty$.
(a.2) The approximation error between the original system model (1) and the truncated model (34) at the given frequency ϖ *satisfies the error bound:*

$$\|G(\jmath\varpi) - G_r(\jmath\varpi)\|_2 \leq 2\sum_{i=r+1}^{n}\sigma_{i\varpi}. \tag{35}$$

(b) Given a linear discrete-time system (1) with a pre-specified operating frequency ϖ, *then for anyone of its Schur stable* ϑ*-dependent extended systems (12) given in* ϑ*-dependent balanced realization with respect to the* ϑ*-dependent Gramian* $\Sigma_\vartheta = \operatorname{diag}(\Sigma_{1\vartheta}, \Sigma_{2\vartheta})$

$$\begin{array}{l} \Sigma_{1\vartheta} = \operatorname{diag}(\sigma_{1\vartheta}, \sigma_{2\vartheta}, \ldots, \sigma_{r\vartheta}), \\ \Sigma_{2\vartheta} = \operatorname{diag}(\sigma_{(r+1)\vartheta}, \sigma_{(r+2)\vartheta}, \ldots, \sigma_{n\vartheta}), \end{array}$$

and $\sigma_{1\vartheta} \geq \sigma_{2\vartheta} \geq \ldots \geq \sigma_{r\vartheta} \geq \sigma_{(r+1)\vartheta} \geq \ldots \geq \sigma_{(n-1)\vartheta} \geq \sigma_{n\vartheta}$, *the desired* r^{th}*-order model* $G_r(e^{\jmath\theta}) := \left[\begin{array}{c|c} A_r & B_r \\ \hline C_r & D_r \end{array}\right]$ *is given by:*

$$\begin{array}{l} A_r = e^{\jmath\vartheta}I - \epsilon Z_r(e^{\jmath\vartheta}I - A_\vartheta)Z_r^T(\epsilon I - Z_r(e^{\jmath\vartheta}I - A_\vartheta)Z_r^T)^{-1}, \\ B_r = \epsilon^{-1}(\epsilon I + e^{\jmath\vartheta}I - A_r)Z_r B_\vartheta, \\ C_r = \epsilon^{-1}C_\vartheta Z_r^T(\epsilon I + e^{\jmath\vartheta}I - A_r), \\ D_r = D_\vartheta - C_r(\epsilon I + e^{\jmath\vartheta}I - A_r)^{-1}B_r, \end{array} \tag{36}$$

if $\vartheta \in [-\pi/2, \pi/2]$, *and*

$$\begin{array}{l} A_r = -e^{\jmath\vartheta}I - Z_r(-e^{\jmath\vartheta}I - A_\vartheta)Z_r^T(\epsilon I - Z_r(-e^{\jmath\vartheta}I - A_\vartheta)Z_r^T)^{-1}, \\ B_r = \epsilon^{-1}(\epsilon I - e^{\jmath\vartheta}I - A_r)Z_r B_\vartheta, \\ C_r = \epsilon^{-1}C_\vartheta Z_r^T(\epsilon I - e^{\jmath\vartheta}I - A_r), \\ D_r = D_\vartheta - C_r(\epsilon I - e^{\jmath\vartheta}I - A_r)^{-1}B_r, \end{array} \tag{37}$$

if $\vartheta \in [-\pi, -\pi/2]$ *or* $\vartheta \in [\pi/2, \pi]$, *where* $Z_r = [I^{r\times r} \;\; 0^{r\times(n-r)}]$. *Furthermore, the truncated model* $G_r(e^{\jmath\theta})$ *possesses the following properties*
(b.1) If $\|G(e^{\jmath\vartheta})\|_2 < +\infty$, *then* $\|G_r(e^{\jmath\vartheta})\|_2 < +\infty$.
(b.2) The approximation error between the given system model and the truncated model at the given frequency ϑ *satisfies the error bound*

$$\|G(e^{\jmath\vartheta}) - G_r(e^{\jmath\vartheta})\|_2 \leq 2\sum_{i=r+1}^{n}\sigma_{i\vartheta}. \tag{38}$$

Proof.
(a.1) Let denote $G_{r\varpi}(\jmath\omega) := \left[\begin{array}{c|c} A_{r\varpi} & B_{r\varpi} \\ \hline C_{r\varpi} & D_{r\varpi} \end{array}\right]$ as the corresponding ϖ-dependent extended system of the truncated reduced-order system $G_r(\jmath\omega) := \left[\begin{array}{c|c} A_r & B_r \\ \hline C_r & D_r \end{array}\right]$, where

$$\left[\begin{array}{c|c} A_{r\vartheta} & B_{r\vartheta} \\ \hline C_{r\vartheta} & D_{r\vartheta} \end{array}\right] = \left[\begin{array}{c|c} \jmath\varpi I - \epsilon(\epsilon I + \jmath\varpi I - A_r)^{-1}(\jmath\varpi I - A_r) & \epsilon(\epsilon I + \jmath\varpi I - A_r)^{-1}B_r \\ \hline \epsilon C_r(\epsilon I + \jmath\varpi I - A_r)^{-1} & D_r + C_r(\epsilon I + \jmath\varpi I - A_r)^{-1}B_r \end{array}\right], \tag{39}$$

According to (34) and (39), we know $A_{r\varpi} = Z_r A_\varpi Z_r^T$, therefore $A_{r\varpi}$ is stable since A_ϖ is stable (see Theorem 7.1 of [7]). Furthermore, we have $\|G_r(\jmath\varpi)\|_2 = \|G_{r\varpi}(\jmath\varpi)\|_2 < \infty$.

(a.2) The detailed proof for $r = n-1$ case will be provided in the sequel, and the $r = n-2, \ldots 1$ cases can be easily completed step by step [7].

