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Near-optimal frequency-weighted interpolatory model reduction

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Abstract

This paper extends an interpolatory framework for weighted-$\mathcal{H}_2$ model reduction to MIMO dynamical systems. A new representation of the weighted-$\mathcal{H}_2$ inner product in MIMO settings is presented together with associated first-order necessary conditions for an optimal weighted-$\mathcal{H}_2$ reduced-order model. Equivalence of these conditions with necessary conditions given by Halevi is shown. An examination of realizations for equivalent weighted-$\mathcal{H}_2$ systems leads to an algorithm that remains tractable for large state-space dimension. Several numerical examples illustrate the effectiveness of this approach and its competitiveness with Frequency Weighted Balanced Truncation and Weighted Iterative Rational Krylov Algorithm.

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1 Introduction

Consider a multiple input/multiple output (MIMO) linear dynamical system having a state-space realization (which will be presumed minimal) given by

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}\,\mathbf{u}(t) \\ \mathbf{y}(t) &= \mathbf{C}\mathbf{x}(t) + \mathbf{D}\,\mathbf{u}(t) \end{aligned} \tag{1}$$

where $\mathbf{A} \in \mathbb{R}^{n\times n}$, $\mathbf{B} \in \mathbb{R}^{n\times m}$, $\mathbf{C} \in \mathbb{R}^{p\times n}$, and $\mathbf{D} \in \mathbb{R}^{p\times m}$ are constant matrices. $\mathbf{x}(t) \in \mathbb{R}^n$, $\mathbf{u}(t) \in \mathbb{R}^m$ and $\mathbf{y}(t) \in \mathbb{R}^p$ are, respectively, the *state*, the *input*, and the *output* of the system. The *transfer function* of this system is $\mathbf{G}(s) = \mathbf{C}(s\mathbf{I}-\mathbf{A})^{-1}\mathbf{B}+\mathbf{D}$. Following common usage, the underlying system will also be denoted by $\mathbf{G}$. The circumstances of interest for us presume very large state-space dimensions relative to the input/output dimensions, $n \gg m, p$. This leads to fundamental difficulties for any task that involves optimization or control of this system. This in turn motivates *model reduction*: finding a reduced order model (ROM),

$$\begin{aligned} \dot{\mathbf{x}}_r(t) &= \mathbf{A}_r\mathbf{x}_r(t) + \mathbf{B}_r\mathbf{u}(t), \\ \mathbf{y}_r(t) &= \mathbf{C}_r\mathbf{x}_r(t) + \mathbf{D}_r\,\mathbf{u}(t) \end{aligned} \tag{2}$$

with an associated transfer function $\mathbf{G}_r(s) = \mathbf{C}_r(s\mathbf{I} - \mathbf{A}_r)^{-1}\mathbf{B}_r + \mathbf{D}_r$ where $\mathbf{A}_r \in \mathbb{R}^{n_r\times n_r}$, $\mathbf{B}_r \in \mathbb{R}^{n_r\times m}$, $\mathbf{C}_r \in \mathbb{R}^{p\times n_r}$, and $\mathbf{D}_r \in \mathbb{R}^{p\times m}$. The goal is to produce a greatly reduced state-space dimension, $n_r \ll n$, yet still assure that $\mathbf{y}_r(t) \approx \mathbf{y}(t)$ over a large class of inputs $\mathbf{u}(t)$. This is accomplished by requiring $\mathbf{G}_r(s)$ to approximate $\mathbf{G}(s)$ very well, in an appropriate sense, which we interpret as making $\mathbf{G}_r(s) - \mathbf{G}(s)$ small with respect to an appropriate system norm.

For example, one may consider approximations that attempt to minimize the $\mathcal{H}_2$-error:

$$\|\mathbf{G} - \mathbf{G}_r\|_{\mathcal{H}_2} \stackrel{\text{def}}{=} \left(\frac{1}{2\pi}\int_{-\infty}^{+\infty} \|\mathbf{G}(\imath\omega) - \mathbf{G}_r(\imath\omega)\|_F^2 \,\mathrm{d}\omega\right)^{1/2}, \tag{3}$$

where $\|\mathbf{M}\|_F^2 = \sum_{i,j}|m_{ij}|^2$ denotes the *Frobenius norm* of the matrix $\mathbf{M}$. Notice that in order to ensure that this error measure is even *finite*, it is necessary that $\mathbf{D}_r = \mathbf{D}$.

"Typical" inputs, $\mathbf{u}(t)$, often will have their power concentrated in known frequency ranges, and so, some frequency ranges will naturally be more important than others with regard to ROM fidelity. This leads in a natural way to consideration of *weighted* system errors designed in such a way so as to enhance accuracy in certain frequency ranges while permitting larger errors at other frequencies.

Consider then, a *weighted* system error

$$\|\mathbf{G}_r - \mathbf{G}\|_{\mathcal{H}_2(W)} \stackrel{\text{def}}{=} \|\left(\mathbf{G}_r(s) - \mathbf{G}(s)\right)\mathbf{W}(s)\|_{\mathcal{H}_2} \tag{4}$$

where $\mathbf{W}(s)$ is a given input weighting ("shaping filter"). One may specify an output weighting as well, however in the interest of clarity and brevity, we do not do this here. For a given system $\mathbf{G} \in \mathcal{H}_2$ the goal will be to construct a reduced system $\mathbf{G}_r \in \mathcal{H}_2$ solving the weighted-$\mathcal{H}_2$ approximation problem:

$$\mathbf{G}_r = \underset{\mathsf{ord}(\tilde{\mathbf{G}})\,\le\, n_r}{\operatorname{argmin}} \|\mathbf{G} - \tilde{\mathbf{G}}\|_{\mathcal{H}_2(W)} \tag{5}$$

Choosing $\mathbf{W}(s)$ to be a transfer function associated with a band-pass filter penalizes approximation errors at frequencies within the passband and while discounting approximation error at frequencies outside the passband.

Another choice of shaping filter arises from controller reduction: Consider a linear dynamical system, $\mathbf{P}$ (the *plant*), with order n_P together with an associated stabilizing controller, $\mathbf{G}$, having order n, that is connected to $\mathbf{P}$ in a feedback loop. Many control design methodologies, such as LQG and $\mathcal{H}_\infty$ methods, lead ultimately to controllers whose order is generically as high as the order of the plant, $n \approx n_P$, see [26, 30] and references therein. Thus, high-order plants will generally lead to high-order controllers. However, high-order controllers are usually undesirable in real-time applications because this typically translates into unduly complex and costly hardware implementation that may suffer degraded performance both in terms of speed and accuracy. Thus, one may prefer to replace $\mathbf{G}$ with a reduced order controller, $\mathbf{G}_r$, having order $n_r \ll n$.

