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FORTRAN 77 Subroutines for the Solution of Skew-Hamiltonian/Hamiltonian Eigenproblems – Part I: Algorithms and Applications

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Abstract

Skew-Hamiltonian/Hamiltonian matrix pencils $\lambda\mathcal{S} - \mathcal{H}$ appear in many applications, including linear quadratic optimal control problems, $\mathcal{H}_\infty$-optimization, certain multi-body systems and many other areas in applied mathematics, physics, and chemistry. In these applications it is necessary to compute certain eigenvalues and/or corresponding deflating subspaces of these matrix pencils. Recently developed methods exploit and preserve the skew-Hamiltonian/Hamiltonian structure and hence increase reliability, accuracy and performance of the computations. In this paper we describe the corresponding algorithms which have been implemented in the style of subroutines of the Subroutine Library in Control Theory (SLICOT). Furthermore we address some of their applications. We describe variants for real and complex problems with versions for factored and unfactored matrices $\mathcal{S}$.

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1 Introduction

In this paper we discuss algorithms for the solution of generalized eigenvalue problems with skew-Hamiltonian/Hamiltonian structure. We are interested in the computation of certain eigenvalues and corresponding deflating subspaces. We have to deal with the following algebraic structures [4].

Definition 1.1. *Let* $\mathcal{J} := \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}$*, where* I_n *is the* $n \times n$ *identity matrix. For brevity of notation, we do not indicate the dimension with the matrix* $\mathcal{J}$ *and use it for all possible values of* n*.*

(i) A matrix $\mathcal{H} \in \mathbb{C}^{2n\times 2n}$ *is* Hamiltonian *if* $(\mathcal{H}\mathcal{J})^H = \mathcal{H}\mathcal{J}$*. The Lie algebra of Hamiltonian matrices in* $\mathbb{C}^{2n\times 2n}$ *is denoted by* $\mathbb{H}_{2n}$*.*

(ii) A matrix $\mathcal{S} \in \mathbb{C}^{2n\times 2n}$ *is* skew-Hamiltonian *if* $(\mathcal{S}\mathcal{J})^H = -\mathcal{S}\mathcal{J}$*. The Jordan algebra of skew-Hamiltonian matrices in* $\mathbb{C}^{2n\times 2n}$ *is denoted by* $\mathbb{SH}_{2n}$*.*

(iii) A matrix pencil $\lambda\mathcal{S} - \mathcal{H} \in \mathbb{C}^{2n\times 2n}$ *is* skew-Hamiltonian/Hamiltonian *if* $\mathcal{S} \in \mathbb{SH}_{2n}$ *and* $\mathcal{H} \in \mathbb{H}_{2n}$*.*

(iv) A matrix $\mathcal{S} \in \mathbb{C}^{2n\times 2n}$ *is* symplectic *if* $\mathcal{S}\mathcal{J}\mathcal{S}^H = \mathcal{J}$*. The Lie group of symplectic matrices in* $\mathbb{C}^{2n\times 2n}$ *is denoted by* $\mathbb{S}_{2n}$*.*

(v) A matrix $\mathcal{U} \in \mathbb{C}^{2n\times 2n}$ *is* unitary symplectic *if* $\mathcal{U}\mathcal{J}\mathcal{U}^H = \mathcal{J}$ *and* $\mathcal{U}\mathcal{U}^H = I_{2n}$*. The compact Lie group of unitary symplectic matrices in* $\mathbb{C}^{2n\times 2n}$ *is denoted by* $\mathbb{US}_{2n}$*.*

Note that a similar definition can be given for real matrices. As a convention, all following considerations also hold for real skew-Hamiltonian/Hamiltonian matrix pencils. Then, all matrices $\cdot^H$ must be replaced by $\cdot^T$, all (skew-)Hermitian matrices become (skew-)symmetric, and unitary matrices become orthogonal. More significant differences to the complex case are explicitly mentioned.

Skew-Hamiltonian/Hamiltonian matrix pencils satisfy certain properties which we will briefly state. Every skew-Hamiltonian/Hamiltonian matrix pencil can be written as $\lambda\mathcal{S} - \mathcal{H} = \lambda \begin{bmatrix} A & D \\ E & A^H \end{bmatrix} - \begin{bmatrix} B & F \\ G & -B^H \end{bmatrix}$ with skew-Hermitian matrices D, E and Hermitian matrices F, G. If λ is a (generalized) eigenvalue of $\lambda\mathcal{S} - \mathcal{H}$, so is also $-\bar{\lambda}$. In other words, eigenvalues which are not purely imaginary, occur in pairs. For real skew-Hamiltonian/Hamiltonian matrix pencils we also have a pairing of complex conjugate eigenvalues, i.e., if λ is an eigenvalue of $\lambda\mathcal{S} - \mathcal{H}$, so are also $\bar{\lambda}$, $-\lambda$, $-\bar{\lambda}$. This leads to eigenvalue pairs $(\lambda, -\lambda)$ if λ is purely real or purely imaginary, or otherwise to eigenvalue quadruples $(\lambda, \bar{\lambda}, -\lambda, -\bar{\lambda})$. The structure of skew-Hamiltonian/Hamiltonian matrix pencils is preserved under *$\mathcal{J}$-congruence transformations*, that is, $\lambda\tilde{\mathcal{S}} - \tilde{\mathcal{H}} := \mathcal{J}\mathcal{P}^H\mathcal{J}^T(\lambda\mathcal{S} - \mathcal{H})\mathcal{P}$ with nonsingular $\mathcal{P}$ is again skew-Hamiltonian/Hamiltonian. If we choose $\mathcal{P}$ unitary, we additionally preserve the condition of the problem. In this way there is hope that we can choose a unitary $\mathcal{J}$-congruence transformation to transform $\lambda\mathcal{S} - \mathcal{H}$ into a condensed form which reveals its eigenvalues and deflating subspaces.

A suitable candidate for this condensed form is the *structured Schur form*, i.e., we compute a unitary matrix $\mathcal{Q}$ such that

$$\mathcal{J}\mathcal{Q}^H\mathcal{J}^T(\lambda\mathcal{S}-\mathcal{H})\mathcal{Q} = \lambda\begin{bmatrix} S_{11} & S_{12} \\ 0 & S_{11}^H \end{bmatrix} - \begin{bmatrix} H_{11} & H_{12} \\ 0 & -H_{11}^H \end{bmatrix}$$

with the subpencil $\lambda S_{11} - H_{11}$ in generalized Schur form, where S_{11} is upper triangular, H_{11} is upper triangular (upper quasi-triangular in the real case), S_{12} is skew-Hermitian, and H_{12} is Hermitian. However, a structured Schur form does *not* necessarily exist. Conditions for the existence are proven in [17, 18] for the complex case or in [26] for the real case. This problem can be circumvented by embedding $\lambda\mathcal{S}-\mathcal{H}$ into a skew-Hamiltonian/Hamiltonian matrix pencil of double dimension in an appropriate way, as explained in Section 3. Throughout this paper we denote by $\Lambda_-(\mathcal{S},\mathcal{H})$, $\Lambda_0(\mathcal{S},\mathcal{H})$, $\Lambda_+(\mathcal{S},\mathcal{H})$ the set of finite eigenvalues of $\lambda\mathcal{S}-\mathcal{H}$ with negative, zero, and positive real parts, respectively. The set of infinite eigenvalues is denoted by $\Lambda_\infty(\mathcal{S},\mathcal{H})$. Multiple eigenvalues are repeated in $\Lambda_-(\mathcal{S},\mathcal{H})$, $\Lambda_0(\mathcal{S},\mathcal{H})$, $\Lambda_+(\mathcal{S},\mathcal{H})$, and $\Lambda_\infty(\mathcal{S},\mathcal{H})$ according to their algebraic multiplicity. The set of all eigenvalues counted according to multiplicity is $\Lambda(\mathcal{S},\mathcal{H})$. Similarly, we denote by $\mathrm{Def}_-(\mathcal{S},\mathcal{H})$, $\mathrm{Def}_0(\mathcal{S},\mathcal{H})$, $\mathrm{Def}_+(\mathcal{S},\mathcal{H})$, and $\mathrm{Def}_\infty(\mathcal{S},\mathcal{H})$ the right deflating subspaces corresponding to $\Lambda_-(\mathcal{S},\mathcal{H})$, $\Lambda_0(\mathcal{S},\mathcal{H})$, $\Lambda_+(\mathcal{S},\mathcal{H})$, and $\Lambda_\infty(\mathcal{S},\mathcal{H})$, respectively.

2 Applications

2.1 Linear-Quadratic Optimal Control

First we consider the continuous-time, infinite horizon, linear-quadratic optimal control problem:
choose a control function $u(t)$ to minimize the cost functional

$$\mathcal{S}_c := \int_{t_0}^{\infty} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}^H \begin{bmatrix} Q & S \\ S^H & R \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} \mathrm{d}t \tag{1}$$

subject to the linear time-invariant *descriptor system*

$$E\dot{x}(t) = Ax(t) + Bu(t), \quad x(t_0) = x^0. \tag{2}$$

Here, $u(t) \in \mathbb{C}^m$ is control input vector, $x(t) \in \mathbb{C}^n$ is the descriptor vector, and E, $A \in \mathbb{C}^{n\times n}$, $B \in \mathbb{C}^{n\times m}$, $Q = Q^H \in \mathbb{C}^{n\times n}$, $R = R^H \in \mathbb{C}^{m\times m}, S \in \mathbb{C}^{n\times m}$. For well-posedness, the $(m+n)\times(m+n)$ weighting matrix

$$\mathcal{R} = \begin{bmatrix} Q & S \\ S^H & R \end{bmatrix}$$

must be Hermitian and positive semidefinite. Typically, in addition to minimizing (1), the control $u(t)$ must make $x(t)$ asymptotically stable. under some conditions, the application of the maximum principle [19, 22] yields as a necessary condition that the control u satisfies the two-point boundary value problem of Euler-Lagrange equations

$$\mathcal{E}_c \begin{bmatrix} \dot{x}(t) \\ \dot{\mu}(t) \\ \dot{u}(t) \end{bmatrix} = \mathcal{A}_c \begin{bmatrix} x(t) \\ \mu(t) \\ u(t) \end{bmatrix}, \quad x(t_0) = x^0, \quad \lim_{t\to\infty} E^H \mu(t) = 0, \tag{3}$$

with the matrix pencil

$$\lambda \mathcal{E}_c - \mathcal{A}_c = \lambda \begin{bmatrix} E & 0 & 0 \\ 0 & -E^H & 0 \\ 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} A & 0 & B \\ Q & A^H & S \\ S^H & B^H & R \end{bmatrix}.$$

Assuming that the matrix R is nonsingular, we can substitute $u(t) = -R^{-1}\left(S^H x(t) + B^H \mu(t)\right)$ and system (3) simplifies to

$$\mathcal{S} \begin{bmatrix} \dot{x}(t) \\ \dot{\mu}(t) \end{bmatrix} = \mathcal{H} \begin{bmatrix} x(t) \\ \mu(t) \end{bmatrix}, \quad x(t_0) = x^0, \quad \lim_{t\to\infty} E^H \mu(t) = 0,$$

with the skew-Hamiltonian/Hamiltonian matrix pencil

$$\lambda \mathcal{S} - \mathcal{H} = \lambda \begin{bmatrix} E & 0 \\ 0 & E^H \end{bmatrix} - \begin{bmatrix} A - BR^{-1}S^H & -BR^{-1}B^H \\ SR^{-1}S^H - Q & -\left(A - BR^{-1}S^H\right)^H \end{bmatrix}. \tag{4}$$

The generalized algebraic Riccati equation associated to the skew-Hamiltonian/Hamiltonian matrix pencil is given by [14]

$$\begin{aligned} 0 &= Q - SR^{-1}S^H + X^H\left(A - BR^{-1}S^H\right) + \left(A - BR^{-1}S^H\right)^H X \\ &\quad - X^H\left(BR^{-1}B^H\right)X, \\ E^H X &= X^H E. \end{aligned} \tag{5}$$

Under certain conditions the optimal control $u_*(t)$ that stabilizes the descriptor system (2) can be constructed by using a stabilizing solution X_* of (5). The matrix X_* can be obtained by computing the deflating subspace of (4) associated to the finite eigenvalues with negative real parts and to some purely imaginary and infinite eigenvalues. Note, that when the matrix R is singular, the problem becomes much more involved. Then, one has to consider so-called (generalized) Lur'e equations instead of Riccati equations. However, there is also a connection between Lur'e equations and skew-Hamiltonian/Hamiltonian and related even matrix pencils [24, 25].

2.2 $\mathcal{H}_\infty$-Optimization

Similar structures as in Subsection 2.1 occur in $\mathcal{H}_\infty$-optimization [16]. Consider a descriptor system of the form

$$\mathbf{P}: \quad \begin{cases} E\dot{x}(t) = Ax(t) + B_1 w(t) + B_2 u(t), \\ \quad z(t) = C_1 x(t) + D_{11} w(t) + D_{12} u(t), \qquad x(t_0) = x^0, \\ \quad y(t) = C_2 x(t) + D_{21} w(t) + D_{22} u(t), \end{cases} \tag{6}$$

where E, $A \in \mathbb{R}^{n\times n}$, $B_i \in \mathbb{R}^{n\times m_i}$, $C_i \in \mathbb{R}^{p_i\times n}$, and $D_{ij} \in \mathbb{R}^{p_i\times m_j}$ for i, $j = 1, 2$. In this system, $x(t) \in \mathbb{R}^n$ is the (generalized) state vector, $u(t) \in \mathbb{R}^{m_2}$ is the control input vector, and $w(t) \in \mathbb{R}^{m_1}$ is an exogenous input that may include noise, linearization errors, and unmodeled dynamics. The vector $y(t) \in \mathbb{R}^{p_2}$ contains measured outputs, while $z(t) \in \mathbb{R}^{p_1}$ is a regulated output or an estimation error.

The $\mathcal{H}_\infty$ control problem is usually formulated in the frequency domain. For this we need the space $\mathcal{H}_\infty^{p\times m}$ which consists of all $\mathbb{C}^{p\times m}$-valued functions that are analytic and bounded in the open right half-plane $\mathbb{C}^+$. For $F \in \mathcal{H}_\infty^{p\times m}$, the $\mathcal{H}_\infty$-norm is defined by

$$\|F\|_{\mathcal{H}_\infty} := \sup_{s\in\mathbb{C}^+} \sigma_{\max}\left(F(s)\right),$$

where $\sigma_{\max}\left(F(s)\right)$ denotes the maximal singular value of the matrix $F(s)$. In robust control, $\|F\|_{\mathcal{H}_\infty}$ is used as a measure of the worst-case influence of the disturbances w on the output z, where in this case F is the transfer function mapping noise or disturbance inputs to error signals [27]. Solving the optimal $\mathcal{H}_\infty$ control problem is the task of designing a dynamic controller

$$\mathbf{K}: \quad \begin{cases} \hat{E}\dot{\hat{x}}(t) = \hat{A}\hat{x}(t) + \hat{B}y(t), \\ \quad u(t) = \hat{C}\hat{x}(t) + \hat{D}y(t), \end{cases} \tag{7}$$

with $\hat{E}$, $\hat{A} \in \mathbb{R}^{N\times N}$, $\hat{B} \in \mathbb{R}^{N\times p_2}$, $\hat{C} \in \mathbb{R}^{m_2\times N}$, $\hat{D} \in \mathbb{R}^{m_2\times p_2}$ such that the closed-loop system resulting from inserting (7) into (6), that is,

$$\begin{aligned} E\dot{x}(t) &= \left(A + B_2\hat{D}Z_1C_2\right)x(t) + B_2Z_2\hat{C}\hat{x}(t) + \left(B_1 + B_2\hat{D}Z_1D_{21}\right)w(t), \\ \hat{E}\dot{\hat{x}}(t) &= \hat{B}Z_1C_2x(t) + \left(\hat{A} + \hat{B}Z_1D_{22}\hat{C}\right)\hat{x}(t) + \hat{B}Z_1D_{21}w(t), \\ z(t) &= \left(C_1 + D_{12}Z_2\hat{D}C_2\right)x(t) + D_{12}Z_2\hat{C}\hat{x}(t) + \left(D_{11} + D_{12}\hat{D}Z_1D_{21}\right)w(t), \end{aligned} \tag{8}$$

with $Z_1 = \left(I_{p_2} - D_{22}\hat{D}\right)^{-1}$, and $Z_2 = \left(I_{m_2} - \hat{D}D_{22}\right)^{-1}$ has the following properties:

(i) System (8) is internally stable, that is, the solution $\begin{bmatrix} x(t) \\ \hat{x}(t) \end{bmatrix}$ of the system with $w \equiv 0$ is asymptotically stable, i.e., $\lim_{t\to\infty} \begin{bmatrix} x(t) \\ \hat{x}(t) \end{bmatrix} = 0$.