The error system model $\mathcal{E}(\jmath\omega)$ between the original high-order system model $G(\jmath\omega)$ and the truncated low-order system model $G_r(\jmath\omega)$ can be represented by

$$\mathcal{E}(\jmath\omega) = G(\jmath\omega) - G_r(\jmath\omega) =: \left[\begin{array}{c|c} \mathcal{A}_e & \mathcal{B}_e \\ \hline \mathcal{C}_e & \mathcal{D}_e \end{array}\right] = \left[\begin{array}{cc|c} A_r & 0 & B_r \\ 0 & A & B \\ \hline -C_r & C & D - D_r \end{array}\right]. \tag{40}$$

The corresponding ϖ-dependent extended error system model $\mathcal{E}_\varpi(\jmath\omega)$ can be represented by

$$\begin{aligned} \mathcal{E}_\varpi(\jmath\omega) &=: \left[\begin{array}{c|c} \mathcal{A}_{e\varpi} & \mathcal{B}_{e\varpi} \\ \hline \mathcal{C}_{e\varpi} & \mathcal{D}_{e\varpi} \end{array}\right] \\ &= \left[\begin{array}{c|c} \jmath\varpi I - \epsilon(\epsilon I + \jmath\varpi I - \mathcal{A}_e)^{-1}(\jmath\varpi I - \mathcal{A}_e) & \epsilon(\epsilon I + \jmath\varpi I - \mathcal{A}_e)^{-1}\mathcal{B}_e \\ \hline \epsilon\mathcal{C}_e(\epsilon I + \jmath\varpi I - \mathcal{A}_e)^{-1} & \mathcal{D}_e + \mathcal{C}_e(\epsilon I + \jmath\varpi I - \mathcal{A}_e)^{-1}\mathcal{B}_e \end{array}\right]. \end{aligned} \tag{41}$$

Combining (42), (40) and (41), we have

$$\begin{aligned} &\mathcal{A}_{e\varpi}\hat{\mathcal{Z}}\Sigma\hat{\mathcal{Z}}^T + \hat{\mathcal{Z}}\Sigma\hat{\mathcal{Z}}^T\mathcal{A}_{e\varpi}^* + \mathcal{B}_{e\varpi}\mathcal{B}_{e\varpi}^* = 0, \\ &\mathcal{A}_{e\varpi}^*\check{\mathcal{Z}}\Sigma\check{\mathcal{Z}}^T + \check{\mathcal{Z}}\Sigma\check{\mathcal{Z}}^T\mathcal{A}_{e\varpi} + \mathcal{C}_{e\varpi}^*\mathcal{C}_{e\varpi} = 0, \end{aligned} \tag{42}$$

where $\hat{\mathcal{Z}} = [Z^T \quad I]^T$ and $\check{\mathcal{Z}} = [-Z^T \quad I]^T$. From the error system $\mathcal{E}(\jmath\omega)$, we can construct a dilated system $\mathscr{E}(\jmath\omega)$ which is an H_∞ performance preserving one with respect to the error system $\mathcal{E}(\jmath\omega)$. Furthermore, the desired dilated system can be constructed as follow:

$$\mathscr{E}(\jmath\omega) = \left[\begin{array}{c|c} \mathscr{A}_e & \mathscr{B}_e \\ \hline \mathscr{C}_e & \mathscr{D}_e \end{array}\right] = \left[\begin{array}{c|cc} \mathcal{A}_e & \mathcal{B}_e & \mathcal{B}_d \\ \hline \mathcal{C}_e & \mathcal{D}_e & \mathcal{D}_{d11} \\ \mathcal{C}_d & \mathcal{D}_{d12} & \mathcal{D}_{d22} \end{array}\right], \tag{43}$$

where $\mathcal{B}_d, \mathcal{C}_d, \mathcal{D}_{d12}, \mathcal{D}_{d21}, \mathcal{D}_{22}$ are auxiliary 'dilated' matrices, and those matrices are constructed as follows:

$$\begin{aligned} \mathcal{B}_d &= -\sigma_{n\varpi}(\epsilon I + \jmath\varpi I - \mathcal{A}_e)\check{\mathcal{Z}}\Sigma^{-1}C_\varpi^*, \\ \mathcal{C}_d{}^* &= -\sigma_{n\varpi}(\epsilon I + \jmath\varpi I - \mathcal{A}_e)^T\hat{\mathcal{Z}}\Sigma^{-1}B_\varpi, \\ \mathcal{D}_{d12} &= -\mathcal{C}_e(\epsilon I + \jmath\varpi I - \mathcal{A}_e)^{-1}\mathcal{B}_d + 2\sigma_{n\varpi}I, \\ \mathcal{D}_{d21} &= -\mathcal{C}_d(\epsilon I + \jmath\varpi I - \mathcal{A}_e)^{-1}\mathcal{B}_e + 2\sigma_{n\varpi}I, \\ \mathcal{D}_{d22} &= -\mathcal{C}_d(\epsilon I + \jmath\varpi I - \mathcal{A}_e)^{-1}\mathcal{B}_d. \end{aligned} \tag{44}$$

Defining the Lyapunov variable $\mathscr{Q} \geq 0$ and $\mathscr{P}$ as follows:

$$\begin{aligned} &\mathscr{Q} = 2\epsilon^{-1}\hat{\mathcal{Z}}\Sigma\hat{\mathcal{Z}}^T + 2\epsilon^{-1}\sigma_{n\varpi}^2\check{\mathcal{Z}}\Sigma^{-1}\check{\mathcal{Z}}^T \geq 0, \\ &\mathscr{P} = \hat{\mathcal{Z}}\Sigma\hat{\mathcal{Z}}^T + \sigma_{n\varpi}^2\check{\mathcal{Z}}\Sigma\check{\mathcal{Z}}^T. \end{aligned} \tag{45}$$