It is often not enough to simply require $\mathbf{G}_r$ to be a good approximation to $\mathbf{G}$. In order to accurately recover closed-loop performance, plant dynamics need to be taken into account during the reduction process. This may be achieved through frequency weighting: Given a stabilizing controller $\mathbf{G}$, if a reduced model, $\mathbf{G}_r$, has the same number of unstable poles as $\mathbf{G}$ and

$$\left\| [\mathbf{G} - \mathbf{G}_r] \cdot \mathbf{P}[\mathbf{I} + \mathbf{P}\mathbf{G}]^{-1} \right\|_{\mathcal{H}_\infty} < 1,$$

then, if $\mathbf{G}_r$ is used to replace $\mathbf{G}$, $\mathbf{G}_r$ will also be a stabilizing controller [1, 30]. Seeking $\mathbf{G}_r$ to minimize the weighted-$\mathcal{H}_2$ error measure (4) is an effective proxy, using $\mathbf{W}(s) = \mathbf{P}(s)[\mathbf{I} + \mathbf{P}(s)\mathbf{G}(s)]^{-1}$. This approach has been considered in [26, 1, 20, 10, 7, 28, 15, 27, 25] and references therein, leading then to variants of frequency-weighted balanced truncation. Related methods in [13] and [24] are tailored instead towards minimizing a similarly weighted $\mathcal{H}_2$ error, as we do here.

2 Optimal approximations in a weighted-$\mathcal{H}_2$ norm.

$\mathcal{H}_\infty$ will denote here the set of $m \times m_w$ matrix-valued functions, $\mathbf{W}(s)$, having entries, $w_{ij}(s)$, that are analytic for s in the open right half plane and uniformly bounded along the imaginary axis, which $\sup_{\omega\in\mathbb{R}} |w_{ij}(\imath\omega)|$ is finite for all i, j. A norm may be defined on $\mathcal{H}_\infty$ as $\|\mathbf{W}\|_{\mathcal{H}_\infty} = \sup_{\omega\in\mathbb{R}} \|\mathbf{W}(\imath\omega)\|_2$, where $\|\mathbf{M}\|_2$ here represents the induced matrix 2-norm. We assume throughout this work that weighting functions are drawn from $\mathcal{H}_\infty$.

For any such weight $\mathbf{W} \in \mathcal{H}_\infty$, denote by $\mathcal{H}_2(W)$ the set of $p \times m$ matrix-valued functions, $\mathbf{G}(s)$, that have components analytic for s in the open right half plane, and such that for each fixed $Re(s) = x > 0$, $\mathbf{G}(x + \imath y)$ is square integrable with respect to $\mathbf{W}$ as a function of $y \in (-\infty, \infty)$ in the sense that

$$\sup_{x>0} \int_{-\infty}^{\infty} \|\mathbf{G}(x + \imath y)\mathbf{W}(x + \imath y)\|_F^2 \, dy < \infty.$$

If $\mathbf{G}$, $\mathbf{H} \in \mathcal{H}_2(W)$ are transfer functions representing real dynamical systems then an inner product may be defined as

$$\begin{aligned}
\langle \mathbf{G},\ \mathbf{H} \rangle_{\mathcal{H}_2(W)} &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathsf{tr}\left(\overline{\mathbf{G}(\imath\omega)\mathbf{W}(\imath\omega)} \mathbf{W}(\imath\omega)^T \mathbf{H}(\imath\omega)^T \right) \,\mathrm{d}\omega \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathsf{tr}\left(\mathbf{G}(-\imath\omega)\mathbf{W}(-\imath\omega)\mathbf{W}(\imath\omega)^T \mathbf{H}(\imath\omega)^T \right) \,\mathrm{d}\omega.
\end{aligned}$$

The associated norm on $\mathcal{H}_2(W)$ is

$$\|\mathbf{G}\|_{\mathcal{H}_2(W)} = \left(\langle \mathbf{G},\ \mathbf{G} \rangle_{\mathcal{H}_2(W)} \right)^{1/2}.$$

$\mathcal{H}_2$ will denote precisely the set $\mathcal{H}_2(W)$ but using the particular choice $\mathbf{W}(s) = \mathbf{I}$ (and with $m = m_w$). Note that $\mathcal{H}_2 \subset \mathcal{H}_2(W)$ and for $\mathbf{G}$, $\mathbf{H} \in \mathcal{H}_2$,

$$\left| \langle \mathbf{G},\ \mathbf{H} \rangle_{\mathcal{H}_2(W)} \right| \leq \|\mathbf{W}\|_{\mathcal{H}_\infty}^2 \ \|\mathbf{G}\|_{\mathcal{H}_2} \ \|\mathbf{H}\|_{\mathcal{H}_2}. \tag{6}$$

In all that follows, we suppose the weight $\mathbf{W} \in \mathcal{H}_\infty$ is a rational function with simple poles at $\{\gamma_1, \ldots, \gamma_{n_w}\}$ and that it has alternative representations given by

$$\mathbf{W}(s) = \mathbf{C}_w \left(s\mathbf{I} - \mathbf{A}_w \right)^{-1} \mathbf{B}_w + \mathbf{D}_w \tag{7}$$

$$\text{and} \quad \mathbf{W}(s) = \sum_{k=1}^{n_w} \frac{\mathbf{e}_k \ \mathbf{f}_k^T}{s - \gamma_k} + \mathbf{D}_w. \tag{8}$$

with $\mathbf{A}_w \in \mathbb{R}^{n_w \times n_w}$, $\mathbf{B}_w \in \mathbb{R}^{n_w \times m_w}$, $\mathbf{C}_w \in \mathbb{R}^{m \times n_w}$, and $\mathbf{D}_w \in \mathbb{R}^{m \times m_w}$. Echoing the setting of [13], our analysis does not require $m = m_w$, though it may be a natural choice. The (matrix-valued) residue of a meromorphic matrix-valued function, $\mathbf{M}(s)$, at a point $\zeta \in \mathbb{C}$ will be denoted as $\mathsf{res}[\mathbf{M}(s), \zeta]$, so for example, with $\mathbf{W}$ as in (8), $\mathsf{res}[\mathbf{W}, \gamma_k] = \mathbf{e}_k \ \mathbf{f}_k^T$.