(ii) The closed-loop transfer function T_{zw} from w to z satisfies $T_{zw} \in \mathcal{H}_\infty^{p_1\times m_1}$ and is minimized in the $\mathcal{H}_\infty$-norm.

Closely related to the optimal $\mathcal{H}_\infty$ control problem is the modified optimal $\mathcal{H}_\infty$ control problem. For a given descriptor system of the form (6) we search the infimum value γ for which there exists an internally stabilizing dynamic controller of the form (7) such that the corresponding closed-loop system (8) satisfies $T_{zw} \in \mathcal{H}_\infty^{p_1\times m_1}$ with $\|T_{zw}\|_{\mathcal{H}_\infty} < \gamma$. For the construction of optimal controllers, one can make use of the following *even* matrix pencils (see [23] for a definition and related software)

$$\lambda N_H - M_H(\gamma) = \left[\begin{array}{cc|ccc} 0 & -\lambda E^T - A^T & 0 & 0 & -C_1^T \\ \lambda E - A & 0 & -B_1 & -B_2 & 0 \\ \hline 0 & -B_1^T & -\gamma^2 I_{m_1} & 0 & -D_{11}^T \\ 0 & -B_2^T & 0 & 0 & -D_{12}^T \\ -C_1 & 0 & -D_{11} & -D_{12} & -I_{p_1} \end{array}\right], \tag{9}$$

and

$$\lambda N_J - M_J(\gamma) = \left[\begin{array}{cc|ccc} 0 & -\lambda E - A & 0 & 0 & -B_1 \\ \lambda E^T - A^T & 0 & -C_1^T & -C_2^T & 0 \\ \hline 0 & -C_1 & -\gamma^2 I_{p_1} & 0 & -D_{11} \\ 0 & -C_2 & 0 & 0 & -D_{12} \\ -B_1^T & 0 & -D_{11}^T & -D_{12}^T & -I_{m_1} \end{array}\right], \tag{10}$$

which can be transformed to skew-Hamiltonian/Hamiltonian structure by using the method used in [3, 26]. Using appropriate deflating subspaces of the matrix pencils (9) and (10) it is possible to state conditions for the existence of an optimal $\mathcal{H}_\infty$ controller. Then we can check if these conditions are fulfilled for a given value of γ. Using a bisection scheme we can iteratively refine γ until a wanted accuracy is achieved (see [16, 5] for details). Note that the transformation to skew-Hamiltonian/Hamiltonian structure is done in order to compute the deflating subspaces in a structure-preserving manner which is still an open problem for even matrix pencils. Finally, when a suboptimal value γ has been found, one can compute the actual controller. The controller formulas are rather cumbersome and are therefore omitted. For details, see [15].

2.3 $\mathcal{L}_\infty$-Norm Computation

Finally, we briefly describe a method to compute the $\mathcal{L}_\infty$-norm of an LTI system using skew-Hamiltonian/Hamiltonian matrix pencils [26, 6, 7]. This norm plays an important role in robust control or model order reduction (see [1, 20, 27] and references therein). Consider a descriptor system

$$E\dot{x}(t) = Ax(t) + Bu(t), \tag{11}$$

$$y(t) = Cx(t) + Du(t), \tag{12}$$

with E, $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}, D \in \mathbb{R}^{p\times m}$, and descriptor vector $x(t) \in \mathbb{R}^n$, control vector $u(t) \in \mathbb{R}^m$, and output vector $y(t) \in \mathbb{R}^p$. For such a system its transfer function is given by

$$G(s) := C\left(sE - A\right)^{-1} B + D,$$

which directly maps inputs to outputs in the frequency domain [10]. We define the space $\mathcal{RL}_\infty^{p\times m}$ of all *proper* rational $p \times m$-matrix-valued transfer functions which are bounded on the imaginary axis. The natural norm of this space is the $\mathcal{L}_\infty$-norm, defined by

$$\|G\|_{\mathcal{L}_\infty} := \sup_{\omega\in\mathbb{R}} \sigma_{\max}\left(G(\mathrm{i}\omega)\right).$$

Consider the skew-Hamiltonian/Hamiltonian matrix pencils

$$\lambda N - M(\gamma) = \lambda \begin{bmatrix} E & 0 \\ 0 & E^T \end{bmatrix} - \begin{bmatrix} A - BR^{-1}D^TC & -\gamma BR^{-1}B^T \\ \gamma C^T S^{-1} C & -A^T + C^T DR^{-1}B^T \end{bmatrix} \tag{13}$$

with the matrices $R = D^TD - \gamma^2 I_m$, and $S = DD^T - \gamma^2 I_p$. It can be shown that if $\lambda E - A$ has no purely imaginary eigenvalue and $\gamma > \min_{\omega\in\mathbb{R}} \sigma_{\max}(G(\mathrm{i}\omega))$ is not a singular value of D, then $\|G\|_{\mathcal{L}_\infty} \geq \gamma$ if and only if $\lambda N - M(\gamma)$ has purely imaginary eigenvalues. In this way we can again use an iterative scheme to improve the value of γ until a wanted accuracy for the $\mathcal{L}_\infty$-norm is achieved.

3 Theory and Algorithm Description

In this section we briefly describe the theory behind the algorithms that we will use. We refer to [4, 2] for a very detailed analysis of the algorithms. We consider complex and real problems separately since there are significant differences in the theory. We also distinguish the cases of unfactored and factored skew-Hamiltonian matrices $\mathcal{S}$. Note that the skew-Hamiltonian matrices in (4), (13) and the skew-Hamiltonian matrices resulting from appropriate transformations of the skew-symmetric matrices in (9), (10) are block-diagonal and hence admit a factorization

$$\mathcal{S} = \mathcal{J}\mathcal{Z}^H\mathcal{J}^T\mathcal{Z}. \tag{14}$$

For example, if $\mathcal{S} = \begin{bmatrix} E & 0 \\ 0 & E^H \end{bmatrix}$, then $\mathcal{Z} = \begin{bmatrix} I & 0 \\ 0 & E^H \end{bmatrix}$. The factorization (14) can be understood as a Cholesky-like decomposition of $\mathcal{S}$ with respect to the indefinite inner product $\langle x, y\rangle := x^H\mathcal{J}y$, since $\mathcal{J}\mathcal{Z}^H\mathcal{J}^T$ is the adjoint of $\mathcal{Z}$ with respect to $\langle\cdot,\cdot\rangle$. We also say that a skew-Hamiltonian matrix $\mathcal{S}$ is *$\mathcal{J}$-semidefinite*, if it admits a factorization of the form (14). Hence, in our implementation we distinguish the cases that the full matrix $\mathcal{S}$ or just its "Cholesky factor" $\mathcal{Z}$ is given. In all cases we apply an embedding strategy to the matrix pencil $\lambda\mathcal{S} - \mathcal{H}$ to avoid the problem of non-existence of a structured Schur form.

3.1 The Complex Case

Let $\lambda\mathcal{S} - \mathcal{H}$ be a given complex skew-Hamiltonian/Hamiltonian matrix pencil with $\mathcal{J}$-semidefinite skew-Hamiltonian part $\mathcal{S} = \mathcal{J}\mathcal{Z}^H\mathcal{J}^T\mathcal{Z}$. We split the skew-Hamiltonian matrix $\mathrm{i}\mathcal{H} =: \mathcal{N} = \mathcal{N}_1 + \mathrm{i}\mathcal{N}_2$, where $\mathcal{N}_1$ is real skew-Hamiltonian and $\mathcal{N}_2$ is real Hamiltonian, i.e.,

$$\mathcal{N}_1 = \begin{bmatrix} F_1 & G_1 \\ H_1 & F_1^T \end{bmatrix}, \quad G_1 = -G_1^T, \quad H_1 = -H_1^T,$$

$$\mathcal{N}_2 = \begin{bmatrix} F_2 & G_2 \\ H_2 & -F_2^T \end{bmatrix}, \quad G_2 = G_2^T, \quad H_2 = H_2^T,$$

and F_j, G_j, $H_j \in \mathbb{R}^{n\times n}$ for $j = 1, 2$. We define the matrices

$$\mathcal{Y}_c = \frac{\sqrt{2}}{2}\begin{bmatrix} I_{2n} & \mathrm{i}I_{2n} \\ I_{2n} & -\mathrm{i}I_{2n} \end{bmatrix}, \quad \mathcal{P} = \begin{bmatrix} I_n & 0 & 0 & 0 \\ 0 & 0 & I_n & 0 \\ 0 & I_n & 0 & 0 \\ 0 & 0 & 0 & I_n \end{bmatrix}, \quad \mathcal{X}_c = \mathcal{Y}_c\mathcal{P}. \tag{15}$$

By using the embedding $\mathcal{B}_\mathcal{N} := \operatorname{diag}\left(\mathcal{N}, \bar{\mathcal{N}}\right)$ we obtain that

$$\mathcal{B}^c_\mathcal{N} := \mathcal{X}_c^H \mathcal{B}_\mathcal{N} \mathcal{X}_c = \left[\begin{array}{cc|cc} F_1 & -F_2 & G_1 & -G_2 \\ F_2 & F_1 & G_2 & G_1 \\ \hline H_1 & -H_2 & F_1^T & F_2^T \\ H_2 & H_1 & -F_2^T & F_1^T \end{array}\right] \tag{16}$$

is a real $4n \times 4n$ skew-Hamiltonian matrix. Similarly, we define

$$\mathcal{B}_\mathcal{Z} := \begin{bmatrix} \mathcal{Z} & 0 \\ 0 & \bar{\mathcal{Z}} \end{bmatrix}, \quad \mathcal{B}_\mathcal{T} := \begin{bmatrix} \mathcal{J}\mathcal{Z}^H\mathcal{J}^T & 0 \\ 0 & \overline{\mathcal{J}\mathcal{Z}^H\mathcal{J}^T} \end{bmatrix}, \quad \mathcal{B}_\mathcal{S} := \begin{bmatrix} \mathcal{S} & 0 \\ 0 & \bar{\mathcal{S}} \end{bmatrix} = \mathcal{B}_\mathcal{T}\mathcal{B}_\mathcal{Z}.$$

It can be shown that

$$\mathcal{B}^c_\mathcal{Z} := \mathcal{X}_c^H \mathcal{B}_\mathcal{Z} \mathcal{X}_c, \quad \mathcal{B}^c_\mathcal{T} := \mathcal{X}_c^H \mathcal{B}_\mathcal{T} \mathcal{X}_c, \quad \mathcal{B}^c_\mathcal{S} := \mathcal{X}_c^H \mathcal{B}_\mathcal{S} \mathcal{X}_c \tag{17}$$

are all real. Hence,

$$\lambda\mathcal{B}^c_\mathcal{S} - \mathcal{B}^c_\mathcal{N} = \mathcal{X}_c^H \left(\lambda\mathcal{B}_\mathcal{S} - \mathcal{B}_\mathcal{N}\right)\mathcal{X}_c = \mathcal{X}_c^H \left(\begin{bmatrix} \lambda\mathcal{S} - \mathcal{N} & 0 \\ 0 & \lambda\bar{\mathcal{S}} - \bar{\mathcal{N}} \end{bmatrix}\right)\mathcal{X}_c$$

is a real $4n \times 4n$ skew-Hamiltonian/skew-Hamiltonian matrix pencil. To compute the eigenvalues of this matrix pencil we can compute the structured decomposition of the following theorem [4].

Theorem 3.1. *Let $\lambda\mathcal{S} - \mathcal{N}$ be a real, regular skew-Hamiltonian/skew-Hamiltonian matrix pencil with $\mathcal{S} = \mathcal{J}\mathcal{Z}^T\mathcal{J}^T\mathcal{Z}$. Then there exist a real orthogonal matrix $\mathcal{Q} \in \mathbb{R}^{2n\times 2n}$ and a real orthogonal symplectic matrix $\mathcal{U} \in \mathbb{R}^{2n\times 2n}$ such that*

$$\begin{aligned} \mathcal{U}^T\mathcal{Z}\mathcal{Q} &= \begin{bmatrix} Z_{11} & Z_{12} \\ 0 & Z_{22} \end{bmatrix}, \\ \mathcal{J}\mathcal{Q}^T\mathcal{J}^T\mathcal{N}\mathcal{Q} &= \begin{bmatrix} N_{11} & N_{12} \\ 0 & N_{11}^T \end{bmatrix}, \end{aligned} \tag{18}$$

where Z_{11} and Z_{22}^T are upper triangular, N_{11} is upper quasi triangular and N_{12} is skew-symmetric. Moreover,

$$\begin{aligned} \mathcal{J}\mathcal{Q}^T\mathcal{J}^T\left(\lambda\mathcal{S} - \mathcal{N}\right)\mathcal{Q} &= \lambda\begin{bmatrix} Z_{22}^TZ_{11} & Z_{22}^TZ_{12} - Z_{12}^TZ_{22} \\ 0 & Z_{11}^TZ_{22} \end{bmatrix} - \begin{bmatrix} N_{11} & N_{12} \\ 0 & N_{11}^T \end{bmatrix} \\ &=: \lambda\begin{bmatrix} S_{11} & S_{12} \\ 0 & S_{11}^T \end{bmatrix} - \begin{bmatrix} N_{11} & N_{12} \\ 0 & N_{11}^T \end{bmatrix} \end{aligned} \tag{19}$$

is a $\mathcal{J}$-congruent skew-Hamiltonian/skew-Hamiltonian matrix pencil.

Proof. See [4]. ∎

By defining

$$\mathcal{B}_{\mathcal{H}} = \begin{bmatrix} \mathcal{H} & 0 \\ 0 & -\bar{\mathcal{H}} \end{bmatrix}, \quad \mathcal{B}_{\mathcal{H}}^c = \mathcal{X}_c^H \mathcal{B}_{\mathcal{H}} \mathcal{X}_c,$$

and using Theorem 3.1 we can compute factorizations

$$\tilde{\mathcal{B}}_{\mathcal{Z}}^c := \mathcal{U}^T \mathcal{B}_{\mathcal{Z}}^c \mathcal{Q} = \begin{bmatrix} \mathcal{Z}_{11} & \mathcal{Z}_{12} \\ 0 & \mathcal{Z}_{22} \end{bmatrix},$$

$$\tilde{\mathcal{B}}_{\mathcal{H}}^c := \mathcal{J}\mathcal{Q}^T\mathcal{J}^T\mathcal{B}_{\mathcal{H}}^c\mathcal{Q} = \mathcal{J}\mathcal{Q}^T\mathcal{J}^T\left(-\mathrm{i}\mathcal{B}_{\mathcal{N}}^c\right)\mathcal{Q} = -\mathrm{i}\tilde{\mathcal{B}}_{\mathcal{N}}^c = \begin{bmatrix} -\mathrm{i}\mathcal{N}_{11} & -\mathrm{i}\mathcal{N}_{12} \\ 0 & -\left(-\mathrm{i}\mathcal{N}_{11}\right)^H \end{bmatrix},$$

where $\lambda\tilde{\mathcal{B}}_{\mathcal{S}}^c - \tilde{\mathcal{B}}_{\mathcal{H}}^c = \mathcal{J}\mathcal{Q}^T\mathcal{J}^T\left(\lambda\mathcal{B}_{\mathcal{S}}^c - \mathcal{B}_{\mathcal{H}}^c\right)\mathcal{Q}$ are $\mathcal{J}$-congruent complex skew-Hamiltonian/Hamiltonian matrix pencils and $\lambda\tilde{\mathcal{B}}_{\mathcal{S}}^c - \tilde{\mathcal{B}}_{\mathcal{H}}^c$ is in a structured quasi-triangular form. Then, the structured Schur form can be obtained by further triangularizing the diagonal 2×2 blocks of $\lambda\tilde{\mathcal{B}}_{\mathcal{S}}^c - \tilde{\mathcal{B}}_{\mathcal{H}}^c$ via a $\mathcal{J}$-congruence transformation. From the symmetry of the eigenvalues if follows that $\Lambda\left(\mathcal{S},\mathcal{H}\right) = \Lambda\left(\mathcal{Z}_{22}^H\mathcal{Z}_{11}, -\mathrm{i}\mathcal{N}_{11}\right)$. Now we can reorder the eigenvalues of $\lambda\tilde{\mathcal{B}}_{\mathcal{S}}^c - \tilde{\mathcal{B}}_{\mathcal{H}}^c$ to the top in order to compute the desired deflating subspaces (corresponding to the eigenvalues with negative real parts). The following theorem makes statements about the deflating subspaces [4].