We conclude the following GKYP-form equality:

$$\begin{bmatrix} \mathscr{A}_e & I \\ \mathscr{C}_e & 0 \end{bmatrix} \begin{bmatrix} -\mathscr{Q} & \mathscr{P} + \jmath\varpi\mathscr{Q} \\ \mathscr{P} - \jmath\varpi\mathscr{Q} & -\varpi^2\mathscr{Q} \end{bmatrix} \begin{bmatrix} \mathscr{A}_e & I \\ \mathscr{C}_e & 0 \end{bmatrix}^* + \begin{bmatrix} \mathscr{B}_e & 0 \\ \mathscr{D}_e & I \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & -(2\sigma_{n\varpi})^2 \end{bmatrix} \begin{bmatrix} \mathscr{B}_e & 0 \\ \mathscr{D}_e & I \end{bmatrix}^* = 0, \tag{46}$$

which can be validated in detail by referring to the proof of Theorem 1 (a.8) and the constructive proof of standard balanced truncation in [7], and the details behind this equation are omitted here for simplicity of presentation. According to the GKYP Lemma, the dilated error systems $\mathscr{E}(\jmath\omega)$ satisfies $\|\mathscr{E}(\jmath\varpi)\|_2 \leq 2\sigma_{n\varpi}$. Therefore the error system $\mathcal{E}(\jmath\omega)$ satisfies $\|\mathcal{E}(\jmath\varpi)\|_2 \leq \|\mathscr{E}(\jmath\varpi)\|_2 \leq 2\sigma_{n\varpi}$.

The proof of the discrete-time counterparts can be fulfilled in a similar way and is therefore omitted here for brevity.

Remark 2: *According to Theorems 2 and 3, it is clear that the approximation error at a given frequency can be asymptotically regulated to be zero by adjusting the scalar* $\epsilon \to 0$. *For the single-frequency model reduction problem, better approximation performance will be obtained with smaller* ϵ*, however, adopting an extremely small* ϵ *in practical implementation may lead to numerical difficulties due to the rapid rank decay of the corresponding frequency-dependent Gramian matrices when* $\epsilon \to 0$. *On the other hand, the proposed FDBT can be expected as an alternative way to deal with interval-type finite frequency (such as* $\omega \in [\varpi_1, \varpi_2]$*) model reduction problems by* $\varpi = (\varpi_1 + \varpi_2)$. *Under such a circumstance, letting the parameter* ϵ *become very small is not meaningful since the approximation performance at the frequency points far from* ϖ. *It is an open question how to obtain an optimal* ε.

Remark 3: *To transform any minimal realization of system (1) into the frequency-dependent balanced realization, one can resort to the* SVD*-based simultaneous diagonalization algorithm [10]. If the given state space realization of (1) is non-minimal, we refer to [39] for the corresponding algorithm. Besides, to overcome the rank-deficient problem of frequency-dependent Gramian matrices which commonly occurs if the dimension of the original system* n *is very large or the parameter* ϵ *is very small, one can implement the proposed FDBT method by combing the techniques adopted for implementing the SBT via low rank approximate Gramians (See [40] and the references therein).*

Remark 4: *There exist other model reduction methods approximating the original system very well at a single frequency. The approximation error at a single frequency* ϖ *can be made zero via moment-matching (MM) model reduction method [41] [42]. Singular perturbation approximation (SPA) [43] [44] can be used for exact approximation in the case that* $\varpi = 0$. *However, the proposed frequency-dependent balanced truncation theoretically provides an alternative way to eliminate the single frequency approximation error, it also presents a new viewpoint on the general frequency-limited model reduction problems.*

Remark 5: *It should be pointed out that the resulting parameter matrices of the reduced order model will become complex even if the original system is described by real data for a pre-specified frequency* $\varpi \neq 0$*, which may lead to difficulty for physical implementation. However, this not a problem if one only uses the reduced-order model to simulate the input-output relationships.*

5 Applications of Frequency-dependent Balanced Truncation

5.1 RLC ladder network

We consider an RLC ladder network [45] as depicted in Fig.1.

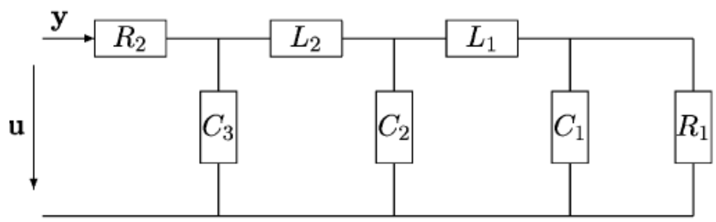

Figure 1: 5^{th} RLC ladder network

The input is the voltage u and the output is the current y as shown in Fig. 1 (see [45] for more details). It is assumed that all the capacitors and inductors have unit value, while $R_1 = 1/2, R_2 =$

1/5. A minimal realization for this circuit system is:

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] = \left[\begin{array}{ccc|ccc} -2 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & -5 & 2 \\ \hline 0 & 0 & 0 & 0 & -2 & 1 \end{array}\right]$$

Now, assume the input signal of this ladder RLC network is a direct current voltage signal (i.e. $\varpi = \omega(u(t){=}0)$. If we choose the adjusting scalar ϵ as $\epsilon = 1$, the corresponding ϖ-dependent extended system will be Hurwitz stable and can be transformed into ϖ-dependent balanced realization as follows :

$$\left[\begin{array}{c|c} A_{\varpi b} & B_{\varpi b} \\ \hline C_{\varpi b} & D_{\varpi b} \end{array}\right] = \left[\begin{array}{ccc|ccc} 1.8768 & 4.0037 & -1.4978 & 1.5619 & 0.2400 & -1.0368 \\ -4.0037 & -7.4976 & 3.1139 & -3.2476 & -0.4991 & 2.2633 \\ 1.4978 & 3.1139 & -2.5871 & 3.3460 & 0.5085 & -0.8755 \\ 1.5619 & 3.2476 & -3.3460 & 2.7219 & 0.3684 & -0.9130 \\ -0.2400 & -0.4991 & 0.5085 & -0.3684 & -1.5140 & 0.1403 \\ \hline -1.0368 & -2.2633 & 0.8755 & -0.9130 & -0.1403 & 0.3973 \end{array}\right]$$

and the corresponding balanced Gramian matrix is:

$$\Sigma = \text{diag}(0.0447, 0.0289, 2.3143, 4.3652 \times 10^{-5}, 6.1003 \times 10^{-8})$$