Notice that the transfer function, $\mathbf{G}$, associated with the system (1) will be in $\mathcal{H}_2(W)$ if and only if $\mathbf{A}$ is stable and $\mathbf{D}\mathbf{D}_w = 0$. For $\mathbf{G} \in \mathcal{H}_2(W)$, define

$$\mathfrak{F}[\mathbf{G}](s) = \mathbf{G}(s)\mathbf{W}(s)\mathbf{W}(-s)^T + \sum_{k=1}^{n_w} \mathbf{G}(-\gamma_k)\mathbf{W}(-\gamma_k) \frac{\mathbf{f}_k \mathbf{e}_k^T}{s + \gamma_k} \tag{9}$$

Lemma 1 *For $\mathfrak{F}$ as defined in (9)*

a. $\mathfrak{F}$ is a bounded linear transformation from $\mathcal{H}_2(W)$ to $\mathcal{H}_2$.

b. For any $\mathbf{G}$, $\mathbf{H} \in \mathcal{H}_2$, $\langle \mathbf{G},\ \mathbf{H} \rangle_{\mathcal{H}_2(W)} = \langle \mathfrak{F}[\mathbf{G}],\ \mathbf{H} \rangle_{\mathcal{H}_2}$. Hence, $\mathfrak{F}$ is a positive-definite, selfadjoint linear operator on $\mathcal{H}_2$.

The proof of this lemma and subsequent arguments employ an elementary result that we list here. It is an immediate corollary to [3, Lemma 1]:

Proposition 2 *Let* $\mathbf{G}_1 \in \mathcal{H}_2$ *and* $\mathbf{G}_2(s) = \frac{\mathbf{c}\mathbf{b}^T}{s-\mu} \in \mathcal{H}_2$. *Then,*

$$\langle \mathbf{G}_1, \mathbf{G}_2 \rangle_{\mathcal{H}_2} = \mathbf{c}^T \overline{\mathbf{G}_1}(-\mu)\mathbf{b} \text{ and } \|\mathbf{G}_2\|_{\mathcal{H}_2} = \frac{\|\mathbf{c}\|\|\mathbf{b}\|}{\sqrt{2|\mathsf{Re}\gamma_k|}}.$$

Proof of Lemma 1: Clearly, $\mathfrak{F}[\mathbf{G}]$ is linear in $\mathbf{G}$. Let $\mathbf{G} \in \mathcal{H}_2(W)$. $\mathbf{G}(s)\mathbf{W}(s)\mathbf{W}(-s)^T$ has simple poles in the right half plane at $-\gamma_1, -\gamma_2, \ldots, -\gamma_{n_w}$, and

$$\begin{aligned}
\mathsf{res}[\mathbf{G}(s)\mathbf{W}(s)\mathbf{W}(-s)^T, -\gamma_k] &= \lim_{s\to-\gamma_k} (s+\gamma_k)\mathbf{G}(s)\mathbf{W}(s)\mathbf{W}(-s)^T \\
&= \mathbf{G}(-\gamma_k)\mathbf{W}(-\gamma_k) \lim_{s\to-\gamma_k} (s+\gamma_k)\mathbf{W}(-s)^T \\
&= -\mathbf{G}(-\gamma_k)\mathbf{W}(-\gamma_k) \lim_{s\to\gamma_k} (s-\gamma_k)\mathbf{W}(s)^T \\
&= -\mathbf{G}(-\gamma_k)\mathbf{W}(-\gamma_k) \cdot \mathsf{res}[\mathbf{W}(s)^T, \gamma_k] \\
&= -\mathbf{G}(-\gamma_k)\mathbf{W}(-\gamma_k)\ \mathbf{f}_k\ \mathbf{e}_k^T.
\end{aligned}$$

Thus $\mathfrak{F}[\mathbf{G}](s)$ is analytic in the right-half plane. To show that $\mathfrak{F}[\mathbf{G}] \in \mathcal{H}_2$, observe first that $\mathbf{G}\cdot\mathbf{W} \in \mathcal{H}_2$ so that for each $k = 1, \ldots, n_w$:

$$\begin{aligned}
\|\mathbf{G}(-\gamma_k)\mathbf{W}(-\gamma_k)\|_2 &= \max_{\mathbf{u},\mathbf{v}} \frac{\mathbf{u}^*\ [\mathbf{G}(-\gamma_k)\mathbf{W}(-\gamma_k)]\ \mathbf{v}}{\|\mathbf{u}\|_2\ \|\mathbf{v}\|_2} \\
&= \max_{\mathbf{u},\mathbf{v}} \frac{1}{\|\mathbf{u}\|_2\ \|\mathbf{v}\|_2} \left\langle \mathbf{G}(s)\mathbf{W}(s),\ \frac{\mathbf{v}\mathbf{u}^*}{s-\gamma_k} \right\rangle_{\mathcal{H}_2} \\
&\leq \|\mathbf{G}\mathbf{W}\|_{\mathcal{H}_2} \cdot \max_{\mathbf{u},\mathbf{v}} \frac{\left\| \frac{\mathbf{v}\mathbf{u}^*}{s-\gamma_k} \right\|_{\mathcal{H}_2}}{\|\mathbf{u}\|\ \|\mathbf{v}\|} = \frac{\|\mathbf{G}\|_{\mathcal{H}_2(W)}}{\sqrt{2\,|\mathsf{Re}\,\gamma_k|}},
\end{aligned}$$

where the final equality follows from Proposition 2. This amounts to the observation that point evaluation in the right half-plane is a continuous map from $\mathcal{H}_2(W)$ to $\mathbb{C}^{m\times p}$. We now calculate

$$\begin{aligned}
\|\mathfrak{F}[\mathbf{G}]\|_{\mathcal{H}_2} &\leq \|\mathbf{W}\|_{\mathcal{H}_\infty} \|\mathbf{G}(s)\mathbf{W}(s)\|_{\mathcal{H}_2} + \sum_{k=1}^{n_w} \|\mathbf{G}(-\gamma_k)\mathbf{W}(-\gamma_k)\frac{\mathbf{f}_k\mathbf{e}_k^T}{s+\gamma_k}\|_{\mathcal{H}_2} \\
&\leq \left(\|\mathbf{W}\|_{\mathcal{H}_\infty} + \sum_{k=1}^{n_w} \frac{\|\mathbf{f}_k\|\ \|\mathbf{e}_k\|}{2\,|\mathsf{Re}\,\gamma_k|} \right) \|\mathbf{G}\|_{\mathcal{H}_2(W)};
\end{aligned}$$

$\mathfrak{F}$ is a bounded linear transformation from $\mathcal{H}_2(W)$ to $\mathcal{H}_2$.

For assertion 1b, suppose first that $\mathbf{H}$ has simple poles $\{\mu_1, \ldots, \mu_\ell\}$. Note that since $\mathfrak{F}[\mathbf{G}](-s)$ is analytic in the left half plane, $\mathfrak{F}[\mathbf{G}](-s)\mathbf{H}(s)^T$ will have poles in the left halfplane exactly at $\{\mu_1, \ldots, \mu_\ell\}$.