Theorem 3.2. *Let $\lambda\mathcal{S} - \mathcal{H} \in \mathbb{C}^{2n\times 2n}$ be a skew-Hamiltonian/Hamiltonian matrix pencil with $\mathcal{J}$-semidefinite skew-Hamiltonian matrix $\mathcal{S} = \mathcal{J}\mathcal{Z}^H\mathcal{J}^T\mathcal{Z}$. Consider the extended matrices $\mathcal{B}_{\mathcal{Z}} = \operatorname{diag}\left(\mathcal{Z},\bar{\mathcal{Z}}\right)$, $\mathcal{B}_{\mathcal{T}} = \operatorname{diag}\left(\mathcal{J}\mathcal{Z}^H\mathcal{J}^T, \overline{\mathcal{J}\mathcal{Z}^H\mathcal{J}^T}\right)$, $\mathcal{B}_{\mathcal{S}} = \mathcal{B}_{\mathcal{T}}\mathcal{B}_{\mathcal{Z}} = \operatorname{diag}\left(\mathcal{S},\bar{\mathcal{S}}\right)$, $\mathcal{B}_{\mathcal{H}} = \operatorname{diag}\left(\mathcal{H},-\bar{\mathcal{H}}\right)$. Let $\mathcal{U}$, $\mathcal{V}$, $\mathcal{W}$ be unitary matrices such that*

$$\mathcal{U}^H\mathcal{B}_{\mathcal{Z}}\mathcal{V} = \begin{bmatrix} \mathcal{Z}_{11} & \mathcal{Z}_{12} \\ 0 & \mathcal{Z}_{22} \end{bmatrix} =: \mathcal{R}_{\mathcal{Z}},$$

$$\mathcal{W}^H\mathcal{B}_{\mathcal{T}}\mathcal{U} = \begin{bmatrix} \mathcal{T}_{11} & \mathcal{T}_{12} \\ 0 & \mathcal{T}_{22} \end{bmatrix} =: \mathcal{R}_{\mathcal{T}},$$

$$\mathcal{W}^H\mathcal{B}_{\mathcal{H}}\mathcal{V} = \begin{bmatrix} \mathcal{H}_{11} & \mathcal{H}_{12} \\ 0 & \mathcal{H}_{22} \end{bmatrix} =: \mathcal{R}_{\mathcal{H}},$$

where $\Lambda_-\left(\mathcal{B}_{\mathcal{S}},\mathcal{B}_{\mathcal{H}}\right) \subset \Lambda\left(\mathcal{T}_{11}\mathcal{Z}_{11},\mathcal{H}_{11}\right)$ and $\Lambda\left(\mathcal{T}_{11}\mathcal{Z}_{11},\mathcal{H}_{11}\right) \cap \Lambda_+\left(\mathcal{B}_{\mathcal{S}},\mathcal{B}_{\mathcal{H}}\right) = \emptyset$. Here, $\mathcal{Z}_{11}$, $\mathcal{T}_{11}$, $\mathcal{H}_{11} \in \mathbb{C}^{m\times m}$. Suppose $\Lambda_-(\mathcal{S},\mathcal{H})$ contains p eigenvalues. If $\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} \in \mathbb{C}^{4n\times m}$ are the first m columns of $\mathcal{V}$, $2p \le m \le 2n-2p$, then there are subspaces $\mathbb{L}_1$ and $\mathbb{L}_2$ such that

$$\operatorname{range} V_1 = \operatorname{Def}_-(\mathcal{S},\mathcal{H}) + \mathbb{L}_1, \quad \mathbb{L}_1 \subseteq \operatorname{Def}_0(\mathcal{S},\mathcal{H}) + \operatorname{Def}_\infty(\mathcal{S},\mathcal{H}),$$

$$\operatorname{range} \overline{V_2} = \operatorname{Def}_+(\mathcal{S},\mathcal{H}) + \mathbb{L}_2, \quad \mathbb{L}_2 \subseteq \operatorname{Def}_0(\mathcal{S},\mathcal{H}) + \operatorname{Def}_\infty(\mathcal{S},\mathcal{H}).$$

If $\Lambda\left(\mathcal{T}_{11}\mathcal{Z}_{11},\mathcal{H}_{11}\right)=\Lambda_{-}\left(\mathcal{B}_{\mathcal{S}},\mathcal{B}_{\mathcal{H}}\right)$*, and* $\begin{bmatrix}U_1\\U_2\end{bmatrix}$, $\begin{bmatrix}W_1\\W_2\end{bmatrix}$ *are the first* m *columns of* $\mathcal{U}$, $\mathcal{W}$*, respectively, then there exist unitary matrices* Q_U, Q_V, Q_W *such that*

$$\begin{aligned}U_1&=\begin{bmatrix}P_U^- & 0\end{bmatrix}Q_U, & U_2&=\begin{bmatrix}0 & \overline{P_U^+}\end{bmatrix}Q_U,\\ V_1&=\begin{bmatrix}P_V^- & 0\end{bmatrix}Q_V, & V_2&=\begin{bmatrix}0 & \overline{P_V^+}\end{bmatrix}Q_V,\\ W_1&=\begin{bmatrix}P_W^- & 0\end{bmatrix}Q_W, & W_2&=\begin{bmatrix}0 & \overline{P_W^+}\end{bmatrix}Q_W,\end{aligned}$$

and the columns of P_V^- *and* $\overline{P_V^+}$ *form orthogonal bases of* $\mathrm{Def}_-(\mathcal{S},\mathcal{H})$ *and* $\mathrm{Def}_+(\mathcal{S},\mathcal{H})$*, respectively. Moreover, the matrices* P_U^-, P_U^+, P_W^-*, and* P_W^+ *have orthonormal columns and the following relations are satisfied*

$$\begin{aligned}\mathcal{Z}P_V^-&=P_U^-\tilde{Z}_{11}, & \mathcal{J}\mathcal{Z}^H\mathcal{J}^TP_U^-&=P_W^-\tilde{T}_{11}, & \mathcal{H}P_V^-&=P_W^-\tilde{H}_{11},\\ \mathcal{Z}\overline{P_V^+}&=\overline{P_U^+}\,\overline{\tilde{Z}_{22}}, & \mathcal{J}\mathcal{Z}^H\mathcal{J}^T\overline{P_U^+}&=\overline{P_W^+}\,\overline{\tilde{T}_{22}}, & \mathcal{H}\overline{P_V^+}&=-\overline{P_W^+}\,\overline{\tilde{H}_{22}}.\end{aligned}$$

Here, $\tilde{Z}_{kk}$, $\tilde{T}_{kk}$*, and* $\tilde{H}_{kk}$, $k=1,2$*, satisfy* $\Lambda\left(\tilde{T}_{11}\tilde{Z}_{11},\tilde{H}_{11}\right)=\Lambda\left(\tilde{T}_{22}\tilde{Z}_{22},\tilde{H}_2\right)=\Lambda_-(\mathcal{S},\mathcal{H})$.

Proof. See [4]. □

So, the algorithm for computing the stable deflating subspaces of a complex skew-Hamiltonian/Hamiltonian matrix pencil $\lambda\mathcal{S}-\mathcal{H}$ with $\mathcal{S}=\mathcal{J}\mathcal{Z}^H\mathcal{J}^T\mathcal{Z}$ is as follows [4].

ALGORITHM 1. *Computation of stable deflating subspaces of complex skew-Hamiltonian/Hamiltonian matrix pencils in factored form*

Input: *Hamiltonian matrix* $\mathcal{H}$ *and the factor* $\mathcal{Z}$ *of* $\mathcal{S}\in\mathbb{SH}_n$.

Output: *Structured Schur form of the extended skew-Hamiltonian/Hamiltonian matrix pencil* $\lambda\mathcal{B}_{\mathcal{S}}^{c}-\mathcal{B}_{\mathcal{H}}^{c}$*, eigenvalues of* $\lambda\mathcal{S}-\mathcal{H}$*, orthonormal bases* P_V^-, P_U^- *of the deflating subspace* $\mathrm{Def}_-\left(\mathcal{S},\mathcal{H}\right)$ *and the companion subspace, respectively, as in Theorem 3.2.*

1: *Set* $\mathcal{N}=\mathrm{i}\mathcal{H}$ *and determine the matrices* $\mathcal{B}_{\mathcal{Z}}^{c},\mathcal{B}_{\mathcal{N}}^{c}$ *as in* (17) *and* (16)*, respectively. Perform Algorithm 2 to compute the factorization*

$$\begin{aligned}\hat{\mathcal{B}}_{\mathcal{Z}}^{c}&=\mathcal{U}^T\mathcal{B}_{\mathcal{Z}}^{c}\mathcal{Q}=\begin{bmatrix}\mathcal{Z}_{11} & \mathcal{Z}_{12}\\ 0 & \mathcal{Z}_{22}\end{bmatrix},\\ \hat{\mathcal{B}}_{\mathcal{N}}^{c}&=\mathcal{J}\mathcal{Q}^T\mathcal{J}^T\mathcal{B}_{\mathcal{N}}^{c}\mathcal{Q}=\begin{bmatrix}\mathcal{N}_{11} & \mathcal{N}_{12}\\ 0 & \mathcal{N}_{11}^T\end{bmatrix},\end{aligned}$$

where $\mathcal{Q}$ *is real orthogonal,* $\mathcal{U}$ *is real orthogonal symplectic,* $\mathcal{Z}_{11}$, $\mathcal{Z}_{22}^T$ *are upper triangular and* $\mathcal{N}_{11}$ *is upper quasi triangular.*

2: *Apply the periodic QZ algorithm [9, 13] to the* 2×2 *diagonal blocks of the matrix pencil* $\lambda\mathcal{Z}_{22}^H\mathcal{Z}_{11}-\mathcal{N}_{11}$ *to determine unitary matrices* Q_1, Q_2, U *such that* $U^H\mathcal{Z}_{11}Q_1$, $Q_2^H\mathcal{Z}_{22}^HU$, $Q_2^H\mathcal{N}_{11}Q_1$ *are all upper triangular. Define* $\hat{\mathcal{U}}:=\mathrm{diag}\left(U,U\right)$, $\hat{\mathcal{Q}}:=\mathrm{diag}\left(Q_1,Q_2\right)$ *and set*

$$\tilde{\mathcal{B}}_{\mathcal{Z}}^{c}=\hat{\mathcal{U}}^H\hat{\mathcal{B}}_{\mathcal{Z}}^{c}\hat{\mathcal{Q}},\quad \tilde{\mathcal{B}}_{\mathcal{N}}^{c}=\mathcal{J}\hat{\mathcal{Q}}^H\mathcal{J}^T\hat{\mathcal{B}}_{\mathcal{N}}^{c}\hat{\mathcal{Q}}.$$

3: *Use Algorithm 3 to determine a unitary matrix $\tilde{\mathcal{Q}}$ and a unitary symplectic matrix $\tilde{U}$ such that*

$$\tilde{U}^H \tilde{\mathcal{B}}^{\mathrm{c}}_{\mathcal{Z}} \tilde{\mathcal{Q}} = \begin{bmatrix} \tilde{Z}_{11} & \tilde{Z}_{12} \\ 0 & \tilde{\mathcal{Z}}_{22} \end{bmatrix},$$

$$\mathcal{J} \tilde{\mathcal{Q}}^H \mathcal{J}^T \left(-\mathrm{i}\tilde{\mathcal{B}}^{\mathrm{c}}_{\mathcal{N}} \right) \tilde{\mathcal{Q}} = \begin{bmatrix} \mathcal{H}_{11} & \mathcal{H}_{12} \\ 0 & -\mathcal{H}_{11}^H \end{bmatrix},$$

where $\tilde{Z}_{11}$, $\tilde{Z}_{22}^H$, $\mathcal{H}_{11}$ are upper triangular such that $\Lambda_- \left(\mathcal{J}(\tilde{\mathcal{B}}^{\mathrm{c}}_{\mathcal{Z}})^H \mathcal{J}^T \tilde{\mathcal{B}}^{\mathrm{c}}_{\mathcal{Z}}, -\mathrm{i}\tilde{\mathcal{B}}^{\mathrm{c}}_{\mathcal{N}} \right)$ is contained in the spectrum of the $2p \times 2p$ leading principal subpencil of $\lambda \tilde{\mathcal{Z}}_{22}^H \tilde{\mathcal{Z}}_{11} - \mathcal{H}_{11}$.

4: *Set $V = \begin{bmatrix} I_{2n} & 0 \end{bmatrix} \mathcal{X}_c \mathcal{Q}\hat{\mathcal{Q}}\tilde{\mathcal{Q}} \begin{bmatrix} I_{2p} \\ 0 \end{bmatrix}$, $U = \begin{bmatrix} I_{2n} & 0 \end{bmatrix} \mathcal{X}_c \mathcal{U}\hat{\mathcal{U}}\tilde{\mathcal{U}} \begin{bmatrix} I_{2p} \\ 0 \end{bmatrix}$ and compute P_V^-, P_U^+, orthogonal bases of* range V *and* range U*, respectively, using any numerically stable orthogonalization scheme.*

Next we briefly discuss the algorithms which are used in Algorithm 1.

ALGORITHM 2. *Computation of a structured matrix factorization for real skew-Hamiltonian/skew-Hamiltonian matrix pencils in factored form*

Input: *A real skew-Hamiltonian matrix $\mathcal{N} \in \mathbb{R}^{2n \times 2n}$ and the factor $\mathcal{Z} \in \mathbb{R}^{2n \times 2n}$ of $\mathcal{S}$.*

Output: *A real orthogonal matrix $\mathcal{Q}$, a real orthogonal symplectic matrix $\mathcal{U}$ and the structured factorization* (18)*.*

1: *Set $\mathcal{Q} = \mathcal{U} = I_{2n}$. By changing the elimination order in the classical RQ decomposition, determine an orthogonal matrix $\mathcal{Q}_1$ such that*

$$\mathcal{Z} := \mathcal{Z}\mathcal{Q}_1 =: \begin{bmatrix} Z_{11} & Z_{12} \\ 0 & Z_{22} \end{bmatrix},$$

where Z_{11}, Z_{22}^T are upper triangular. Update $\mathcal{N} = \mathcal{J}\mathcal{Q}_1^T \mathcal{J}^T \mathcal{N} \mathcal{Q}_1$, $\mathcal{Q} := \mathcal{Q}\mathcal{Q}_1$.