Adopting the proposed frequency-dependent balanced truncation method, we can easily obtain the parameter matrices of the reduced-order model with any pre-assigned order $r, r < n$. For example, we present the detailed 4^{th} reduced-order model as follows:

$$\left[\begin{array}{c|c} A_4 & B_4 \\ \hline C_4 & D_4 \end{array}\right] = \left[\begin{array}{cc|ccc} 1.8538 & 3.9561 & -1.4492 & 1.5268 & -1.0234 \\ -3.9561 & -7.3985 & 3.0130 & -3.1744 & 2.2354 \\ 1.4492 & 3.0130 & -2.4843 & 3.2715 & -0.8471 \\ 1.5268 & 3.1744 & -3.2715 & 2.6679 & -0.8924 \\ \hline -1.0234 & -2.2354 & 0.8471 & -0.8924 & 0.9922 \end{array}\right].$$

To show the advantage of the proposed method, the approximation error bound and the actual approximation error obtained by frequency-dependent balanced truncation (FDBT) and standard balanced truncation [5] (SBT) are all listed in the Table I. The sharpness of improving the approximation (H_∞) performance by ϖ-dependent balanced truncation can be obviously verified.

Table 1: Comparison of *FDBT (letting $\epsilon = 1$)* and *SBT*

r	FDBT		SBT	
	error bound	*actual error*$\|E(\jmath\varpi)\|$	*error bound*	*actual error*$\|E(\jmath\varpi)\|$
4	1.2201×10^{-7}	1.2201×10^{-7}	0.0006	0.0006
3	8.7426×10^{-5}	8.7182×10^{-5}	0.1752	0.1740
2	5.5028×10^{-4}	3.7568×10^{-4}	0.3914	0.0421
1	0.0584	0.0582	0.6311	0.1975

As revealed by Theorem 2, the frequency-dependent Gramian matrices $W_{\varpi c}, W_{\varpi c}$ and Σ_ϖ can be scaled from 0 to the standard Gramian matrices W_c, W_o and Σ by adjusting the parameter ϵ, to show this conclusion more clearly, the plots of frequency-dependent error bound versus scalar ϵ are depicted in Fig. 2.

5.2 Butterworth Filters

In this example, we will approximate four types of Butterworth filters [47] by means of SBT, FDBT, SPA and moment matching. These are:

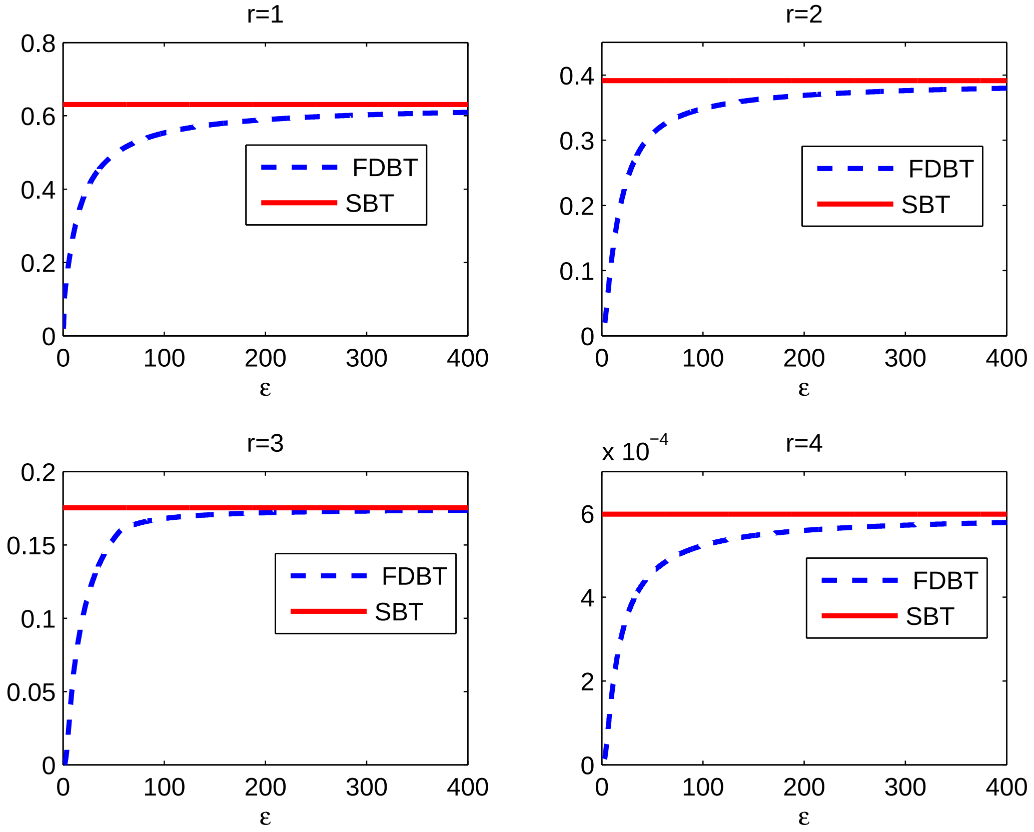

Figure 2: error bounds derived by FDBT and SBT

1. Σ_{BF2}–Continuous-time bandstop Butterworth filter of order 100 with the cutoff frequency being $90 - 110 rad/sec$, which can be generated by MATLAB command:
`[A,B,C,D]= butter(50,[90 110],'stop','s').`