For any $R > 0$, define a semicircular contour in the left halfplane:

$$\mathcal{C}_R = \{z\,|\,z = \imath\omega \text{ with } \omega \in [-R, R]\} \cup \left\{ z \,\middle|\, z = R\,e^{\imath\theta} \text{ with } \theta \in [\frac{\pi}{2}, \frac{3\pi}{2}] \right\}.$$

For R large enough, the region bounded by $\mathcal{C}_R$ contains $\{\mu_1, \ldots, \mu_\ell\}$. Using the Residue Theorem and linearity of the trace, we find

$$
\begin{aligned}
&\langle \mathfrak{F}[\mathbf{G}],\ \mathbf{H}\rangle_{\mathcal{H}_2} = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \mathsf{tr}\left(\mathfrak{F}[\mathbf{G}](-\imath\omega)\ \mathbf{H}(\imath\omega)^T\right)\,\mathrm{d}\omega \\
&= \lim_{R\to\infty}\frac{1}{2\pi\imath}\int_{\mathcal{C}_R} \mathsf{tr}\left(\mathfrak{F}[\mathbf{G}](-s)\ \mathbf{H}(s)^T\right)\,\mathrm{d}\omega \\
&= \sum_{k=1}^{\ell} \mathsf{tr}\left(\mathsf{res}[\mathfrak{F}[\mathbf{G}](-s)\mathbf{H}(s)^T, \mu_k]\right) \\
&= \sum_{k=1}^{\ell} \mathsf{tr}\left(\mathfrak{F}[\mathbf{G}](-\mu_k)\mathsf{res}[\mathbf{H}, \mu_k]^T\right) \\
&= \sum_{k=1}^{\ell} \mathsf{tr}\left(\mathbf{G}(-\mu_k)\mathbf{W}(-\mu_k)\mathbf{W}(\mu_k)^T\mathsf{res}[\mathbf{H}, \mu_k]^T\right) \\
&\quad + \sum_{k=1}^{\ell}\sum_{i=1}^{n_w} \mathsf{tr}\left(\mathbf{G}(-\gamma_i)\mathbf{W}(-\gamma_i)\frac{\mathbf{f}_i\mathbf{e}_i^T}{-\mu_k+\gamma_i}\ \mathsf{res}[\mathbf{H}, \mu_k]^T\right) \\
&= \sum_{k=1}^{\ell} \mathsf{tr}\left(\mathbf{G}(-\mu_k)\mathbf{W}(-\mu_k)\mathbf{W}(\mu_k)^T\mathsf{res}[\mathbf{H}, \mu_k]^T\right) \\
&\quad + \sum_{i=1}^{n_w} \mathsf{tr}\left(\mathbf{G}(-\gamma_i)\mathbf{W}(-\gamma_i)\mathbf{f}_i\mathbf{e}_i^T\sum_{k=1}^{\ell}\frac{\mathsf{res}[\mathbf{H}, \mu_k]^T}{\gamma_i-\mu_k}\right)
\end{aligned}
$$

Since $\mathbf{H}$ has simple poles and is in $\mathcal{H}_2$, $\sum_{k=1}^{\ell}\frac{\mathsf{res}[\mathbf{H},\mu_k]^T}{s-\mu_k} = \mathbf{H}(s)^T$. Note that $\{\mu_1, \ldots, \mu_\ell\} \cup \{\gamma_1, \ldots, \gamma_{n_w}\}$ is precisely the set of poles in the left half plane for the meromorphic function $\mathbf{G}(-s)\mathbf{W}(-s)\mathbf{W}(s)^T\mathbf{H}(s)^T$.

So, we continue:

$$
\begin{aligned}
&\langle \mathfrak{F}[\mathbf{G}],\ \mathbf{H}\rangle_{\mathcal{H}_2} \\
&\quad= \sum_{k=1}^{\ell} \mathsf{tr}\left(\mathbf{G}(-\mu_k)\mathbf{W}(-\mu_k)\mathbf{W}(\mu_k)^T\mathsf{res}[\mathbf{H}, \mu_k]^T\right) \\
&\qquad + \sum_{i=1}^{n_w} \mathsf{tr}\left(\mathbf{G}(-\gamma_i)\mathbf{W}(-\gamma_i)\mathsf{res}[\mathbf{W}, \gamma_i]^T\mathbf{H}(\gamma_i)^T\right) \\
&\quad= \lim_{R\to\infty}\frac{1}{2\pi\imath}\int_{\mathcal{C}_R} \mathsf{tr}\left(\mathbf{G}(-s)\mathbf{W}(-s)\mathbf{W}(s)^T\mathbf{H}(s)^T\right)\,\mathrm{d}s \\
&\quad= \frac{1}{2\pi}\int_{-\infty}^{+\infty} \mathsf{tr}\left(\mathbf{G}(-\imath\omega)\ \mathbf{W}(-\imath\omega)\mathbf{W}(\imath\omega)^T\mathbf{H}(\imath\omega)^T\right)\,\mathrm{d}\omega \\
&\quad= \langle \mathbf{G},\ \mathbf{H}\rangle_{\mathcal{H}_2(W)}
\end{aligned}
$$

This remains true independent of whether $\mathbf{H}$ has simple poles or not: Take a sequence, $\mathbf{H}_k$, converging to $\mathbf{H}$ in $\mathcal{H}_2$ with each $\mathbf{H}_k$ having simple poles. Then, appeal to the continuity of the expressions $\langle \mathbf{G},\ \mathbf{H}_k\rangle_{\mathcal{H}_2(W)} = \langle \mathfrak{F}[\mathbf{G}],\ \mathbf{H}_k\rangle_{\mathcal{H}_2}$ with respect to the $\mathcal{H}_2$ norm.

$\mathfrak{F}$ is positive-definite and selfadjoint on $\mathcal{H}_2$ because, for $\mathbf{G}$, $\mathbf{H} \in \mathcal{H}_2$,

$$\langle \mathfrak{F}[\mathbf{G}],\ \mathbf{H}\rangle_{\mathcal{H}_2} = \langle \mathbf{G},\ \mathbf{H}\rangle_{\mathcal{H}_2(W)} = \overline{\langle \mathbf{H},\ \mathbf{G}\rangle_{\mathcal{H}_2(W)}} = \overline{\langle \mathfrak{F}[\mathbf{H}],\ \mathbf{G}\rangle_{\mathcal{H}_2}} = \langle \mathbf{G},\ \mathfrak{F}[\mathbf{H}]\rangle_{\mathcal{H}_2}$$

and $\langle \mathfrak{F}[\mathbf{G}],\ \mathbf{G}\rangle_{\mathcal{H}_2} = \langle \mathbf{G},\ \mathbf{G}\rangle_{\mathcal{H}_2(W)} > 0 \quad \text{if} \quad \mathbf{G} \neq 0.$ $\Box$

Given state-space realizations for $\mathbf{W} \in \mathcal{H}_\infty$ and $\mathbf{G} \in \mathcal{H}_2(W)$, one may obtain an explicit state-space realization for $\mathfrak{F}[\mathbf{G}](s)$.