2: *Compute an orthogonal matrix $\mathcal{Q}_1$ and an orthogonal symplectic matrix $\mathcal{U}_1$ such that*

$$\mathcal{Z} : = \mathcal{U}_1^T \mathcal{Z} \mathcal{Q}_1 =: \begin{bmatrix} Z_{11} & Z_{12} \\ 0 & Z_{22} \end{bmatrix},$$

$$\mathcal{N} : = \mathcal{J}\mathcal{Q}_1^T \mathcal{J}^T \mathcal{N} \mathcal{Q}_1 =: \begin{bmatrix} N_{11} & N_{12} \\ 0 & N_{11}^T \end{bmatrix},$$

where Z_{11}, Z_{22}^T are upper triangular and N_{11} is upper Hessenberg. Update $\mathcal{Q} := \mathcal{Q}\mathcal{Q}_1$ and $\mathcal{U} := \mathcal{U}\mathcal{U}_1$. This step is performed by using a sequence of orthogonal and orthogonal symplectic matrices to annihilate the elements in $\mathcal{N}$ in a specific order without destroying the structure of $\mathcal{Z}$ (see [4] for details).

3: *Apply the periodic QZ algorithm [9, 13] to the matrix pencil $\lambda Z_{22}^T Z_{11} - N_{11}$ to determine orthogonal matrices Q_1, Q_2, U such that $U^T Z_{11} Q_1$, $Q_2^T Z_{22}^T U$ are both upper triangular and $Q_2^T N_{11} Q_1$ is upper quasi triangular. Set $\mathcal{U}_1 :=$ diag(U, U), $\mathcal{Q}_1 :=$ diag(Q_1, Q_2). Update $\mathcal{Z} := \mathcal{U}_1^T \mathcal{Z} \mathcal{Q}_1$, $\mathcal{N} := \mathcal{J}\mathcal{Q}_1^T \mathcal{J}^T \mathcal{N} \mathcal{Q}_1$, $\mathcal{Q} := \mathcal{Q}\mathcal{Q}_1$, $\mathcal{U} := \mathcal{U}\mathcal{U}_1$.*

After performing Algorithm 2 the eigenvalues of the complex skew-Hamiltonian/Hamiltonian matrix pencil $\lambda\mathcal{S} - \mathcal{H}$ can be determined by the diagonal 1×1 and 2×2 blocks of the matrices Z_{11}, Z_{22}, and N_{11}.

Next, we describe the eigenvalue reordering technique to reorder the finite, stable eigenvalues to the top of the matrix pencil, which enables us to compute the corresponding deflating subspaces.

ALGORITHM 3. *Eigenvalue reordering for complex skew-Hamiltonian/Hamiltonian matrix pencils in factored form*

Input: *Regular* $2n \times 2n$ *complex skew-Hamiltonian/Hamiltonian matrix pencil* $\lambda\mathcal{S} - \mathcal{H}$ *with* $\mathcal{S} = \mathcal{J}\mathcal{Z}^H\mathcal{J}^T\mathcal{Z}$, $\mathcal{Z} = \begin{bmatrix} Z & W \\ 0 & T \end{bmatrix}$, $\mathcal{H} = \begin{bmatrix} H & D \\ 0 & -H^H \end{bmatrix}$ *with upper triangular* Z, T^H *and* H.

Output: *A unitary matrix* $\mathcal{Q}$, *a unitary symplectic matrix* $\mathcal{U}$, *and the transformed matrices* $\mathcal{U}^H\mathcal{Z}\mathcal{Q}$, $\mathcal{J}\mathcal{Q}^H\mathcal{J}^T\mathcal{H}\mathcal{Q}$ *which have still the same triangular form as* $\mathcal{Z}$ *and* $\mathcal{H}$, *respectively, but the eigenvalues in* $\Lambda_-(\mathcal{S},\mathcal{H})$ *are reordered such that they occur in the leading principal subpencil of* $\mathcal{J}\mathcal{Q}^H\mathcal{J}^T(\lambda\mathcal{S} - \mathcal{H})\mathcal{Q}$.

1: *Set* $\mathcal{Q} = \mathcal{U} = I_{2n}$. *Reorder the eigenvalues in the subpencil* $\lambda T^H Z - H$.

a) *Determine unitary matrices* Q_1, Q_2, Q_3 *such that* $T^H := Q_3^H T^H Q_2$, $Z := Q_2^H Z Q_1$, $H := Q_3^H H Q_1$ *are still upper triangular but the* m_- *eigenvalues with negative real part are reordered to the top of* $\lambda T^H Z - H$. *Set* $\mathcal{Q}_1 := \operatorname{diag}(Q_1, Q_3)$, $\mathcal{U}_1 := \operatorname{diag}(Q_2, Q_2)$ *and update* $\mathcal{Q} := \mathcal{Q}\mathcal{Q}_1$, $\mathcal{U} := \mathcal{U}\mathcal{U}_1$.

b) *Determine unitary matrices* Q_1, Q_2, Q_3 *such that* $T^H := Q_3^H T^H Q_2$, $Z := Q_2^H Z Q_1$, $H := Q_3^H H Q_1$ *are still upper triangular but the* m_+ *eigenvalues with positive real part are reordered to the bottom of* $\lambda T^H Z - H$. *Set* $\mathcal{Q}_1 := \operatorname{diag}(Q_1, Q_3)$, $\mathcal{U}_1 := \operatorname{diag}(Q_2, Q_2)$ *and update* $\mathcal{Q} := \mathcal{Q}\mathcal{Q}_1$, $\mathcal{U} := \mathcal{U}\mathcal{U}_1$.

2: *Reorder the remaining* $n - m_+ + 1$ *eigenvalues with negative real parts which are now in the bottom right subpencil of* $\lambda\mathcal{S} - \mathcal{H}$. *Determine a unitary matrix* $\mathcal{Q}_1$ *and a unitary symplectic matrix* $\mathcal{U}_1$ *such that the eigenvalues of top left subpencil of* $\lambda\mathcal{S} - \mathcal{H}$ *with positive real parts and those of the bottom right subpencil of* $\lambda\mathcal{S} - \mathcal{H}$ *with negative real parts are interchanged. Update* $\mathcal{Q} := \mathcal{Q}\mathcal{Q}_1$, $\mathcal{U} := \mathcal{U}\mathcal{U}_1$.

If the matrix $\mathcal{S}$ is not given in factored form, we can use the following algorithm for the computation of the deflating subspaces [4].

ALGORITHM 4. *Computation of stable deflating subspaces of complex skew-Hamiltonian/Hamiltonian matrix pencils in unfactored form*

Input: *Complex skew-Hamiltonian/Hamiltonian matrix pencil* $\lambda\mathcal{S} - \mathcal{H}$.

Output: *Structured Schur form of the extended skew-Hamiltonian/Hamiltonian matrix pencil* $\lambda\mathcal{B}_{\mathcal{S}}^{c} - \mathcal{B}_{\mathcal{H}}^{c}$, *eigenvalues of* $\lambda\mathcal{S} - \mathcal{H}$, *orthonormal basis* P_V^- *of the deflating subspace* $\operatorname{Def}_-(\mathcal{S},\mathcal{H})$, *as in Theorem 3.2.*

1: *Set* $\mathcal{N} = \mathrm{i}\mathcal{H}$ *and determine the matrices* $\mathcal{B}_{\mathcal{S}}^{c}, \mathcal{B}_{\mathcal{N}}^{c}$ *as in* (17) *and* (16), *respectively. Perform*

Algorithm 5 to compute the factorization

$$\hat{\mathcal{B}}_{\mathcal{S}}^{c} = \mathcal{J}\mathcal{Q}^T\mathcal{J}^T\mathcal{B}_{\mathcal{S}}^{c}\mathcal{Q} = \begin{bmatrix} \mathcal{S}_{11} & \mathcal{S}_{12} \\ 0 & \mathcal{S}_{11}^T \end{bmatrix},$$
$$\hat{\mathcal{B}}_{\mathcal{N}}^{c} = \mathcal{J}\mathcal{Q}^T\mathcal{J}^T\mathcal{B}_{\mathcal{N}}^{c}\mathcal{Q} = \begin{bmatrix} \mathcal{N}_{11} & \mathcal{N}_{12} \\ 0 & \mathcal{N}_{11}^T \end{bmatrix},$$

where $\mathcal{Q}$ is real orthogonal, $\mathcal{S}_{11}$ is upper triangular and $\mathcal{N}_{11}$ is upper quasi triangular.

2: *Apply the QZ algorithm [11] to the 2×2 diagonal blocks of the matrix pencil $\lambda\mathcal{S}_{11} - \mathcal{N}_{11}$ to determine unitary matrices Q_1, Q_2 such that $Q_2^H\mathcal{S}_{11}Q_1$, $Q_2^H\mathcal{N}_{11}Q_1$ are both upper triangular. Define $\hat{\mathcal{Q}} := \operatorname{diag}(Q_1, Q_2)$ and set*

$$\tilde{\mathcal{B}}_{\mathcal{S}}^{c} = \mathcal{J}\hat{\mathcal{Q}}^H\mathcal{J}^T\hat{\mathcal{B}}_{\mathcal{S}}^{c}\hat{\mathcal{Q}}, \quad \tilde{\mathcal{B}}_{\mathcal{N}}^{c} = \mathcal{J}\hat{\mathcal{Q}}^H\mathcal{J}^T\hat{\mathcal{B}}_{\mathcal{N}}^{c}\hat{\mathcal{Q}}.$$

3: *Use Algorithm 6 to determine a unitary matrix $\tilde{\mathcal{Q}}$ such that*

$$\mathcal{J}\tilde{\mathcal{Q}}^H\mathcal{J}^T\tilde{\mathcal{B}}_{\mathcal{S}}^{c}\tilde{\mathcal{Q}} = \begin{bmatrix} \tilde{\mathcal{S}}_{11} & \tilde{\mathcal{S}}_{12} \\ 0 & \tilde{\mathcal{S}}_{11}^H \end{bmatrix},$$
$$\mathcal{J}\tilde{\mathcal{Q}}^H\mathcal{J}^T\left(-\mathrm{i}\tilde{\mathcal{B}}_{\mathcal{N}}^{c}\right)\tilde{\mathcal{Q}} = \begin{bmatrix} \mathcal{H}_{11} & \mathcal{H}_{12} \\ 0 & -\mathcal{H}_{11}^H \end{bmatrix},$$

where $\tilde{S}_{11}$, $\mathcal{H}_{11}$ are upper triangular such that $\Lambda_-\left(\tilde{\mathcal{B}}_{\mathcal{S}}^{c}, -\mathrm{i}\tilde{\mathcal{B}}_{\mathcal{N}}^{c}\right)$ is contained in the spectrum of the $2p\times 2p$ leading principal subpencil of $\lambda\tilde{\mathcal{S}}_{11} - \mathcal{H}_{11}$.

4: *Set $V = \begin{bmatrix} I_{2n} & 0 \end{bmatrix} \mathcal{X}_c\mathcal{Q}\hat{\mathcal{Q}}\tilde{\mathcal{Q}} \begin{bmatrix} I_{2p} \\ 0 \end{bmatrix}$ and compute P_V^-, an orthogonal basis of $\operatorname{range} V$, using any numerically stable orthogonalization scheme.*

Now we present the algorithm for the computation of the structured matrix factorization for complex matrix pencils in unfactored form.

ALGORITHM 5. *Computation of a structured matrix factorization for real skew-Hamiltonian/skew-Hamiltonian matrix pencils in unfactored form*

Input: *A real skew-Hamiltonian/skew-Hamiltonian matrix pencil $\lambda\mathcal{S} - \mathcal{N}$.*
Output: *A real orthogonal matrix $\mathcal{Q}$ and the structured factorization* (19).

1: *Set $\mathcal{Q} = I_{2n}$. Reduce $\mathcal{S}$ to skew-Hamiltonian triangular form, i.e., determine an orthogonal matrix $\mathcal{Q}_1$ such that*

$$\mathcal{S} := \mathcal{J}\mathcal{Q}_1^T\mathcal{J}^T\mathcal{S}\mathcal{Q}_1 = \begin{bmatrix} S_{11} & S_{12} \\ 0 & S_{11}^T \end{bmatrix}$$

with an upper triangular matrix S_{11}. Update $\mathcal{N} := \mathcal{J}\mathcal{Q}_1^T\mathcal{J}^T\mathcal{N}\mathcal{Q}_1$, $\mathcal{Q} := \mathcal{Q}\mathcal{Q}_1$. This step is performed by applying a sequence of Householder reflections and Givens rotations in a specific order, see [4] for details.

2: Reduce $\mathcal{N}$ to skew-Hamiltonian Hessenberg form. Determine an orthogonal matrix $\mathcal{Q}_1$ such that

$$\mathcal{S} := \mathcal{J}\mathcal{Q}_1^T\mathcal{J}^T\mathcal{S}\mathcal{Q}_1 = \begin{bmatrix} S_{11} & S_{12} \\ 0 & S_{11}^T \end{bmatrix},$$
$$\mathcal{N} := \mathcal{J}\mathcal{Q}_1^T\mathcal{J}^T\mathcal{N}\mathcal{Q}_1 = \begin{bmatrix} N_{11} & N_{12} \\ 0 & N_{11}^T \end{bmatrix},$$

where S_{11} is upper triangular and N_{11} is upper Hessenberg. Update $\mathcal{Q} := \mathcal{Q}\mathcal{Q}_1$. This step is performed by applying an appropriate sequence of Givens rotations to annihilate the elements in $\mathcal{N}$ in a specific order without destroying the structure of $\mathcal{S}$, for details see [4].

3: Apply the QZ algorithm to the matrix pencil $\lambda S_{11} - N_{11}$ to determine orthogonal matrices Q_1 and Q_2 such that $Q_2^T S_{11} Q_1$ is upper triangular and $Q_2^T N_{11} Q_1$ is upper quasi triangular. Set $\mathcal{Q}_1 := \operatorname{diag}(Q_1, Q_2)$ and update $\mathcal{S} := \mathcal{J}\mathcal{Q}_1^T\mathcal{J}^T\mathcal{S}\mathcal{Q}_1$, $\mathcal{N} := \mathcal{J}\mathcal{Q}_1^T\mathcal{J}^T\mathcal{N}\mathcal{Q}_1$, $\mathcal{Q} := \mathcal{Q}\mathcal{Q}_1$.

Again, similar to the factored case, the eigenvalues are determined by the diagonal 1×1 and 2×2 blocks of S_{11} and N_{11}. Also, the following eigenvalue reordering routine is similar to the one of the factored case.

ALGORITHM 6. *Eigenvalue reordering for complex skew-Hamiltonian/Hamiltonian matrix pencils in unfactored form*

Input: *Regular $2n \times 2n$ complex skew-Hamiltonian/Hamiltonian matrix pencil $\lambda\mathcal{S} - \mathcal{H}$ of the form $\mathcal{S} = \begin{bmatrix} S & W \\ 0 & S^H \end{bmatrix}$, $\mathcal{H} = \begin{bmatrix} H & D \\ 0 & -H^H \end{bmatrix}$, with upper triangular S, H.*

Output: *A unitary matrix $\mathcal{Q}$ and the transformed matrices $\mathcal{J}\mathcal{Q}^H\mathcal{J}^T\mathcal{S}\mathcal{Q}$, $\mathcal{J}\mathcal{Q}^H\mathcal{J}^T\mathcal{H}\mathcal{Q}$ which have still the same triangular form as $\mathcal{S}$ and $\mathcal{H}$, respectively, but the eigenvalues in $\Lambda_-(\mathcal{S},\mathcal{H})$ are reordered such that they occur in the leading principal subpencil of $\mathcal{J}\mathcal{Q}^H\mathcal{J}^T(\lambda\mathcal{S} - \mathcal{H})\mathcal{Q}$.*

1: Set $\mathcal{Q} = I_{2n}$. Reorder the eigenvalues in the subpencil $\lambda S - H$.

a) Determine unitary matrices Q_1, Q_2 such that $S := Q_2^H S Q_1$, $H := Q_2^H H Q_1$, are still upper triangular but the m_- eigenvalues with negative real part are reordered to the top of $\lambda S - H$. Set $\mathcal{Q}_1 := \operatorname{diag}(Q_1, Q_2)$ and update $\mathcal{Q} := \mathcal{Q}\mathcal{Q}_1$.

b) Determine unitary matrices Q_1, Q_2 such that $S := Q_2^H S Q_1$, $H := Q_2^H H Q_1$, are still upper triangular but the m_+ eigenvalues with positive real part are reordered to the bottom of $\lambda S - H$. Set $\mathcal{Q}_1 := \operatorname{diag}(Q_1, Q_2)$ and update $\mathcal{Q} := \mathcal{Q}\mathcal{Q}_1$.