2. Σ_{BF3}–Discrete-time lowpass Butterworth filter of order 100 with the cutoff frequency being $0.1\pi rad/sec$, which can be generated using the MATLAB command:
`[A,B,C,D]=butter(100, 0.1).`

3. Σ_{BF4}–Discrete-time bandpass Butterworth filter of order 200 with the cutoff frequency being $0.2\pi - 0.4\pi rad/sec$, which can be generated using the MATLAB command:
`[A,B,C,D]=butter(100,[0.2 0.4]).`

For those cases, the approximation performance over the cut-off frequency intervals are all very important. Although the proposed single frequency FDBT is only designed for the single frequency case, it can also be treated as an alternative way to solve the model reduction problems in this example by letting $\varpi = 0$ for Σ_{BF1}, $\varpi = 100$ for Σ_{BF2}, $\vartheta = 0$ for Σ_{BF3} and $\vartheta = 0.3\pi$ for Σ_{BF4}. We first compute the standard Hankel singular values σ_i and frequency-dependent Hankel singular values $\sigma_{\varpi i}$ ($\sigma_{\vartheta i}$) of the four Butterworth filters.

As Fig. 3 illustrates, the standard Hankel singular values σ_i stay constant at the beginning and only start to decay until the order gets greater than $\hat{r}$ ($\hat{r} = 25$ for $\Sigma_{BF1}, \Sigma_{BF2}, \Sigma_{BF3}$ and $\hat{r} = 50$ for Σ_{BF4}). In contrast, the decay rate of the frequency-dependent Hankel singular values $\sigma_{\varpi i}$ ($\sigma_{\vartheta i}$) is fast even for orders smaller than $\hat{r}$, which implies that better approximation performance at the assigned frequency point ϖ (ϑ) can be obtained by FDBT compared with SBT.

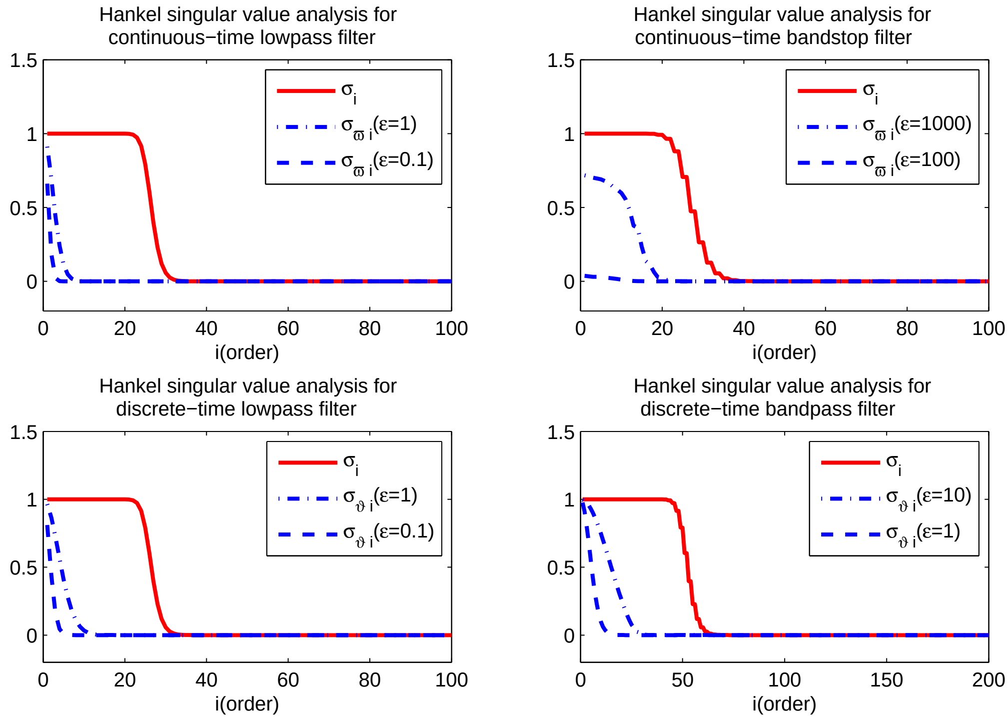

Figure 3: Hankel singular values analysis for the four filters

Next we approximate the continuous-time lowpass Butterworth filter Σ_{BF1} with a model of order 20 via SBT, SPA, FDBT, moment matching (MM) by expanding the transfer function $G(\jmath\omega)$ around ω=0 and multi-point MM by expanding the transfer function $G(\jmath\omega)$ around $\omega_1 = -1, \omega_2 = 0, \omega_3 = 1$ (The moment matching method is implemented via an Arnoldi procedure [41] throughout the paper). The sigma plots of the original filter system and the reduced systems are depicted in Fig. 4. Fig. 4 reveals the reduced systems generated by FDBT ($\epsilon = 0.1$) and MM are very close to each other and they are both much better than the reduced systems obtained via SBT and SPA. The parameter ϵ has an impact on the quality of approximation. However, as discussed in Remark 2, there is no systematic way to decide the most appropriate ϵ in this example. Thus, one has to pick a satisfactory ϵ by trial and error. In the example, using multi-point MM does not improve the overall approximation performance significantly compared with MM.

Next we approximate the continuous-time bandpass Butterworth filter Σ_{BF2} with a model of order 20 via SBT, FDBT and moment matching (MM) by expanding the transfer function $G(\jmath\omega)$ around ω=100 [41], multi-point MM by expanding $G(\jmath\omega)$ around $\omega_1 = 90, \omega_2 = 100, \omega_3 = 110$ and multi-point MM by expanding $G(\jmath\omega)$ around $\omega_1 = 80, \omega_2 = 100, \omega_3 = 120$. The sigma plots of the original filter system and the reduced systems are depicted in Fig. 5. Fig. 5 shows that the reduced systems generated by FDBT ($\epsilon = 100$) matches the original system very well while SBT failed again as expected. In this example, the approximation performance via FDBT ($\epsilon = 100$) is even much better than the MM and multi-point MM.