Lemma 3 *Suppose $\mathbf{W} \in \mathcal{H}_\infty$ has simple poles at $\{\gamma_1, \ldots, \gamma_p\}$ and $\mathbf{G} \in \mathcal{H}_2(W)$. Suppose further that $\mathbf{W}(s)$ has a realization as given in (7) and $\mathbf{G}(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}$ from (1).*

Then $\mathfrak{F}[\mathbf{G}](s)$ as defined in (9) has a realization given by

$$\begin{aligned} \mathfrak{F}[\mathbf{G}](s) &= \mathfrak{C}_{\mathfrak{F}}(s\mathbf{I} - \mathcal{A}_{\mathfrak{F}})^{-1}\mathcal{B}_{\mathfrak{F}} \qquad (10)\\ &= \underbrace{\begin{bmatrix} \mathbf{C} & \mathbf{D}\mathbf{C}_w \end{bmatrix}}_{\mathfrak{C}_{\mathfrak{F}}} \left(s\mathbf{I} - \underbrace{\begin{bmatrix} \mathbf{A} & \mathbf{B}\mathbf{C}_w \\ \mathbf{0} & \mathbf{A}_w \end{bmatrix}}_{\mathcal{A}_{\mathfrak{F}}} \right)^{-1} \underbrace{\begin{bmatrix} \mathbf{Z}\mathbf{C}_w^T + \mathbf{B}\mathbf{D}_w\mathbf{D}_w^T \\ \mathbf{P}_w\mathbf{C}_w^T + \mathbf{B}_w\mathbf{D}_w^T \end{bmatrix}}_{\mathcal{B}_{\mathfrak{F}}}, \end{aligned}$$

where $\mathbf{P}_w$ and $\mathbf{Z}$ solve, respectively,

$$\mathbf{A}_w\mathbf{P}_w + \mathbf{P}_w\mathbf{A}_w^T + \mathbf{B}_w\mathbf{B}_w^T = \mathbf{0} \quad \textit{and} \qquad (11)$$

$$\mathbf{A}\mathbf{Z} + \mathbf{Z}\mathbf{A}_w^T + \mathbf{B}(\mathbf{C}_w\mathbf{P}_w + \mathbf{D}_w\mathbf{B}_w^T) = \mathbf{0}. \qquad (12)$$

Proof We evaluate (10) in two parts. Note first that since $\mathbf{G} \in \mathcal{H}_2(W)$, $\mathbf{D}\mathbf{D}_w = \mathbf{0}$. We may directly compute a realization of $\mathbf{G}(s) \cdot \mathbf{W}(s)$:

$$\begin{aligned} &\begin{bmatrix} \mathbf{C} & \mathbf{D}\mathbf{C}_w \end{bmatrix} \begin{bmatrix} s\mathbf{I} - \mathbf{A} & -\mathbf{B}\mathbf{C}_w \\ \mathbf{0} & s\mathbf{I} - \mathbf{A}_w \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{B}\mathbf{D}_w \\ \mathbf{B}_w \end{bmatrix} \\ &\quad = \begin{bmatrix} \mathbf{C} & \mathbf{D}\mathbf{C}_w \end{bmatrix} \begin{bmatrix} (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}\mathbf{W}(s) \\ (s\mathbf{I} - \mathbf{A}_w)^{-1}\mathbf{B}_w \end{bmatrix} = \mathbf{G}(s)\mathbf{W}(s). \qquad (13) \end{aligned}$$

$\mathbf{A}_w$ has distinct eigenvalues by hypothesis; assume that its eigenvalue decomposition is given as $\mathbf{A}_w = \mathbf{U}\boldsymbol{\Gamma}\mathbf{U}^{-1}$. Postmultiply (11) with $\mathbf{U}^{-T}$:

$$\mathbf{A}_w\tilde{\mathbf{P}}_w + \tilde{\mathbf{P}}_w\boldsymbol{\Gamma} + \mathbf{B}_w\tilde{\mathbf{F}} = \mathbf{0},$$

where $\mathbf{P}_w\mathbf{U}^{-T} = \tilde{\mathbf{P}}_w = [\tilde{\mathbf{p}}_1,\, \tilde{\mathbf{p}}_2,\, \ldots,\, \tilde{\mathbf{p}}_{n_w}]$ and $\mathbf{B}_w^T\mathbf{U}^{-T} = \tilde{\mathbf{F}} = \left[\tilde{\mathbf{f}}_1,\, \tilde{\mathbf{f}}_2,\, \ldots,\, \tilde{\mathbf{f}}_{n_w}\right]$. In particular, for each column of $\tilde{\mathbf{P}}_w$, we have $\tilde{\mathbf{p}}_k = (-\gamma_k\mathbf{I} - \mathbf{A}_w)^{-1}\mathbf{B}_w\tilde{\mathbf{f}}_k$. Defining $\tilde{\mathbf{E}} = \mathbf{C}_w\mathbf{U} = [\tilde{\mathbf{e}}_1,\, \tilde{\mathbf{e}}_2,\, \ldots,\, \tilde{\mathbf{e}}_{n_w}]$, we have

$$\mathbf{P}_w\mathbf{C}_w^T = \mathbf{P}_w\mathbf{U}^{-T}\mathbf{U}^T\mathbf{C}_w^T = \tilde{\mathbf{P}}_w\tilde{\mathbf{E}}^T = \sum_{k=1}^{n_w}(-\gamma_k\mathbf{I} - \mathbf{A}_w)^{-1}\mathbf{B}_w\tilde{\mathbf{f}}_k\tilde{\mathbf{e}}_k^T.$$