2: Reorder the remaining $n - m_+ + 1$ eigenvalues with negative real parts which are now in the bottom right subpencil of $\lambda\mathcal{S} - \mathcal{H}$. Determine a unitary matrix $\mathcal{Q}_1$ such that the eigenvalues of top left subpencil of $\lambda\mathcal{S} - \mathcal{H}$ with positive real parts and those of the bottom right subpencil of $\lambda\mathcal{S} - \mathcal{H}$ with negative real parts are interchanged. Update $\mathcal{Q} := \mathcal{Q}\mathcal{Q}_1$.

3.2 The Real Case

We also briefly recall the theory for the real case which has some significant differences compared to the complex case. For a very detailed description we refer to [2]. Let $\lambda\mathcal{S} - \mathcal{H}$ be a real skew-Hamiltonian/Hamiltonian matrix pencil with $\mathcal{J}$-semidefinite skew-Hamiltonian part $\mathcal{S} = \mathcal{J}\mathcal{Z}^T\mathcal{J}^T\mathcal{Z}$ where $\mathcal{Z} = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix}$, $\mathcal{H} = \begin{bmatrix} F & G \\ H & -F^T \end{bmatrix}$. We introduce the orthogonal matrices

$$\mathcal{Y}_r = \frac{\sqrt{2}}{2}\begin{bmatrix} I_{2n} & I_{2n} \\ -I_{2n} & I_{2n} \end{bmatrix}, \quad \mathcal{X}_r = \mathcal{Y}_r\mathcal{P}$$

with $\mathcal{P}$ as in (15). Now we define the double-sized matrices

$$\begin{aligned}
\mathcal{B}_{\mathcal{Z}} &:= \begin{bmatrix} \mathcal{Z} & 0 \\ 0 & \mathcal{Z} \end{bmatrix}, \\
\mathcal{B}_{\mathcal{T}} &:= \begin{bmatrix} \mathcal{J}\mathcal{Z}^T\mathcal{J}^T & 0 \\ 0 & \mathcal{J}\mathcal{Z}^T\mathcal{J}^T \end{bmatrix} = \begin{bmatrix} \mathcal{J} & 0 \\ 0 & \mathcal{J} \end{bmatrix} \mathcal{B}_{\mathcal{Z}}^T \begin{bmatrix} \mathcal{J} & 0 \\ 0 & \mathcal{J} \end{bmatrix}^T, \\
\mathcal{B}_{\mathcal{S}} &:= \begin{bmatrix} \mathcal{S} & 0 \\ 0 & \mathcal{S} \end{bmatrix} = \mathcal{B}_{\mathcal{T}}\mathcal{B}_{\mathcal{Z}}, \\
\mathcal{B}_{\mathcal{H}} &:= \begin{bmatrix} \mathcal{H} & 0 \\ 0 & -\mathcal{H} \end{bmatrix}.
\end{aligned}$$

Furthermore, we define

$$\begin{aligned}
\mathcal{B}_{\mathcal{Z}}^r &:= \mathcal{X}_r^T\mathcal{B}_{\mathcal{Z}}\mathcal{X}_r = \begin{bmatrix} Z_{11} & 0 & Z_{12} & 0 \\ 0 & Z_{11} & 0 & Z_{12} \\ Z_{21} & 0 & Z_{22} & 0 \\ 0 & Z_{21} & 0 & Z_{22} \end{bmatrix}, \\
\mathcal{B}_{\mathcal{T}}^r &:= \mathcal{X}_r^T\mathcal{B}_{\mathcal{T}}\mathcal{X}_r = \mathcal{J}\left(\mathcal{B}_{\mathcal{Z}}^r\right)^T\mathcal{J}^T, \\
\mathcal{B}_{\mathcal{S}}^r &:= \mathcal{X}_r^T\mathcal{B}_{\mathcal{S}}\mathcal{X}_r = \mathcal{J}\left(\mathcal{B}_{\mathcal{Z}}^r\right)^T\mathcal{J}^T\mathcal{B}_{\mathcal{Z}}^r, \\
\mathcal{B}_{\mathcal{H}}^r &:= \mathcal{X}_r^T\mathcal{B}_{\mathcal{H}}\mathcal{X}_r = \left[\begin{array}{cc|cc} 0 & F & 0 & G \\ F & 0 & G & 0 \\ \hline 0 & H & 0 & -F^T \\ H & 0 & -F^T & 0 \end{array}\right].
\end{aligned}$$

It can be easily observed, that the $4n \times 4n$ matrix pencil $\lambda\mathcal{B}_{\mathcal{S}}^r - \mathcal{B}_{\mathcal{H}}^r$ is again real skew-Hamiltonian/Hamiltonian. For the computation of the eigenvalues of $\lambda\mathcal{S} - \mathcal{H}$ we apply the following structured matrix factorization which is also often referred to as *generalized symplectic URV decomposition* [2].

Theorem 3.3. *Let $\lambda\mathcal{S} - \mathcal{H}$ be a real skew-Hamiltonian/Hamiltonian matrix pencil with $\mathcal{S} = \mathcal{J}\mathcal{Z}^T\mathcal{J}^T\mathcal{Z}$. Then there exist orthogonal matrices $\mathcal{Q}_1$, $\mathcal{Q}_2$ and orthogonal* *symplectic matrices $\mathcal{U}_1, \mathcal{U}_2$ such that*

$$\begin{aligned}
\mathcal{Q}_1^T \left(\mathcal{J}\mathcal{Z}^T\mathcal{J}^T\right)\mathcal{U}_1 &= \begin{bmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{bmatrix}, \\
\mathcal{U}_2^T \mathcal{Z}\mathcal{Q}_2 &= \begin{bmatrix} Z_{11} & Z_{12} \\ 0 & Z_{22} \end{bmatrix}, \\
\mathcal{Q}_1^T \mathcal{H}\mathcal{Q}_2 &= \begin{bmatrix} H_{11} & H_{12} \\ 0 & H_{22} \end{bmatrix},
\end{aligned} \tag{20}$$

with the formal matrix product $T_{11}^{-1}H_{11}Z_{11}^{-1}Z_{22}^{-T}H_{22}^TT_{22}^{-T}$ in real periodic Schur form [9, 13], where T_{11}, Z_{11}, H_{11}, T_{22}^T, Z_{22}^T are upper triangular and H_{22}^T is upper quasi triangular.

Proof. The proof is constructive, see [2]. □

By using Theorem 3.3 (with the same notation) we get the following factorization of the embedded matrix pencil $\lambda\mathcal{B}_{\mathcal{S}}^r - \mathcal{B}_{\mathcal{H}}^r$ with factored matrix $\mathcal{B}_{\mathcal{S}}^r$. We can compute an orthogonal matrix $\tilde{\mathcal{Q}}_1$ and an orthogonal symplectic matrix $\tilde{\mathcal{U}}$ such that

$$\begin{aligned}
\tilde{\mathcal{U}}^T\mathcal{B}_{\mathcal{Z}}^r\tilde{\mathcal{Q}} &= \left[\begin{array}{cc|cc} T_{22}^T & 0 & -T_{12}^T & 0 \\ 0 & Z_{11} & 0 & Z_{12} \\ \hline 0 & 0 & T_{11}^T & 0 \\ 0 & 0 & 0 & Z_{22} \end{array}\right] =: \begin{bmatrix} \tilde{\mathcal{Z}}_{11} & \tilde{\mathcal{Z}}_{12} \\ 0 & \tilde{\mathcal{Z}}_{22} \end{bmatrix}, \\
\mathcal{J}\tilde{\mathcal{Q}}^T\mathcal{J}^T\mathcal{B}_{\mathcal{H}}^r\tilde{\mathcal{Q}} &= \left[\begin{array}{cc|cc} 0 & H_{11} & 0 & H_{12} \\ -H_{22}^T & 0 & H_{12} & 0 \\ \hline 0 & 0 & 0 & H_{22} \\ 0 & 0 & -H_{11}^T & 0 \end{array}\right] =: \begin{bmatrix} \tilde{\mathcal{H}}_{11} & \tilde{\mathcal{H}}_{12} \\ 0 & -\tilde{\mathcal{H}}_{11}^T \end{bmatrix},
\end{aligned} \tag{21}$$

where $\tilde{\mathcal{Q}} = \mathcal{P}^T \begin{bmatrix} \mathcal{J}\mathcal{Q}_1\mathcal{J}^T & 0 \\ 0 & \mathcal{Q}_2 \end{bmatrix} \mathcal{P}$, $\tilde{\mathcal{U}} = \mathcal{P}^T \begin{bmatrix} \mathcal{U}_1 & 0 \\ 0 & \mathcal{U}_2 \end{bmatrix} \mathcal{P}$. From the condensed form (21) we can immediately get the eigenvalues of $\lambda\mathcal{S} - \mathcal{H}$ as

$$\Lambda(\mathcal{S},\mathcal{H}) = \Lambda\left(\tilde{\mathcal{Z}}_{22}^T\tilde{\mathcal{Z}}_{11}, \tilde{\mathcal{H}}_{11}\right) = \pm\mathrm{i}\sqrt{\Lambda\left(T_{11}^{-1}H_{11}Z_{11}^{-1}Z_{22}^{-T}H_{22}^TT_{22}^{-T}\right)}. \tag{22}$$

Note that all matrices of the product are upper triangular, except H_{22}^T which is upper quasi triangular. Hence, the eigenvalue information can be extracted directly from the diagonal 1×1 or 2×2 blocks of the main diagonals. Note that the finite, simple, purely imaginary eigenvalues of the initial matrix pencil correspond to the positive eigenvalues of the generalized matrix product. Hence, these eigenvalues can be computed without any error in their real parts. This leads to a high robustness in algorithms which require these eigenvalues, e.g., in the $\mathcal{L}_\infty$-norm computation [6]. However, if two purely imaginary eigenvalues are very close they might still be slightly perturbed from imaginary axis. This essentially depends on the Kronecker structure of a close-by skew-Hamiltonian/Hamiltonian matrix pencil with double purely imaginary eigenvalues. This problem is similar to the Hamiltonian matrix case, see [21].

To compute the deflating subspaces we are interested in, it is necessary to compute the structured Schur form of the embedded matrix pencils $\lambda\mathcal{B}^r_{\mathcal{S}} - \mathcal{B}^r_{\mathcal{H}}$. This can be done by computing a finite number of similarity transformations to the subpencil $\lambda\tilde{\mathcal{Z}}_{22}^T\tilde{\mathcal{Z}}_{11} - \tilde{\mathcal{H}}_{11}$ to put $\tilde{\mathcal{H}}_{11}$ into upper quasi triangular form. That is, we compute orthogonal matrices $\mathcal{Q}_3$, $\mathcal{Q}_4$, $\mathcal{U}_3$ such that

$$\mathcal{H}_{11} = \mathcal{Q}_3^T\tilde{\mathcal{H}}_{11}\mathcal{Q}_4, \quad \mathcal{Z}_{11} = \mathcal{U}_3^T\tilde{\mathcal{Z}}_{11}\mathcal{Q}_4, \quad \mathcal{Z}_{22} = \mathcal{U}_3^T\tilde{\mathcal{Z}}_{22}\mathcal{Q}_3,$$

where $\mathcal{Z}_{11}$, $\mathcal{Z}_{22}^T$ are upper triangular and $\mathcal{H}_{11}$ is upper quasi triangular. By setting $\mathcal{Q} = \tilde{\mathcal{Q}}\begin{bmatrix}\mathcal{Q}_4 & 0\\ 0 & \mathcal{Q}_3\end{bmatrix}$, $\mathcal{U} = \tilde{\mathcal{U}}\begin{bmatrix}\mathcal{U}_3 & 0\\ 0 & \mathcal{U}_3\end{bmatrix}$, $\mathcal{Z}_{12} = \mathcal{U}_3^T\tilde{\mathcal{Z}}_{12}\mathcal{Q}_3$, and $\mathcal{H}_{12} = \mathcal{Q}_3^T\tilde{\mathcal{H}}_{12}\mathcal{Q}_3$ we obtain the structured Schur form of $\lambda\mathcal{B}^r_{\mathcal{S}} - \mathcal{B}^r_{\mathcal{H}}$ as $\lambda\tilde{\mathcal{B}}^r_{\mathcal{S}} - \tilde{\mathcal{B}}^r_{\mathcal{H}}$ with $\tilde{\mathcal{B}}^r_{\mathcal{S}} = \mathcal{J}\left(\tilde{\mathcal{B}}^r_{\mathcal{Z}}\right)^T\mathcal{J}^T\tilde{\mathcal{B}}^r_{\mathcal{Z}}$ and

$$\begin{aligned}
\tilde{\mathcal{B}}^r_{\mathcal{Z}} &:= \mathcal{U}^T\mathcal{B}^r_{\mathcal{Z}}\mathcal{Q} = \begin{bmatrix}\mathcal{Z}_{11} & \mathcal{Z}_{12}\\ 0 & \mathcal{Z}_{22}\end{bmatrix},\\
\tilde{\mathcal{B}}^r_{\mathcal{H}} &:= \mathcal{J}\mathcal{Q}^T\mathcal{J}^T\mathcal{B}^r_{\mathcal{H}}\mathcal{Q} = \begin{bmatrix}\mathcal{H}_{11} & \mathcal{H}_{12}\\ 0 & -\mathcal{H}_{11}^T\end{bmatrix}.
\end{aligned}$$

Now we can reorder the eigenvalues of $\lambda\tilde{\mathcal{B}}^r_{\mathcal{S}} - \tilde{\mathcal{B}}^r_{\mathcal{H}}$ to the top in order to compute the desired deflating subspaces which is similar to the complex case. Then, for the deflating subspaces we find a similar result as Theorem 3.2 which we do not state here for brevity.

If the matrix $\mathcal{S}$ is not given in factored form, we need the following slightly modified version of Theorem 3.3 from [2].

Theorem 3.4. *Let $\lambda\mathcal{S} - \mathcal{H}$ be a real skew-Hamiltonian/Hamiltonian matrix pencil. Then there exist orthogonal matrices $\mathcal{Q}_1$, $\mathcal{Q}_2$ such that*

$$\begin{aligned}
\mathcal{Q}_1^T\mathcal{S}\mathcal{J}\mathcal{Q}_1\mathcal{J}^T &= \begin{bmatrix}S_{11} & S_{12}\\ 0 & S_{11}^T\end{bmatrix} \in \mathbb{SH}_{2n},\\
\mathcal{J}\mathcal{Q}_2^T\mathcal{J}^T\mathcal{S}\mathcal{Q}_2 &= \begin{bmatrix}T_{11} & T_{12}\\ 0 & T_{11}^T\end{bmatrix} =: \mathcal{T} \in \mathbb{SH}_{2n},\\
\mathcal{Q}_1^T\mathcal{H}\mathcal{Q}_2 &= \begin{bmatrix}H_{11} & H_{12}\\ 0 & H_{22}\end{bmatrix},
\end{aligned} \tag{23}$$

with the formal matrix product $S_{11}^{-1}H_{11}T_{11}^{-1}H_{22}^T$ in real periodic Schur form, where S_{11}, T_{11}, H_{11} are upper triangular and H_{22}^T is upper quasi triangular.