To approximate the discrete-time lowpass filter Σ_{BF3} with a model of order 20, SBT, FDBT and moment matching (expanding the transfer function $G(e^{\jmath\theta})$ around θ=0) are adopted here. The sigma plots of the original system and the reduced systems are depicted in Fig. 6. As Fig. 6 shows, the FDBT and MM yield acceptable approximation performance while SBT fails again. For the discrete-time bandpass filter Σ_{BF4} case, we assume the desired reduced order is 50 which is comparably larger then the reduced order in the above three cases. We apply SBT, FDBT and MM (expanding the transfer function $G(e^{\jmath\theta})$ around $\theta = 0.3\pi$) to compute the reduced models.

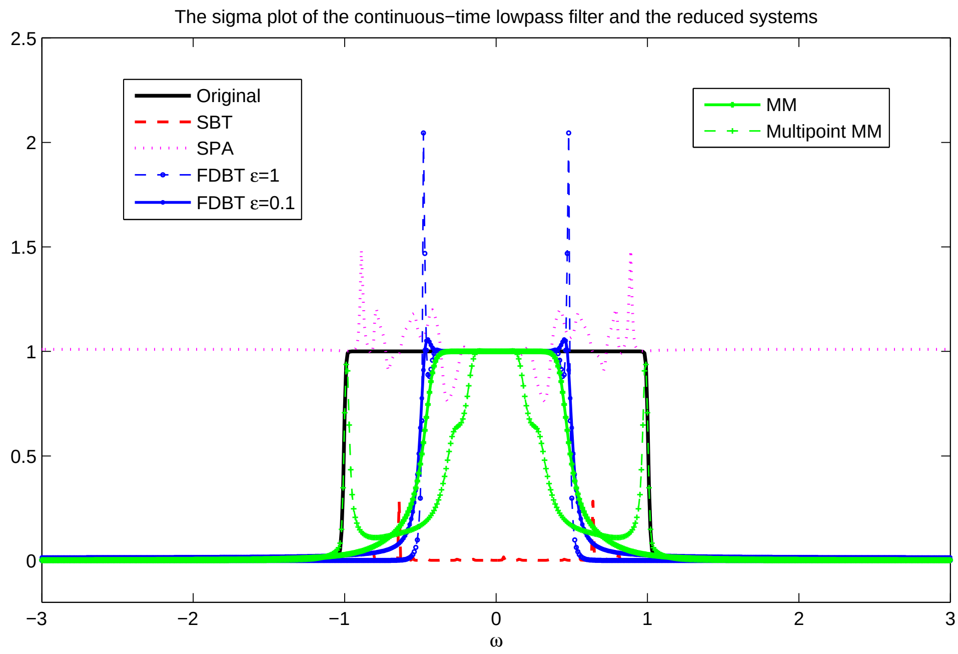

Figure 4: σ_{max} plot of original system and reduced systems of the continuous-time lowpass filter Σ_{BF1}

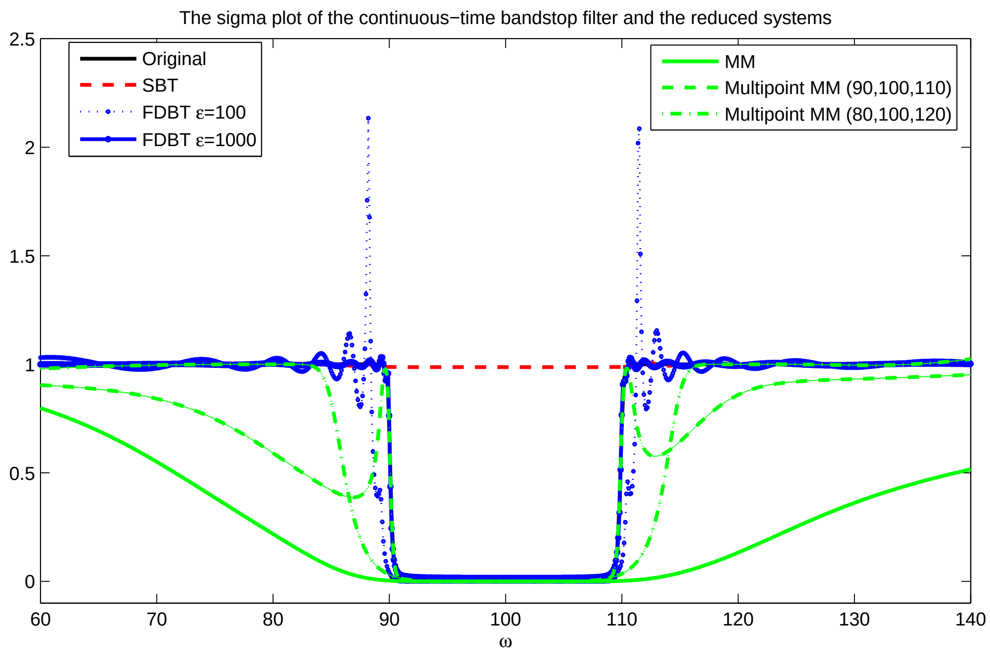

Figure 5: σ_{max} plot of original system and reduced systems of the continuous-time bandstop filter Σ_{BF2}

The sigma plots of the original system and the reduced systems are depicted in Fig. 7. As revealed by Fig. 7, SBT in this case performs better than in the above cases due to the reduced-order r being larger. FDBT ($\epsilon = 10$) yields the best approximation performance while MM also matches the original system well.

The results in this example indicate that FDBT is a promising alternative to solve the interval-type finite frequency model reduction problems.