We follow the same development for (12); postmultiplication with $\mathbf{U}^{-T}$ yields

$$\mathbf{A}\tilde{\mathbf{Z}} + \tilde{\mathbf{Z}}\boldsymbol{\Gamma} + \mathbf{B}(\mathbf{C}_w\tilde{\mathbf{P}}_w + \mathbf{D}_w\tilde{\mathbf{F}}) = \mathbf{0},$$

where $\tilde{\mathbf{Z}} = \mathbf{Z}\,\mathbf{U}^{-T} = [\tilde{\mathbf{z}}_1,\, \tilde{\mathbf{z}}_2,\, \ldots,\, \tilde{\mathbf{z}}_{n_w}]$. Note that

$$\mathbf{C}_w\tilde{\mathbf{p}}_k + \mathbf{D}_w\tilde{\mathbf{f}}_k = \mathbf{W}(-\gamma_k)\tilde{\mathbf{f}}_k$$

so that $\tilde{\mathbf{z}}_k = (-\gamma_k\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}\mathbf{W}(-\gamma_k)\tilde{\mathbf{f}}_k$, and drawing all together, we obtain

$$\mathbf{Z}\mathbf{C}_w^T = \mathbf{Z}\mathbf{U}^{-T}\mathbf{U}^T\mathbf{C}_w^T = \tilde{\mathbf{Z}}\tilde{\mathbf{E}}^T = \sum_{k=1}^{n_w}(-\gamma_k\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}\mathbf{W}(-\gamma_k)\tilde{\mathbf{f}}_k\tilde{\mathbf{e}}_k^T.$$

With these expressions, the remaining contribution to (10) becomes

$$\begin{aligned}
&\begin{bmatrix}\mathbf{C} & \mathbf{D}\mathbf{C}_w\end{bmatrix}\begin{bmatrix}s\mathbf{I} - \mathbf{A} & -\mathbf{B}\mathbf{C}_w \\ \mathbf{0} & s\mathbf{I} - \mathbf{A}_w\end{bmatrix}^{-1}\begin{bmatrix}\mathbf{Z}\mathbf{C}_w^T \\ \mathbf{P}_w\mathbf{C}_w^T\end{bmatrix} \\
&= \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{Z}\mathbf{C}_w^T + \mathbf{G}(s)\mathbf{C}_w(s\mathbf{I} - \mathbf{A}_w)^{-1}\mathbf{P}_w\mathbf{C}_w^T \\
&= \sum_{k=1}^{n_w}\mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}(-\gamma_k\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}\mathbf{W}(-\gamma_k)\tilde{\mathbf{f}}_k\tilde{\mathbf{e}}_k^T \\
&\quad + \sum_{k=1}^{n_w}\mathbf{G}(s)\mathbf{C}_w(s\mathbf{I} - \mathbf{A}_w)^{-1}(-\gamma_k\mathbf{I} - \mathbf{A}_w)^{-1}\mathbf{B}_w\tilde{\mathbf{f}}_k\tilde{\mathbf{e}}_k^T
\end{aligned}$$

The following easily verified resolvent identity allows further simplification:

$$(s\mathbf{I} - \mathbf{A})^{-1}\,(-\gamma_k\mathbf{I} - \mathbf{A})^{-1} = \frac{1}{s + \gamma_k}\,(-\gamma_k\mathbf{I} - \mathbf{A})^{-1} - \frac{1}{s + \gamma_k}\,(s\mathbf{I} - \mathbf{A})^{-1}. \tag{14}$$

Which then yields,

$$\begin{aligned}
\ldots &= \sum_{k=1}^{n_w}\frac{1}{s + \gamma_k}\left(\mathbf{G}(-\gamma_k) - \mathbf{G}(s)\right)\mathbf{W}(-\gamma_k)\tilde{\mathbf{f}}_k\tilde{\mathbf{e}}_k^T \\
&\qquad + \sum_{k=1}^{n_w}\frac{1}{s + \gamma_k}\mathbf{G}(s)\left(\mathbf{W}(-\gamma_k) - \mathbf{W}(s)\right)\tilde{\mathbf{f}}_k\tilde{\mathbf{e}}_k^T \\
&= \sum_{k=1}^{n_w}\mathbf{G}(-\gamma_k)\mathbf{W}(-\gamma_k)\frac{\tilde{\mathbf{f}}_k\tilde{\mathbf{e}}_k^T}{s + \gamma_k} - \mathbf{G}(s)\mathbf{W}(s)\sum_{k=1}^{n_w}\frac{\tilde{\mathbf{f}}_k\tilde{\mathbf{e}}_k^T}{s + \gamma_k}
\end{aligned}$$

Postmultiplying (13) with $\mathbf{D}_w^T$ and combining with this last expression gives

$$\begin{aligned}
&\begin{bmatrix}\mathbf{C} & \mathbf{D}\mathbf{C}_w\end{bmatrix}\begin{bmatrix}s\mathbf{I} - \mathbf{A} & -\mathbf{B}\mathbf{C}_w \\ \mathbf{0} & s\mathbf{I} - \mathbf{A}_w\end{bmatrix}^{-1}\begin{bmatrix}\mathbf{Z}\mathbf{C}_w^T + \mathbf{B}\mathbf{D}_w\mathbf{D}_w^T \\ \mathbf{P}_w\mathbf{C}_w^T + \mathbf{B}_w\mathbf{D}_w^T\end{bmatrix} \\
&\quad = \mathbf{G}(s)\mathbf{W}(s)\left(\sum_{k=1}^{n_w}\frac{\tilde{\mathbf{f}}_k\tilde{\mathbf{e}}_k^T}{-s - \gamma_k} + \mathbf{D}_w^T\right) + \sum_{k=1}^{n_w}\mathbf{G}(-\gamma_k)\mathbf{W}(-\gamma_k)\frac{\tilde{\mathbf{f}}_k\tilde{\mathbf{e}}_k^T}{s + \gamma_k} \\
&\quad = \mathfrak{F}[\mathbf{G}](s). \quad \square
\end{aligned}$$

Lemma 4 *Suppose* $\mathbf{M}_1$ *and* $\mathbf{M}_2$ *are stable matrices. The unique solution,* $\mathbb{X}$*, to the Sylvester equation*

$$\mathbf{M}_1\mathbb{X} + \mathbb{X}\mathbf{M}_2 + \mathbf{N} = \mathbf{0},$$

is given by

$$\mathbb{X} = \frac{1}{2\pi}\int_{-\infty}^{+\infty} (-\imath\omega\mathbf{I} - \mathbf{M}_1)^{-1}\mathbf{N}(\imath\omega\mathbf{I} - \mathbf{M}_2)^{-1}\,\mathrm{d}\omega$$

Lemma 5 *For* $\mathfrak{F}$ *as defined in (9) and any* $\mathbf{G}$, $\mathbf{H} \in \mathcal{H}_2(W)$*,*

a. $\langle \mathfrak{F}[\mathbf{G}],\ \mathbf{D}_H\rangle_{\mathcal{H}_2} = \frac{1}{2}\,\langle \mathbf{G},\ \mathbf{D}_H\rangle_{\mathcal{H}_2(W)}$

b. $\langle \mathfrak{F}[\mathbf{G}],\ \mathbf{H}\rangle_{\mathcal{H}_2} = \langle \mathbf{G},\ \mathbf{H}\rangle_{\mathcal{H}_2(W)} - \frac{1}{2}\,\langle \mathbf{G},\ \mathbf{D}_H\rangle_{\mathcal{H}_2(W)}$.