Proof. The proof is done by construction, see [2]. □

Then we can compute an orthogonal matrix $\tilde{\mathcal{Q}}$ such that

$$\begin{aligned}
\mathcal{J}\tilde{\mathcal{Q}}^T\mathcal{J}^T\mathcal{B}_{\mathcal{S}}^r\tilde{\mathcal{Q}} &= \left[\begin{array}{cc|cc} S_{11} & 0 & S_{12} & 0 \\ 0 & T_{11} & 0 & T_{12} \\ \hline 0 & 0 & S_{11}^T & 0 \\ 0 & 0 & 0 & T_{11}^T \end{array}\right] =: \begin{bmatrix} \tilde{\mathcal{S}}_{11} & \tilde{\mathcal{S}}_{12} \\ 0 & \tilde{\mathcal{S}}_{11}^T \end{bmatrix}, \\
\mathcal{J}\tilde{\mathcal{Q}}^T\mathcal{J}^T\mathcal{B}_{\mathcal{H}}^r\tilde{\mathcal{Q}} &= \left[\begin{array}{cc|cc} 0 & H_{11} & 0 & H_{12} \\ -H_{22}^T & 0 & H_{12} & 0 \\ \hline 0 & 0 & 0 & H_{22} \\ 0 & 0 & -H_{11}^T & 0 \end{array}\right] =: \begin{bmatrix} \tilde{\mathcal{H}}_{11} & \tilde{\mathcal{H}}_{12} \\ 0 & -\tilde{\mathcal{H}}_{11}^T \end{bmatrix},
\end{aligned} \tag{24}$$

with $\tilde{\mathcal{Q}} = \mathcal{P}^T \begin{bmatrix} \mathcal{J}Q_1\mathcal{J}^T & 0 \\ 0 & \mathcal{Q}_2 \end{bmatrix} \mathcal{P}$. The spectrum of $\lambda\mathcal{S} - \mathcal{H}$ is given by

$$\Lambda(\mathcal{S}, \mathcal{H}) = \pm \mathrm{i}\sqrt{\Lambda\left(S_{11}^{-1}H_{11}T_{11}^{-1}H_{22}^T\right)}$$

which can be determined by evaluating the entries on the 1×1 and 2×2 diagonal blocks of the matrices only. To put the matrix pencil formed of the matrices in (24) into structured Schur form we have to triangularize $\lambda\tilde{\mathcal{S}}_{11} - \tilde{\mathcal{H}}_{11}$, i.e., we determine orthogonal matrices $\mathcal{Q}_3$ and $\mathcal{Q}_4$ such that

$$\mathcal{S}_{11} = \mathcal{Q}_4^T\tilde{\mathcal{S}}_{11}\mathcal{Q}_3, \quad \mathcal{H}_{11} = \mathcal{Q}_4^T\tilde{\mathcal{H}}_{11}\mathcal{Q}_3$$

are upper triangular and upper quasi triangular, respectively. By setting the matrices $\mathcal{Q} = \tilde{\mathcal{Q}} \begin{bmatrix} \mathcal{Q}_3 & 0 \\ 0 & \mathcal{Q}_4 \end{bmatrix}$, $\mathcal{S}_{12} = \mathcal{Q}_4^T\tilde{\mathcal{S}}_{12}\mathcal{Q}_4$, and $\mathcal{H}_{12} = \mathcal{Q}_4^T\tilde{\mathcal{H}}_{12}\mathcal{Q}_4$, we obtain the structured Schur form as

$$\begin{aligned}
\tilde{\mathcal{B}}_{\mathcal{S}}^r &:= \mathcal{J}\mathcal{Q}^T\mathcal{J}^T\mathcal{B}_{\mathcal{S}}\mathcal{Q} = \begin{bmatrix} \mathcal{S}_{11} & \mathcal{S}_{12} \\ 0 & \mathcal{S}_{11}^T \end{bmatrix}, \\
\tilde{\mathcal{B}}_{\mathcal{H}}^r &:= \mathcal{J}\mathcal{Q}^T\mathcal{J}^T\mathcal{B}_{\mathcal{H}}\mathcal{Q} = \begin{bmatrix} \mathcal{H}_{11} & \mathcal{H}_{12} \\ 0 & -\mathcal{H}_{11}^T \end{bmatrix}.
\end{aligned}$$

By properly reordering the eigenvalues we can compute the desired deflating subspaces as explained above. As for the complex case we give a brief description of the used algorithms for the real case from [2].

ALGORITHM 7. *Computation of stable deflating subspaces of real skew-Hamiltonian/Hamiltonian matrix pencil in factored form*

Input: *Real Hamiltonian matrix $\mathcal{H}$ and the factor $\mathcal{Z}$ of $\mathcal{S}$.*

Output: *Structured Schur form of the extended skew-Hamiltonian/Hamiltonian matrix pencil $\lambda\mathcal{B}_{\mathcal{S}}^r - \mathcal{B}_{\mathcal{H}}^r$, eigenvalues of $\lambda\mathcal{S} - \mathcal{H}$, orthonormal bases P_V^-, P_U^- of the deflating subspace* $\mathrm{Def}_-(\mathcal{S}, \mathcal{H})$ *and the companion subspace, respectively, as in Theorem 3.2.*

1: *Apply Algorithm 8 to the matrices* $\mathcal{Z}$, $\mathcal{J}\mathcal{Z}^T\mathcal{J}^T$ *and* $\mathcal{H}$*, and determine orthogonal matrices* $\mathcal{Q}_1$, $\mathcal{Q}_2$ *and orthogonal symplectic matrices* $\mathcal{U}_1, \mathcal{U}_2$ *such that*

$$\mathcal{Q}_1^T\left(\mathcal{J}\mathcal{Z}^T\mathcal{J}^T\right)\mathcal{U}_1 = \begin{bmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{bmatrix},$$
$$\mathcal{U}_2^T\mathcal{Z}\mathcal{Q}_2 = \begin{bmatrix} Z_{11} & Z_{12} \\ 0 & Z_{22} \end{bmatrix},$$
$$\mathcal{Q}_1^T\mathcal{H}\mathcal{Q}_2 = \begin{bmatrix} H_{11} & H_{12} \\ 0 & H_{22} \end{bmatrix},$$

with the formal matrix product $T_{11}^{-1}H_{11}Z_{11}^{-1}Z_{22}^{-T}H_{22}^TT_{22}^{-T}$ *in real periodic Schur form, where* T_{11}, Z_{11}, H_{11}, T_{22}^T, Z_{22}^T *are upper triangular and* H_{22}^T *is upper quasi triangular.*

2: *Apply Algorithm 9 to determine orthogonal matrices* $\mathcal{Q}_3$, $\mathcal{Q}_4$, $\mathcal{U}_3$ *such that the matrices* $\mathcal{Z}_{11} = \mathcal{U}_3^T\begin{bmatrix} T_{22}^T & 0 \\ 0 & Z_{11} \end{bmatrix}\mathcal{Q}_4$ *and* $\mathcal{Z}_{22} = \mathcal{U}_3^T\begin{bmatrix} T_{11}^T & 0 \\ 0 & Z_{22} \end{bmatrix}\mathcal{Q}_3$ *are upper triangular and* $\mathcal{H}_{11} = \mathcal{Q}_3^T\begin{bmatrix} 0 & H_{11} \\ -H_{22}^T & 0 \end{bmatrix}\mathcal{Q}_4$ *is upper quasi triangular.*

3: *Update*

$$\mathcal{Z}_{12} := \mathcal{U}_3^T\begin{bmatrix} -T_{12}^T & 0 \\ 0 & Z_{12} \end{bmatrix}\mathcal{Q}_3, \quad \mathcal{H}_{12} := \mathcal{Q}_3^T\begin{bmatrix} 0 & H_{12} \\ H_{12}^T & 0 \end{bmatrix}\mathcal{Q}_3,$$

and set

$$\tilde{\mathcal{B}}_{\mathcal{Z}}^r = \begin{bmatrix} \mathcal{Z}_{11} & \mathcal{Z}_{12} \\ 0 & \mathcal{Z}_{22} \end{bmatrix}, \quad \tilde{\mathcal{B}}_{\mathcal{H}}^r = \begin{bmatrix} \mathcal{H}_{11} & \mathcal{H}_{12} \\ 0 & -\mathcal{H}_{11}^T \end{bmatrix}.$$

Apply the real eigenvalue reordering method in Algorithm 10 to the pair $\left(\tilde{\mathcal{B}}_{\mathcal{Z}}^r, \tilde{\mathcal{B}}_{\mathcal{H}}^r\right)$ *to determine an orthogonal matrix* $\hat{\mathcal{Q}}$ *and an orthogonal symplectic matrix* $\hat{\mathcal{U}}$ *such that* $\hat{\mathcal{U}}^T\tilde{\mathcal{B}}_{\mathcal{Z}}^r\hat{\mathcal{Q}}$, $\mathcal{J}\hat{\mathcal{Q}}^T\mathcal{J}^T\tilde{\mathcal{B}}_{\mathcal{H}}^r\hat{\mathcal{Q}}$ *are in structured triangular form and* $\Lambda_-\left(\mathcal{J}\left(\tilde{\mathcal{B}}_{\mathcal{Z}}^r\right)^T\mathcal{J}^T\tilde{\mathcal{B}}_{\mathcal{Z}}^r, \tilde{\mathcal{B}}_{\mathcal{H}}^r\right)$ *is contained in the leading* $2p \times 2p$ *principal subpencil of* $\lambda\mathcal{Z}_{22}^T\mathcal{Z}_{11} - \mathcal{H}_{11}$.

4: *Set*

$$V = \begin{bmatrix} I_{2n} & 0 \end{bmatrix}\left(\mathcal{Y}_r\begin{bmatrix} \mathcal{J}\mathcal{Q}_1\mathcal{J}^T & 0 \\ 0 & \mathcal{Q}_2 \end{bmatrix}\mathcal{P}\begin{bmatrix} \mathcal{Q}_3 & 0 \\ 0 & \mathcal{Q}_4 \end{bmatrix}\hat{\mathcal{Q}}\right)\begin{bmatrix} I_{2p} \\ 0 \end{bmatrix},$$
$$U = \begin{bmatrix} I_{2n} & 0 \end{bmatrix}\left(\mathcal{Y}_r\begin{bmatrix} \mathcal{U}_1 & 0 \\ 0 & \mathcal{U}_2 \end{bmatrix}\mathcal{P}\begin{bmatrix} \mathcal{U}_3 & 0 \\ 0 & \mathcal{U}_3 \end{bmatrix}\hat{\mathcal{U}}\right)\begin{bmatrix} I_{2p} \\ 0 \end{bmatrix}$$

and compute P_V^-, P_U^-*, orthogonal bases of* range V *and* range U*, respectively, using any numerically stable orthogonalization scheme.*

The next algorithm describes the computation of the generalized symplectic URV decomposition which can, e.g., be used to compute the eigenvalues of a real skew-Hamiltonian/Hamiltonian matrix pencil in factored form.

ALGORITHM 8. *Generalized symplectic URV decomposition*

Input: *A real* $2n \times 2n$ *matrix pencil* $\lambda\mathcal{T}\mathcal{Z} - \mathcal{H}$.

Output: *Orthogonal matrices* $\mathcal{Q}_1$, $\mathcal{Q}_2$, *orthogonal symplectic matrices* $\mathcal{U}_1, \mathcal{U}_2$ *and the structured factorization* (20).

1: *Set* $\mathcal{Q}_1 = \mathcal{Q}_2 = \mathcal{U}_1 = \mathcal{U}_2 = I_{2n}$. *By using different elimination orders in QR and RQ like decompositions, determine orthogonal matrices* $\tilde{\mathcal{Q}}_1$ *and* $\tilde{\mathcal{Q}}_2$ *such that*

$$\mathcal{T} := \tilde{\mathcal{Q}}_1^T \mathcal{T} =: \begin{bmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{bmatrix}, \quad \mathcal{Z} := \mathcal{Z}\tilde{\mathcal{Q}}_2 =: \begin{bmatrix} Z_{11} & Z_{12} \\ 0 & Z_{22} \end{bmatrix},$$

where T_{11}, T_{22}^T, Z_{11}, Z_{22}^T *are* $n \times n$ *and upper triangular. Update* $\mathcal{H} = \tilde{\mathcal{Q}}_1^T \mathcal{H} \tilde{\mathcal{Q}}_2$, $\mathcal{Q}_1 := \mathcal{Q}_1 \tilde{\mathcal{Q}}_1$, $\mathcal{Q}_2 := \mathcal{Q}_2 \tilde{\mathcal{Q}}_2$.

2: *Compute orthogonal matrices* $\tilde{\mathcal{Q}}_1$, $\tilde{\mathcal{Q}}_2$ *and orthogonal symplectic matrices* $\tilde{\mathcal{U}}_1$, $\tilde{\mathcal{U}}_2$ *such that*

$$\begin{aligned} \mathcal{T} &:= \tilde{\mathcal{Q}}_1^T \mathcal{T} \tilde{\mathcal{U}}_1 =: \begin{bmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{bmatrix}, \\ \mathcal{Z} &:= \tilde{\mathcal{U}}_2^T \mathcal{Z} \tilde{\mathcal{Q}}_2 =: \begin{bmatrix} Z_{11} & Z_{12} \\ 0 & Z_{22} \end{bmatrix}, \\ \mathcal{H} &:= \tilde{\mathcal{Q}}_1^T \mathcal{H} \tilde{\mathcal{Q}}_2 =: \begin{bmatrix} H_{11} & H_{12} \\ 0 & H_{22} \end{bmatrix}, \end{aligned}$$

where T_{11}, T_{22}^T, Z_{11}, Z_{22}^T, H_{11} *are upper triangular and* H_{22}^T *is upper Hessenberg. Update* $\mathcal{Q}_1 := \mathcal{Q}_1 \tilde{\mathcal{Q}}_1$, $\mathcal{Q}_2 := \mathcal{Q}_2 \tilde{\mathcal{Q}}_2$, $\mathcal{U}_1 := \mathcal{U}_1 \tilde{\mathcal{U}}_1$, *and* $\mathcal{U}_2 := \mathcal{U}_2 \tilde{\mathcal{U}}_2$. *This step is performed by using a sequence of orthogonal and orthogonal symplectic matrices to annihilate the elements in* $\mathcal{H}$ *in a specific order without destroying the structure of* $\mathcal{T}$ *and* $\mathcal{Z}$ *(see [2] for details).*

3: *Apply the periodic QZ algorithm [9, 13] to the formal matrix product*

$$T_{11}^{-1} H_{11} Z_{11}^{-1} Z_{22}^{-T} H_{22}^T T_{22}^{-T}$$

to determine orthogonal matrices V_1, V_2, V_3, V_4, V_5, V_6 *such that* $V_2^T T_{11} V_1$, $V_2^T H_{11} V_3$, $V_4^T Z_{11} V_3$, $\left(V_4^T Z_{22} V_5\right)^T$, $\left(V_6^T T_{22} V_1\right)^T$ *are all upper triangular and* $\left(V_6^T H_{22} V_5\right)^T$ *is upper quasi triangular. Set*

$$\tilde{\mathcal{Q}}_1 := \mathrm{diag}\,(V_2, V_6)\,, \quad \tilde{\mathcal{Q}}_2 := \mathrm{diag}\,(V_3, V_5)\,, \quad \tilde{\mathcal{U}}_1 := \mathrm{diag}\,(V_1, V_1)\,, \quad \tilde{\mathcal{U}}_2 := \mathrm{diag}\,(V_4, V_4)\,,$$

and update $\mathcal{T} := \tilde{\mathcal{Q}}_1^T \mathcal{T} \tilde{\mathcal{U}}_1$, $\mathcal{Z} := \tilde{\mathcal{U}}_2^T \mathcal{Z} \tilde{\mathcal{Q}}_2$, $\mathcal{H} := \tilde{\mathcal{Q}}_1^T \mathcal{H} \tilde{\mathcal{Q}}_2$, $\mathcal{Q}_1 := \mathcal{Q}_1 \tilde{\mathcal{Q}}_1$, $\mathcal{Q}_2 := \mathcal{Q}_2 \tilde{\mathcal{Q}}_2$, $\mathcal{U}_1 := \mathcal{U}_1 \tilde{\mathcal{U}}_1$, $\mathcal{U}_2 := \mathcal{U}_2 \tilde{\mathcal{U}}_2$.

Note that the algorithm above applies to any (unstructured) matrix pencil of the form $\lambda \mathcal{T}\mathcal{Z} - \mathcal{H}$, but the application of the eigenvalue formula (22) requires the structural assumption that the pencil is skew-Hamiltonian/Hamiltonian. Next we present the triangularization procedure needed for Step 2 of Algorithm 7.