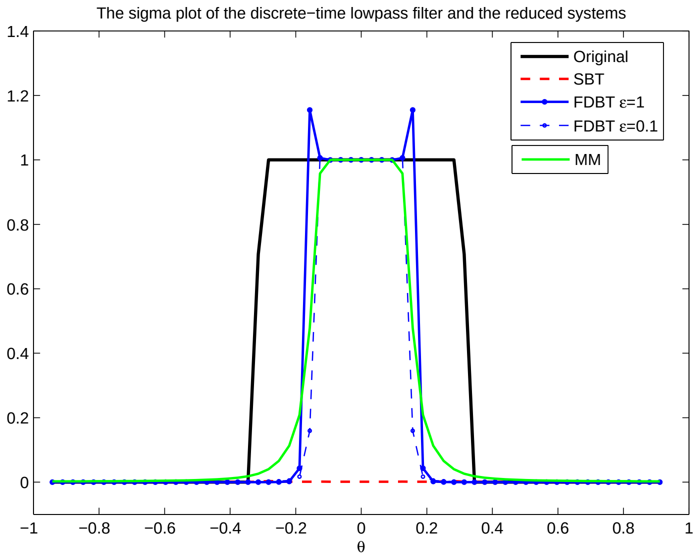

Figure 6: σ_{max} plot of original system and reduced systems of the discrete-time lowpass filter Σ_{BF3}

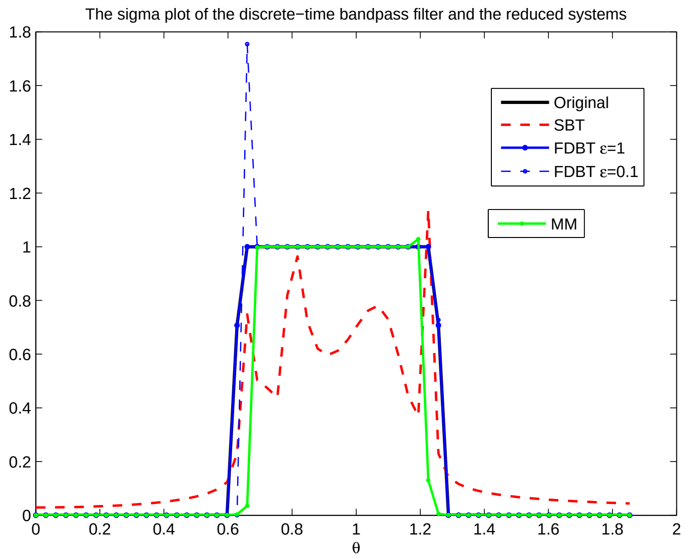

Figure 7: σ_{max} plot of original system and reduced systems of the discrete-time bandpass filter Σ_{BF4}

5.3 CD Player

This system describes the dynamics of a rotating arm compact disc mechanism. The model has 120 states with 2 inputs and 2 outputs. The interesting frequency range here is around $\varpi = 200 rad/sec$ [47]. Standard Hankel singular values σ_i and the corresponding frequency-dependent Hankel singular values $\sigma_{\varpi i}$ are illustrated by Fig. 8.

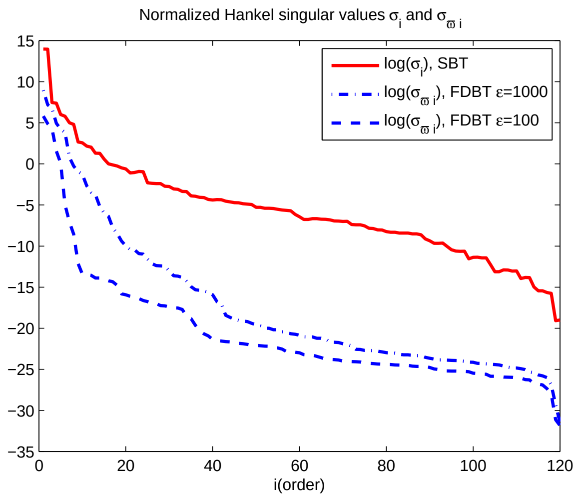

Figure 8: Normalized Hankel singular values σ_i and $\sigma_{\varpi i}$ of the CD player example.

We approximate the system with a model of order 12 via SBT, FDBT and moment matching (expand the transfer function $G(\jmath\omega)$ around $\omega = 200$). As seen from Fig. 9, all the reduced systems match the original system well and FDBT and MM can further improve approximation performance around the pre-specified frequency point $\varpi = 200$ with a similar level.

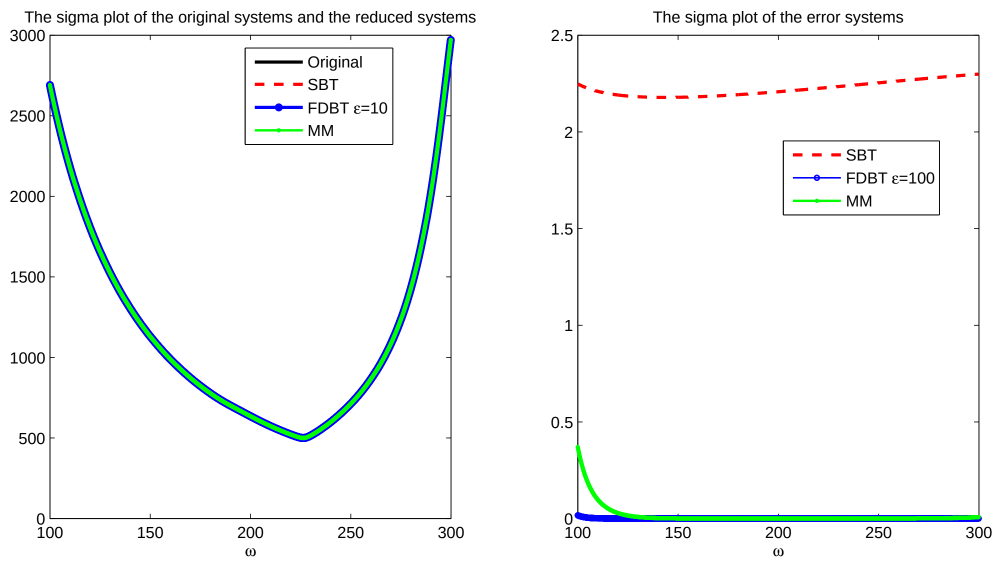

Figure 9: σ_{max} plot of the reduced and error systems of the CD player.