Proof We may decompose $\mathbf{H}$ as $\mathbf{H}(s) = \mathbf{H}_0(s) + \mathbf{D}_H$ with $\mathbf{H}_0 \in \mathcal{H}_2$. Since $\mathbf{G}, \mathbf{H} \in \mathcal{H}_2(W)$, $\mathbf{D}_H \cdot \mathbf{D}_w = \mathbf{0}$ and $\mathbf{D} \cdot \mathbf{D}_w = \mathbf{0}$. Using the realization of $\mathbf{GW}$ in (13), we calculate

$$\begin{aligned}
\langle \mathbf{G},\ \mathbf{D}_H\rangle_{\mathcal{H}_2(W)} &= \langle \mathbf{GW},\ \mathbf{D}_H\mathbf{W}\rangle_{\mathcal{H}_2} \\
&= \frac{1}{2\pi}\int_{-\infty}^{+\infty} \mathsf{tr}\left(\mathbf{G}(-\imath\omega)\mathbf{W}(-\imath\omega)\ \mathbf{W}(\imath\omega)^T\mathbf{D}_H^T\right)\,\mathrm{d}\omega \\
&= \mathsf{tr}\left(\begin{bmatrix}\mathbf{C} & \mathbf{DC}_w\end{bmatrix}\mathbb{X}\mathbf{C}_w^T\mathbf{D}_H^T\right)
\end{aligned}$$

where

$$\mathbb{X} = \frac{1}{2\pi}\int_{-\infty}^{+\infty}(-\imath\omega\mathbf{I} - \boldsymbol{\mathcal{A}}_{\mathfrak{F}})^{-1}\begin{bmatrix}\mathbf{BD}_w\\ \mathbf{B}_w\end{bmatrix}\mathbf{B}_w^T(\imath\omega\mathbf{I} - \mathbf{A}_w^T)^{-1}\,\mathrm{d}\omega$$

From Lemma 4, this $\mathbb{X}$ is the unique solution to the Sylvester equation

$$\boldsymbol{\mathcal{A}}_{\mathfrak{F}}\mathbb{X} + \mathbb{X}\ \mathbf{A}_w^T + \begin{bmatrix}\mathbf{BD}_w\mathbf{B}_w^T\\ \mathbf{B}_w\mathbf{B}_w^T\end{bmatrix} = \mathbf{0}.$$

Recalling (11) and (12), $\mathbb{X}$ evidently may be expressed as $\mathbb{X} = \begin{bmatrix}\mathbf{Z}\\ \mathbf{P}_w\end{bmatrix}$. Thus, $\langle \mathbf{G},\ \mathbf{D}_H\rangle_{\mathcal{H}_2(W)} = \mathsf{tr}\left(\mathbf{CZC}_w^T\mathbf{D}_H^T + \mathbf{DC}_w\mathbf{P}_w\mathbf{C}_w^T\mathbf{D}_H^T\right)$.

Conversely, we may use (10), take account that $\mathbf{D}_w^T\mathbf{D}_H^T = \mathbf{0}$, and calculate:

$$\begin{aligned}
\langle \mathfrak{F}[\mathbf{G}],\ \mathbf{D}_H\rangle_{\mathcal{H}_2} &= \frac{1}{2\pi}\int_{-\infty}^{+\infty}\mathsf{tr}\left(\boldsymbol{\mathfrak{C}}_{\mathfrak{F}}(-\imath\omega\mathbf{I} - \boldsymbol{\mathcal{A}}_{\mathfrak{F}})^{-1}\boldsymbol{\mathfrak{B}}_{\mathfrak{F}}\mathbf{D}_H^T\right)\,\mathrm{d}\omega \\
&= \mathsf{tr}\left(\boldsymbol{\mathfrak{C}}_{\mathfrak{F}}\left(\frac{1}{2\pi}\int_{-\infty}^{+\infty}(-\imath\omega\mathbf{I} - \boldsymbol{\mathcal{A}}_{\mathfrak{F}})^{-1}\,\mathrm{d}\omega\right)\begin{bmatrix}\mathbf{ZC}_w^T\\ \mathbf{P}_w\mathbf{C}_w^T\end{bmatrix}\mathbf{D}_H^T\right),
\end{aligned}$$

where the integral limit is to be interpreted as a principal value. Because the matrix $\boldsymbol{\mathcal{A}}_{\mathfrak{F}}$ is stable, the integral reduces to $\pi\mathbf{I}$, so we have:

$$\langle \mathfrak{F}[\mathbf{G}],\ \mathbf{D}_H\rangle_{\mathcal{H}_2} = \frac{1}{2}\,\mathsf{tr}\left(\mathbf{CZC}_w^T\mathbf{D}_H^T + \mathbf{DC}_w\mathbf{P}_w\mathbf{C}_w^T\mathbf{D}_H^T\right) = \frac{1}{2}\,\langle \mathbf{G},\ \mathbf{D}_H\rangle_{\mathcal{H}_2(W)}$$

Part (b) is shown similarly. We omit the details. □

2.1 Interpolatory weighted-$\mathcal{H}_2$ optimality conditions

The feasible set for (5) consists of all stable transfer functions in $\mathcal{H}_{2(W)}$ having order n_r or less. This is a nonconvex set, hence as a practical matter, finding a global minimizer is extremely difficult and so, instead, one typically seeks efficient *local* minimizers. Methods proposed in [13] and [24] may be used to find local minimizers to (5). However, these methods require solving a sequence of large-scale Lyapunov or Riccati equations and so, rapidly become computationally intractable as system order, n, and shaping filter order, n_w, increase.

We approach (5) instead within an interpolatory framework similar to that developed in [2]. Computational complexity for interpolatory methods grows more slowly with increasing n and n_w, hence much larger problems are feasible. In contrast to the (SISO) results of [2], we are able to treat general MIMO settings (including feedthrough terms). Furthermore, the heuristic algorithm derived in [2] is improved upon here with an iterative correction process that produces near-optimal reduced models (approaching true optimality as reduction order n_r grows).

We first derive interpolatory conditions that necessarily hold for a reduced system, $\mathbf{G}_r$, assuming it solves (5).