ALGORITHM 9. *Triangularization procedure for special matrix pencils in factored form*

Input: *A real matrix pencil* $\lambda \mathcal{A}\mathcal{B} - \mathcal{D} = \lambda \begin{bmatrix} A_{11} & 0 \\ 0 & A_{22} \end{bmatrix} \begin{bmatrix} B_{11} & 0 \\ 0 & B_{22} \end{bmatrix} - \begin{bmatrix} 0 & D_{12} \\ D_{21} & 0 \end{bmatrix}$ *where the formal matrix product* $A_{11}^{-1} D_{12} B_{22}^{-1} A_{22}^{-1} D_{21} B_{11}^{-1}$ *is in real periodic Schur form with upper triangular* A_{11}, A_{22}, B_{11}, B_{22}, D_{12} *and upper quasi triangular* D_{21}.

Output: *Orthogonal matrices* $\mathcal{Q}_1$, $\mathcal{Q}_2$, $\mathcal{Q}_3$ *such that* $\mathcal{Q}_3^T\mathcal{A}\mathcal{Q}_2$, $\mathcal{Q}_2^T\mathcal{B}\mathcal{Q}_1$ *are upper triangular and* $\mathcal{Q}_3^T\mathcal{D}\mathcal{Q}_1$ *is upper quasi triangular.*

1: *Apply the periodic eigenvalue reordering method introduced in [12] to the formal matrix product*

$$A_{11}^{-1}D_{12}B_{22}^{-1}A_{22}^{-1}D_{21}B_{11}^{-1}$$

to determine orthogonal matrices V_1, V_2, V_3, V_4, V_5, V_6 *such that* $V_2^TA_{11}V_1$, $V_2^TD_{12}V_3$, $V_4^TB_{22}V_3$, $V_5^TA_{22}V_4$, $V_5^TD_{21}V_6$, $V_1^TB_{11}V_6$ *keep their upper (quasi) triangular structure but they can be partitioned into* 2×2 *blocks with the last diagonal blocks corresponding to all nonpositive eigenvalues of the formal product, and the first diagonal blocks corresponding to the other eigenvalues.*

2: *Set* $\mathcal{Q}_1 := \operatorname{diag}(V_6, V_3)$, $\mathcal{Q}_2 := \operatorname{diag}(V_1, V_4)$, $\mathcal{Q}_3 := \operatorname{diag}(V_2, V_5)$, *and update*

$$\mathcal{A} := \mathcal{Q}_3^T\mathcal{A}\mathcal{Q}_2 =: \left[\begin{array}{cc|cc} A_{11} & A_{12} & 0 & 0 \\ 0 & A_{22} & 0 & 0 \\ \hline 0 & 0 & A_{33} & A_{34} \\ 0 & 0 & 0 & A_{44} \end{array}\right],$$

$$\mathcal{B} := \mathcal{Q}_2^T\mathcal{B}\mathcal{Q}_1 =: \left[\begin{array}{cc|cc} B_{11} & B_{12} & 0 & 0 \\ 0 & B_{22} & 0 & 0 \\ \hline 0 & 0 & B_{33} & B_{34} \\ 0 & 0 & 0 & B_{44} \end{array}\right],$$

$$\mathcal{D} := \mathcal{Q}_3^T\mathcal{D}\mathcal{Q}_1 =: \left[\begin{array}{cc|cc} 0 & 0 & D_{13} & D_{14} \\ 0 & 0 & 0 & D_{24} \\ \hline D_{31} & D_{32} & 0 & 0 \\ 0 & D_{42} & 0 & 0 \end{array}\right],$$

where $A_{22}^{-1}D_{24}B_{44}^{-1}A_{44}^{-1}D_{42}B_{22}^{-1}$ *has only nonpositive real eigenvalues.*

3: *Let* $\mathcal{P}$ *be an appropriate permutation matrix such that*

$$\mathcal{A} := \mathcal{P}^T\mathcal{A}\mathcal{P} = \left[\begin{array}{cc|cc} A_{11} & 0 & A_{12} & 0 \\ 0 & A_{33} & 0 & A_{34} \\ \hline 0 & 0 & A_{22} & 0 \\ 0 & 0 & 0 & A_{44} \end{array}\right] =: \begin{bmatrix} \tilde{A} & * \\ 0 & \hat{A} \end{bmatrix},$$

$$\mathcal{B} := \mathcal{P}^T\mathcal{B}\mathcal{P} = \left[\begin{array}{cc|cc} B_{11} & 0 & B_{12} & 0 \\ 0 & B_{33} & 0 & B_{34} \\ \hline 0 & 0 & B_{22} & 0 \\ 0 & 0 & 0 & B_{44} \end{array}\right] =: \begin{bmatrix} \tilde{B} & * \\ 0 & \hat{B} \end{bmatrix},$$

$$\mathcal{D} := \mathcal{P}^T\mathcal{D}\mathcal{P} = \left[\begin{array}{cc|cc} 0 & D_{13} & 0 & D_{14} \\ D_{31} & 0 & D_{32} & 0 \\ \hline 0 & 0 & 0 & D_{24} \\ 0 & 0 & D_{42} & 0 \end{array}\right] =: \begin{bmatrix} \tilde{D} & * \\ 0 & \hat{D} \end{bmatrix},$$

and update $\mathcal{Q}_1 := \mathcal{Q}_1\mathcal{P}$, $\mathcal{Q}_2 := \mathcal{Q}_2\mathcal{P}$, $\mathcal{Q}_3 := \mathcal{Q}_3\mathcal{P}$.

4: *Triangularize* $\lambda\tilde{A}\tilde{B} - \tilde{D}$, *i.e., compute orthogonal matrices* $\tilde{\mathcal{Q}}_1$, $\tilde{\mathcal{Q}}_2$, $\tilde{\mathcal{Q}}_3$ *such that* $\mathcal{A} := \tilde{\mathcal{Q}}_3^T\mathcal{A}\tilde{\mathcal{Q}}_2 =: \begin{bmatrix} \tilde{A} & * \\ 0 & \hat{A} \end{bmatrix}$, $\mathcal{B} := \tilde{\mathcal{Q}}_2^T\mathcal{B}\tilde{\mathcal{Q}}_1 =: \begin{bmatrix} \tilde{B} & * \\ 0 & \hat{B} \end{bmatrix}$, $\mathcal{D} := \tilde{\mathcal{Q}}_3^T\mathcal{D}\tilde{\mathcal{Q}}_1 =: \begin{bmatrix} \tilde{D} & * \\ 0 & \hat{D} \end{bmatrix}$ *with upper triangular* $\tilde{A}$, $\tilde{B}$, *upper quasi triangular* $\tilde{D}$ *and unchanged* $\hat{A}$, $\hat{B}$, $\hat{D}$. *Update* $\mathcal{Q}_1 = \mathcal{Q}_1\tilde{\mathcal{Q}}_1$, $\mathcal{Q}_2 = \mathcal{Q}_2\tilde{\mathcal{Q}}_2$, $\mathcal{Q}_3 = \mathcal{Q}_3\tilde{\mathcal{Q}}_3$.

5: Triangularize $\lambda\hat{A}\hat{B}-\hat{D}$ with an appropriate permutation matrix $\hat{\mathcal{P}}$, i.e., $\mathcal{A} := \hat{\mathcal{P}}^T\mathcal{A}\hat{\mathcal{P}} =: \begin{bmatrix}\tilde{A} & * \\ 0 & \hat{A}\end{bmatrix}$, $\mathcal{B} := \hat{\mathcal{P}}^T\mathcal{B}\hat{\mathcal{P}} =: \begin{bmatrix}\tilde{B} & * \\ 0 & \hat{B}\end{bmatrix}$, $\mathcal{D} := \hat{\mathcal{P}}^T\mathcal{D}\hat{\mathcal{P}} =: \begin{bmatrix}\tilde{D} & * \\ 0 & \hat{D}\end{bmatrix}$ with upper triangular $\hat{A}$, $\hat{B}$, upper quasi triangular $\hat{D}$ and unchanged $\tilde{A}$, $\tilde{B}$, $\tilde{D}$. Update $\mathcal{Q}_1 = \mathcal{Q}_1\hat{\mathcal{P}}$, $\mathcal{Q}_2 = \mathcal{Q}_2\hat{\mathcal{P}}$, $\mathcal{Q}_3 = \mathcal{Q}_3\hat{\mathcal{P}}$.

Note, that the separation of the nonpositive from the other eigenvalues of the formal matrix product $A_{11}^{-1}D_{12}B_{22}^{-1}A_{22}^{-1}D_{21}B_{11}^{-1}$ is performed in order to avoid perturbations of the purely imaginary eigenvalues of skew-Hamiltonian/Hamiltonian matrix pencils. This follows from the connection of the nonpositive eigenvalues of the matrix product and the matrix pencil $\lambda\mathcal{A}\mathcal{B}-\mathcal{D}$ similar to (22). When the nonpositive eigenvalues are separated, the triangularization of the corresponding part of $\lambda\mathcal{A}\mathcal{B}-\mathcal{D}$ can be done by only applying permutation matrices. When the matrix pencil is triangularized we apply the following eigenvalue reordering algorithm.

ALGORITHM 10. *Eigenvalue reordering for real skew-Hamiltonian/Hamiltonian matrix pencils in factored form*

Input: *Regular $2n\times 2n$ real skew-Hamiltonian/Hamiltonian matrix pencil $\lambda\mathcal{S}-\mathcal{H}$ with $\mathcal{S} = \mathcal{J}\mathcal{Z}^T\mathcal{J}^T\mathcal{Z}$, $\mathcal{Z} = \begin{bmatrix}Z & W \\ 0 & T\end{bmatrix}$, $\mathcal{H} = \begin{bmatrix}H & D \\ 0 & -H^T\end{bmatrix}$ with upper triangular Z and T^T and upper quasi triangular H.*

Output: *An orthogonal matrix $\mathcal{Q}$, an orthogonal symplectic matrix $\mathcal{U}$, and the transformed matrices $\mathcal{U}^T\mathcal{Z}\mathcal{Q}$, $\mathcal{J}\mathcal{Q}^T\mathcal{J}^T\mathcal{H}\mathcal{Q}$ which have still the same triangular form as $\mathcal{Z}$ and $\mathcal{H}$, respectively, but the eigenvalues in $\Lambda_-(\mathcal{S},\mathcal{H})$ are reordered such that they occur in the leading principal subpencil of $\mathcal{J}\mathcal{Q}^T\mathcal{J}^T(\lambda\mathcal{S}-\mathcal{H})\mathcal{Q}$.*

1: Set $\mathcal{Q}=\mathcal{U}=I_{2n}$. Reorder the eigenvalues in the subpencil $\lambda T^T Z - H$.

a) Determine orthogonal matrices Q_1, Q_2, Q_3 such that $T^T := Q_3^T T^T Q_2$, $Z := Q_2^T Z Q_1$, $H := Q_3^T H Q_1$ are still upper (quasi) triangular but the m_- eigenvalues with negative real part are reordered to the top of $\lambda T^T Z - H$. Set $\mathcal{Q}_1 := \operatorname{diag}(Q_1, Q_3)$, $\mathcal{U}_1 := \operatorname{diag}(Q_2, Q_2)$ and update $\mathcal{Q} := \mathcal{Q}\mathcal{Q}_1$, $\mathcal{U} := \mathcal{U}\mathcal{U}_1$.

b) Determine orthogonal matrices Q_1, Q_2, Q_3 such that $T^T := Q_3^T T^T Q_2$, $Z := Q_2^T Z Q_1$, $H := Q_3^T H Q_1$ are still upper (quasi) triangular but the m_+ eigenvalues with positive real part are reordered to the bottom of $\lambda T^T Z - H$. Set $\mathcal{Q}_1 := \operatorname{diag}(Q_1, Q_3)$, $\mathcal{U}_1 := \operatorname{diag}(Q_2, Q_2)$ and update $\mathcal{Q} := \mathcal{Q}\mathcal{Q}_1$, $\mathcal{U} := \mathcal{U}\mathcal{U}_1$.

2: Reorder the remaining $n-m_+ + 1$ eigenvalues with negative real parts which are now in the bottom right subpencil of $\lambda\mathcal{S}-\mathcal{H}$. Determine an orthogonal matrix $\mathcal{Q}_1$ and an orthogonal symplectic matrix $\mathcal{U}_1$ such that the eigenvalues of top left subpencil of $\lambda\mathcal{S}-\mathcal{H}$ with positive real parts and those of the bottom right subpencil of $\lambda\mathcal{S}-\mathcal{H}$ with negative real parts are interchanged. Update $\mathcal{Q} := \mathcal{Q}\mathcal{Q}_1$, $\mathcal{U} := \mathcal{U}\mathcal{U}_1$.

In case that we have to deal with skew-Hamiltonian/Hamiltonian matrix pencils $\lambda\mathcal{S}-\mathcal{H}$ with unfactored matrix $\mathcal{S}$ we use the following algorithms.

ALGORITHM 11. *Computation of stable deflating subspaces of real skew-Hamiltonian/Hamiltonian matrix pencil in unfactored form*

Input: *Real skew-Hamiltonian/Hamiltonian matrix pencil* $\lambda\mathcal{S} - \mathcal{H}$.
Output: *Structured Schur form of the extended skew-Hamiltonian/Hamiltonian matrix pencil* $\lambda\mathcal{B}^r_{\mathcal{S}} - \mathcal{B}^r_{\mathcal{H}}$, *eigenvalues of* $\lambda\mathcal{S} - \mathcal{H}$, *orthonormal basis* P_V^- *of the deflating subspace* $\mathrm{Def}_-(\mathcal{S},\mathcal{H})$ *as in Theorem 3.2.*

1: *Apply Algorithm 12 to the matrices* $\mathcal{S}$ *and* $\mathcal{H}$ *and determine orthogonal matrices* $\mathcal{Q}_1$, $\mathcal{Q}_2$ *such that*

$$\mathcal{Q}_1^T \mathcal{S}\mathcal{J}\mathcal{Q}_1\mathcal{J}^T = \begin{bmatrix} S_{11} & S_{12} \\ 0 & S_{11}^T \end{bmatrix} \in \mathbb{SH}_{2n},$$
$$\mathcal{J}\mathcal{Q}_2^T\mathcal{J}^T\mathcal{S}\mathcal{Q}_2 = \begin{bmatrix} T_{11} & T_{12} \\ 0 & T_{11}^T \end{bmatrix} \in \mathbb{SH}_{2n},$$
$$\mathcal{Q}_1^T\mathcal{H}\mathcal{Q}_2 = \begin{bmatrix} H_{11} & H_{12} \\ 0 & H_{22} \end{bmatrix},$$

with the formal matrix product $S_{11}^{-1}H_{11}T_{11}^{-1}H_{22}^T$ *in real periodic Schur form, where* S_{11}, T_{11}, H_{11} *are upper triangular and* H_{22}^T *is upper quasi triangular.*

2: *Apply Algorithm 13 to determine orthogonal matrices* $\mathcal{Q}_3$, $\mathcal{Q}_4$ *such that the matrix* $\mathcal{S}_{11} = \mathcal{Q}_4^T \begin{bmatrix} S_{11} & 0 \\ 0 & T_{11} \end{bmatrix} \mathcal{Q}_3$ *is upper triangular and* $\mathcal{H}_{11} = \mathcal{Q}_4^T \begin{bmatrix} 0 & H_{11} \\ -H_{22}^T & 0 \end{bmatrix} \mathcal{Q}_3$ *is upper quasi triangular.*

3: *Update*

$$\mathcal{S}_{12} := \mathcal{Q}_4^T \begin{bmatrix} S_{12} & 0 \\ 0 & T_{12} \end{bmatrix} \mathcal{Q}_4, \quad \mathcal{H}_{12} := \mathcal{Q}_4^T \begin{bmatrix} 0 & H_{12} \\ H_{12}^T & 0 \end{bmatrix} \mathcal{Q}_4,$$

and set

$$\tilde{\mathcal{B}}^r_{\mathcal{S}} = \begin{bmatrix} \mathcal{S}_{11} & \mathcal{S}_{12} \\ 0 & \mathcal{S}_{11}^T \end{bmatrix}, \quad \tilde{\mathcal{B}}^r_{\mathcal{H}} = \begin{bmatrix} \mathcal{H}_{11} & \mathcal{H}_{12} \\ 0 & -\mathcal{H}_{11}^T \end{bmatrix}.$$

Apply the real eigenvalue reordering method in Algorithm 14 to the pair $\left(\tilde{\mathcal{B}}^r_{\mathcal{S}}, \tilde{\mathcal{B}}^r_{\mathcal{H}}\right)$ *to determine an orthogonal matrix* $\hat{\mathcal{Q}}$ *such that* $\mathcal{J}\hat{\mathcal{Q}}^T\mathcal{J}^T\left(\lambda\tilde{\mathcal{B}}^r_{\mathcal{S}} - \tilde{\mathcal{B}}^r_{\mathcal{H}}\right)\hat{\mathcal{Q}}$ *is in structured Schur form and* $\Lambda_-\left(\tilde{\mathcal{B}}^r_{\mathcal{S}}, \tilde{\mathcal{B}}^r_{\mathcal{H}}\right)$ *is contained in the leading* $2p \times 2p$ *principal subpencil of* $\lambda\mathcal{S}_{11} - \mathcal{H}_{11}$.