5.4 ISS (International Space Station)

This is a model of component 1r (Russian service module) of the International Space Station. It has 270 states, 3 inputs and 3 outputs. The interesting frequency range is $[0.5,\ 100] rad/sec$ [46], therefore we select $\varpi = 50$ for applying FDBT. The Hankel singular values of the original system and the corresponding frequency-dependent extended systems are illustrated in Fig. 10.

We approximate the system with a model of order 15 via SBT, FDBT and moment matching (expanding the transfer function $G(\jmath\omega)$ around $\omega = 50$). As seen from Fig. 11, FDBT not only performs better than SBT as expected, but also yields better approximation performance around the selected frequency point $\varpi = 50$ compared with MM in this example.

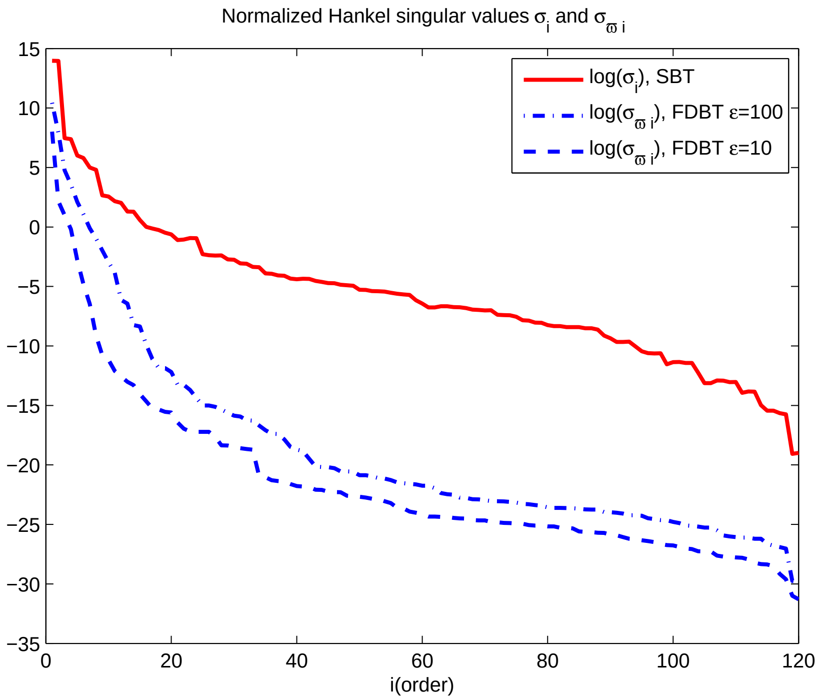

Figure 10: Normalized Hankel singular values σ_i and $\sigma_{\varpi i}$ of the ISS example.

6 Conclusions and Future Work

This paper is mainly dedicated to generalize the frequency-independent standard balanced truncation method to a frequency-dependent one. Under the special case that the operating frequency is a single value, the generalization was successful fulfilled by the proposed frequency-dependent balanced truncation. The results also point to the possibility and directions to further develop a more general frequency-dependent balanced truncation which can be used to solve the more general frequency-limited model reduction problems, in which the pre-specified operating frequency belongs to a known low/middle/high frequency range.

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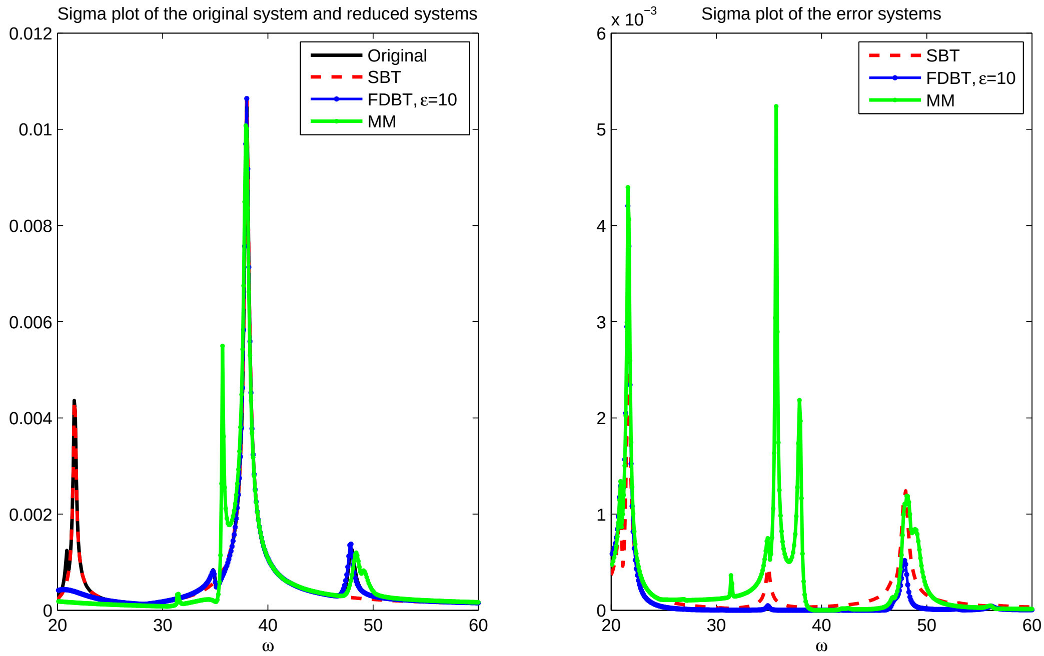

Figure 11: σ_{max} of the frequency response of the original system and the reduced systems of the ISS example.

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