4: *Set*

$$V = \begin{bmatrix} I_{2n} & 0 \end{bmatrix} \left(\mathcal{Y}_r \begin{bmatrix} \mathcal{J}\mathcal{Q}_1\mathcal{J}^T & 0 \\ 0 & \mathcal{Q}_2 \end{bmatrix} \mathcal{P} \begin{bmatrix} \mathcal{Q}_3 & 0 \\ 0 & \mathcal{Q}_4 \end{bmatrix} \hat{\mathcal{Q}} \right) \begin{bmatrix} I_{2p} \\ 0 \end{bmatrix},$$

and compute P_V^-, *orthogonal basis of* $\mathrm{range}\, V$, *using any numerically stable orthogonalization scheme.*

The following algorithm is used to compute a structured matrix pencil decomposition which is similar to the generalized symplectic URV decomposition.

ALGORITHM 12. *Variant of the generalized symplectic URV decomposition for unfactored real skew-Hamiltonian/Hamiltonian matrix pencils*

Input: *A real $2n \times 2n$ skew-Hamiltonian/Hamiltonian matrix pencil $\lambda\mathcal{S} - \mathcal{H}$.*
Output: *Orthogonal matrices $\mathcal{Q}_1$, $\mathcal{Q}_2$ and the structured factorization (23).*

1: *Set $\mathcal{Q}_1 = \mathcal{Q}_2 = I_{2n}$. Reduce $\mathcal{S}$ to skew-Hamiltonian triangular form, i.e., determine an orthogonal matrix $\tilde{\mathcal{Q}}_1$ such that*

$$\mathcal{S} := \tilde{\mathcal{Q}}_1^T \mathcal{S} \mathcal{J} \tilde{\mathcal{Q}}_1 \mathcal{J}^T = \begin{bmatrix} S_{11} & S_{12} \\ 0 & S_{11}^T \end{bmatrix}$$

with an upper triangular matrix S_{11}. Update $\mathcal{H} := \tilde{\mathcal{Q}}_1^T \mathcal{H} \mathcal{J} \tilde{\mathcal{Q}}_1 \mathcal{J}^T$, $\mathcal{Q}_1 := \mathcal{Q}_1 \tilde{\mathcal{Q}}_1$. This step is performed by applying a sequence of Householder reflections and Givens rotations in a specific order, see [2] for details.

2: *Set $\mathcal{T} := \mathcal{S}$, $\mathcal{Q}_2 := \mathcal{J}\mathcal{Q}_1\mathcal{J}^T$. Perform eliminations in $\mathcal{H}$, i.e., compute orthogonal matrices $\tilde{\mathcal{Q}}_1$, $\tilde{\mathcal{Q}}_2$ such that*

$$\begin{aligned} \mathcal{S} &:= \tilde{\mathcal{Q}}_1^T \mathcal{S} \mathcal{J} \tilde{\mathcal{Q}}_1 \mathcal{J}^T = \begin{bmatrix} S_{11} & S_{12} \\ 0 & S_{11}^T \end{bmatrix} \in \mathbb{SH}_{2n}, \\ \mathcal{T} &:= \mathcal{J} \tilde{\mathcal{Q}}_2^T \mathcal{J}^T \mathcal{T} \tilde{\mathcal{Q}}_2 = \begin{bmatrix} T_{11} & T_{12} \\ 0 & T_{11}^T \end{bmatrix} \in \mathbb{SH}_{2n}, \\ \mathcal{H} &:= \tilde{\mathcal{Q}}_1^T \mathcal{H} \tilde{\mathcal{Q}}_2 = \begin{bmatrix} H_{11} & H_{12} \\ 0 & H_{22} \end{bmatrix} \end{aligned}$$

where S_{11}, T_{11}, H_{11} are upper triangular and H_{22}^T is upper Hessenberg. Update $\mathcal{Q}_1 := \mathcal{Q}_1 \tilde{\mathcal{Q}}_1$, $\mathcal{Q}_2 := \mathcal{Q}_2 \tilde{\mathcal{Q}}_2$. This step is performed by applying an appropriate sequence of Givens rotations to annihilate the elements in $\mathcal{H}$ in a specific order without destroying the structure of $\mathcal{S}$ and $\mathcal{T}$, for details see [2].

3: *Apply the periodic QZ algorithm [9, 13] to the formal matrix product*

$$S_{11}^{-1} H_{11} T_{11}^{-1} H_{22}^T$$

to determine orthogonal matrices V_1, V_2, V_3, V_4 such that $V_1^T S_{11} V_3$, $V_1^T H_{11} V_4$, $V_2^T T_{11} V_4$, are all upper triangular and $\left(V_3^T H_{22} V_2\right)^T$ is upper quasi triangular. Set

$$\tilde{\mathcal{Q}}_1 := \operatorname{diag}(V_1, V_3), \quad \tilde{\mathcal{Q}}_2 := \operatorname{diag}(V_4, V_2),$$

and update $\mathcal{S} := \tilde{\mathcal{Q}}_1^T \mathcal{S} \mathcal{J} \tilde{\mathcal{Q}}_1 \mathcal{J}^T$, $\mathcal{T} := \mathcal{J} \tilde{\mathcal{Q}}_2^T \mathcal{J}^T \mathcal{T} \tilde{\mathcal{Q}}_2$, $\mathcal{H} := \tilde{\mathcal{Q}}_1^T \mathcal{H} \tilde{\mathcal{Q}}_2$, $\mathcal{Q}_1 := \mathcal{Q}_1 \tilde{\mathcal{Q}}_1$, $\mathcal{Q}_2 := \mathcal{Q}_2 \tilde{\mathcal{Q}}_2$.

Now we present the triangularization algorithm. All remarks which have been made for the factored case analogously hold for the unfactored case.

ALGORITHM 13. *Triangularization procedure for special matrix pencils in unfactored form*

Input: *A real matrix pencil* $\lambda\mathcal{A}-\mathcal{B}=\lambda\begin{bmatrix}A_{11} & 0\\ 0 & A_{22}\end{bmatrix}-\begin{bmatrix}0 & B_{12}\\ B_{21} & 0\end{bmatrix}$ *where the formal matrix product* $A_{11}^{-1}B_{12}A_{22}^{-1}B_{21}$ *is in real periodic Schur form with upper triangular* A_{11}, A_{22}, B_{12} *and upper quasi triangular* B_{21}.

Output: *Orthogonal matrices* $\mathcal{Q}_1$, $\mathcal{Q}_2$ *such that* $\mathcal{Q}_2^T\mathcal{A}\mathcal{Q}_1$ *is upper triangular and* $\mathcal{Q}_2^T\mathcal{B}\mathcal{Q}_1$ *is upper quasi triangular.*

1: *Apply the periodic eigenvalue reordering method introduced in [12] to the formal matrix product*

$$A_{11}^{-1}B_{12}A_{22}^{-1}B_{21}$$

to determine orthogonal matrices V_1, V_2, V_3, V_4 *such that* $V_2^TA_{11}V_1$, $V_2^TB_{12}V_3$, $V_4^TA_{22}V_3$, $V_4^TB_{21}V_1$, *keep their upper (quasi) triangular structure but they can be partitioned into* 2×2 *blocks with the last diagonal blocks corresponding to all nonpositive real eigenvalues of the formal product, and the first diagonal blocks corresponding to the other eigenvalues.*

2: *Set* $\mathcal{Q}_1 := \mathrm{diag}(V_1,V_3)$, $\mathcal{Q}_2 := \mathrm{diag}(V_2,V_4)$, *and update*

$$\mathcal{A} := \mathcal{Q}_2^T\mathcal{A}\mathcal{Q}_1 =: \left[\begin{array}{cc|cc} A_{11} & A_{12} & 0 & 0\\ 0 & A_{22} & 0 & 0\\ \hline 0 & 0 & A_{33} & A_{34}\\ 0 & 0 & 0 & A_{44}\end{array}\right],$$

$$\mathcal{B} := \mathcal{Q}_2^T\mathcal{B}\mathcal{Q}_1 =: \left[\begin{array}{cc|cc} 0 & 0 & B_{13} & B_{14}\\ 0 & 0 & 0 & B_{24}\\ \hline B_{31} & B_{32} & 0 & 0\\ 0 & B_{42} & 0 & 0\end{array}\right],$$

where $A_{22}^{-1}B_{24}A_{44}^{-1}B_{42}$ *has only nonpositive real eigenvalues.*

3: *Let* $\mathcal{P}$ *be an appropriate permutation matrix such that*

$$\mathcal{A} := \mathcal{P}^T\mathcal{A}\mathcal{P} = \left[\begin{array}{cc|cc} A_{11} & 0 & A_{12} & 0\\ 0 & A_{33} & 0 & A_{34}\\ \hline 0 & 0 & A_{22} & 0\\ 0 & 0 & 0 & A_{44}\end{array}\right] =: \begin{bmatrix}\tilde{A} & *\\ 0 & \hat{A}\end{bmatrix},$$

$$\mathcal{B} := \mathcal{P}^T\mathcal{B}\mathcal{P} = \left[\begin{array}{cc|cc} 0 & B_{13} & 0 & B_{14}\\ B_{31} & 0 & B_{32} & 0\\ \hline 0 & 0 & 0 & B_{24}\\ 0 & 0 & B_{42} & 0\end{array}\right] =: \begin{bmatrix}\tilde{B} & *\\ 0 & \hat{B}\end{bmatrix},$$

and update $\mathcal{Q}_1 := \mathcal{Q}_1\mathcal{P}$, $\mathcal{Q}_2 := \mathcal{Q}_2\mathcal{P}$.

4: *Triangularize* $\lambda\tilde{A}-\tilde{B}$, *i.e., compute orthogonal matrices* $\tilde{\mathcal{Q}}_1$, $\tilde{\mathcal{Q}}_2$ *such that* $\mathcal{A} := \tilde{\mathcal{Q}}_2^T\mathcal{A}\tilde{\mathcal{Q}}_1 =: \begin{bmatrix}\tilde{A} & *\\ 0 & \hat{A}\end{bmatrix}$, $\mathcal{B} := \tilde{\mathcal{Q}}_2^T\mathcal{B}\tilde{\mathcal{Q}}_1 =: \begin{bmatrix}\tilde{B} & *\\ 0 & \hat{B}\end{bmatrix}$ *with upper triangular* $\tilde{A}$, *upper quasi triangular* $\tilde{B}$, *and unchanged* $\hat{A}$, $\hat{B}$. *Update* $\mathcal{Q}_1 = \mathcal{Q}_1\tilde{\mathcal{Q}}_1$, $\mathcal{Q}_2 = \mathcal{Q}_2\tilde{\mathcal{Q}}_2$.

5: Triangularize $\lambda\hat{A} - \hat{B}$ with an appropriate permutation matrix $\hat{\mathcal{P}}$, i.e., $\mathcal{A} := \hat{\mathcal{P}}^T \mathcal{A} \hat{\mathcal{P}} =: \begin{bmatrix} \tilde{A} & * \\ 0 & \hat{A} \end{bmatrix}$, $\mathcal{B} := \hat{\mathcal{P}}^T \mathcal{B} \hat{\mathcal{P}} =: \begin{bmatrix} \tilde{B} & * \\ 0 & \hat{B} \end{bmatrix}$ with upper triangular $\hat{A}$, upper quasi triangular $\hat{B}$ and unchanged $\tilde{A}$, $\tilde{B}$. Update $\mathcal{Q}_1 = \mathcal{Q}_1 \hat{\mathcal{P}}$, $\mathcal{Q}_2 = \mathcal{Q}_2 \hat{\mathcal{P}}$.

Finally, we describe the reordering of the eigenvalues.

ALGORITHM 14. *Eigenvalue reordering for real skew-Hamiltonian/Hamiltonian matrix pencils in unfactored form*

Input: *Regular $2n \times 2n$ real skew-Hamiltonian/Hamiltonian matrix pencil $\lambda\mathcal{S} - \mathcal{H}$ of the form $\mathcal{S} = \begin{bmatrix} S & W \\ 0 & S^T \end{bmatrix}$, $\mathcal{H} = \begin{bmatrix} H & D \\ 0 & -H^T \end{bmatrix}$, with upper triangular S an upper quasi triangular H.*

Output: *An orthogonal matrix $\mathcal{Q}$ and the transformed matrices $\mathcal{J}\mathcal{Q}^T\mathcal{J}^T\mathcal{S}\mathcal{Q}$, $\mathcal{J}\mathcal{Q}^T\mathcal{J}^T\mathcal{H}\mathcal{Q}$ which have still the same (quasi) triangular form as $\mathcal{S}$ and $\mathcal{H}$, respectively, but the eigenvalues in $\Lambda_-(\mathcal{S}, \mathcal{H})$ are reordered such that they occur in the leading principal subpencil of $\mathcal{J}\mathcal{Q}^T\mathcal{J}^T(\lambda\mathcal{S} - \mathcal{H})\mathcal{Q}$.*

1: Set $\mathcal{Q} = I_{2n}$. Reorder the eigenvalues in the subpencil $\lambda S - H$.

a) Determine orthogonal matrices Q_1, Q_2 such that $S := Q_2^T S Q_1$, $H := Q_2^T H Q_1$, are still upper (quasi) triangular but the m_- eigenvalues with negative real part are reordered to the top of $\lambda S - H$. Set $\mathcal{Q}_1 := \mathrm{diag}(Q_1, Q_2)$ and update $\mathcal{Q} := \mathcal{Q}\mathcal{Q}_1$.

b) Determine orthogonal matrices Q_1, Q_2 such that $S := Q_2^T S Q_1$, $H := Q_2^T H Q_1$, are still upper (quasi) triangular but the m_+ eigenvalues with positive real part are reordered to the bottom of $\lambda S - H$. Set $\mathcal{Q}_1 := \mathrm{diag}(Q_1, Q_2)$ and update $\mathcal{Q} := \mathcal{Q}\mathcal{Q}_1$.

2: Reorder the remaining $n - m_+ + 1$ eigenvalues with negative real parts which are now in the bottom right subpencil of $\lambda\mathcal{S} - \mathcal{H}$. Determine an orthogonal matrix $\mathcal{Q}_1$ such that the eigenvalues of top left subpencil of $\lambda\mathcal{S} - \mathcal{H}$ with positive real parts and those of the bottom right subpencil of $\lambda\mathcal{S} - \mathcal{H}$ with negative real parts are interchanged. Update $\mathcal{Q} := \mathcal{Q}\mathcal{Q}_1$.

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4 Conclusion

We have presented algorithms which can be used to compute the eigenvalues and deflating subspaces of skew-Hamiltonian/Hamiltonian matrix pencils in a structure-preserving way which may lead to higher accuracy, reliability and computational performance. Applications which are based on matrix pencils of this structure have been introduced to show the importance of our considerations. In Part II of this paper [8] we describe details of the implementation in the style of SLICOT subroutines. We furthermore present results of some numerical experiments in order to show the superiority of our method compared to standard approaches.

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