Theorem 6 *Suppose that $\mathbf{G}_r \in \mathcal{H}_2(W)$ is a solution to (5). Suppose further that $\mathbf{G}_r$ has only simple poles, $\{\lambda_1, \ldots, \lambda_{n_r}\}$ and is represented as:*

$$\mathbf{G}_r(s) = \mathbf{C}_r\,(s\mathbf{I} - \mathbf{A}_r)^{-1}\,\mathbf{B}_r + \mathbf{D}_r = \sum_{k=1}^{n_r} \frac{\mathbf{c}_k\,\mathbf{b}_k^T}{s-\lambda_k} + \mathbf{D}_r \tag{15}$$

where $\mathbf{A}_r \in \mathbb{R}^{n_r\times n_r}$ and $\mathbf{B}_r \in \mathbb{R}^{n_r\times m}$, and $\mathbf{C}_r \in \mathbb{R}^{p\times n_r}$. Then $\mathbf{G}_r$ must satisfy for each $k = 1, \ldots, n_r$,

$$\mathfrak{F}[\mathbf{G}](-\lambda_k)\mathbf{b}_k = \mathfrak{F}[\mathbf{G}_r](-\lambda_k)\mathbf{b}_k \tag{16a}$$

$$\mathbf{c}_k^T\mathfrak{F}[\mathbf{G}](-\lambda_k) = \mathbf{c}_k^T\mathfrak{F}[\mathbf{G}_r](-\lambda_k),\ \ \textit{and} \tag{16b}$$

$$\mathbf{c}_k^T\mathfrak{F}'[\mathbf{G}](-\lambda_k)\mathbf{b}_k = \mathbf{c}_k^T\mathfrak{F}'[\mathbf{G}_r](-\lambda_k)\mathbf{b}_k. \tag{16c}$$

where $\mathfrak{F}$ is defined in (9) and $\mathfrak{F}'[\,\cdot\,](s) = \frac{d}{ds}\mathfrak{F}[\,\cdot\,](s)$.

(Theorem 7 will provide one additional condition.)

Proof Pick an arbitrary vector $\mathbf{g} \in \mathbb{C}^p$ with $\|\mathbf{g}\| = 1$ and an index k with $1 \le k \le n_r$. Suppose that

$$\left\langle \mathbf{G} - \mathbf{G}_r,\ \frac{\mathbf{g}\mathbf{b}_k^T}{s-\lambda_k} \right\rangle_{\mathcal{H}_{2(W)}} = \alpha_0 \neq 0.$$

Define $\theta_0 = \arg(\alpha_0)$ and for arbitrary $\varepsilon > 0$, define a perturbation to $\mathbf{G}_r$ as

$$\widetilde{\mathbf{G}}_r^{(\varepsilon)}(s) = \frac{\mathbf{c}_k + \varepsilon\, e^{-\imath\theta_0}\mathbf{g}}{s-\lambda_k}\mathbf{b}_k^T + \sum_{i\neq k}\frac{\mathbf{c}_i\mathbf{b}_i^T}{s-\lambda_i}.$$

Then, using (6) and Proposition 2, we obtain

$$\|\mathbf{G}_r - \widetilde{\mathbf{G}}_r^{(\varepsilon)}\|_{\mathcal{H}_2(W)} = \left\| \frac{-\varepsilon\, e^{-\imath\theta_0}}{s-\lambda_k}\mathbf{g}\mathbf{b}_k^T \right\|_{\mathcal{H}_2(W)} \leq \|\mathbf{W}\|_{\mathcal{H}_\infty} \frac{\|\mathbf{b}_k\|\varepsilon}{\sqrt{2|\mathsf{Re}(\lambda_k)|}}.$$

Thus, $\|\mathbf{G}_r(s) - \widetilde{\mathbf{G}}_r^{(\varepsilon)}(s)\|_{\mathcal{H}_2(W)} = \mathcal{O}(\varepsilon)$ as $\varepsilon \to 0$. Since $\mathbf{G}_r$ solves (5),

$$\begin{aligned}\|\mathbf{G} - \mathbf{G}_r\|^2_{\mathcal{H}_2(W)} &\leq \|\mathbf{G} - \widetilde{\mathbf{G}}_r^{(\varepsilon)}\|^2_{\mathcal{H}_2(W)} \leq \|(\mathbf{G} - \mathbf{G}_r) + (\mathbf{G}_r - \widetilde{\mathbf{G}}_r^{(\varepsilon)})\|^2_{\mathcal{H}_2(W)} \\ &\leq \|\mathbf{G} - \mathbf{G}_r\|^2_{\mathcal{H}_2(W)} + 2\mathsf{Re}\left\langle \mathbf{G} - \mathbf{G}_r,\, \mathbf{G}_r - \widetilde{\mathbf{G}}_r^{(\varepsilon)} \right\rangle_{\mathcal{H}_2(W)} + \|\mathbf{G}_r - \widetilde{\mathbf{G}}_r^{(\varepsilon)}\|^2_{\mathcal{H}_2(W)}.\end{aligned}$$

Thus,

$$0 \leq 2\ \mathsf{Re}\left\langle \mathbf{G} - \mathbf{G}_r,\, \mathbf{G}_r - \widetilde{\mathbf{G}}_r^{(\varepsilon)} \right\rangle_{\mathcal{H}_2(W)} + \|\mathbf{G}_r - \widetilde{\mathbf{G}}_r^{(\varepsilon)}\|^2_{\mathcal{H}_2(W)}.$$

This implies that $0 \leq -\varepsilon|\alpha_0| + \mathcal{O}(\varepsilon^2)$, which then leads to a contradiction; it must be that $\alpha_0 = 0$. But then

$$0 = \left\langle \mathbf{G} - \mathbf{G}_r,\, \frac{\mathbf{g}\mathbf{b}_k^T}{s-\lambda_k} \right\rangle_{\mathcal{H}_2(W)} = \left\langle \mathfrak{F}[\mathbf{G} - \mathbf{G}_r],\, \frac{\mathbf{g}\mathbf{b}_k^T}{s-\lambda_k} \right\rangle_{\mathcal{H}_2} = \mathbf{g}^T \left(\mathfrak{F}[\mathbf{G} - \mathbf{G}_r](-\lambda_k)\right)\mathbf{b}_k,$$

using Proposition 2) and since $\mathbf{g}$ was chosen arbitrarily, we must have

$$0 = \mathfrak{F}[\mathbf{G} - \mathbf{G}_r](-\lambda_k)\mathbf{b}_k = \mathfrak{F}[\mathbf{G}](-\lambda_k)\mathbf{b}_k - \mathfrak{F}[\mathbf{G}_r](-\lambda_k)\mathbf{b}_k$$

which confirms (16a). (16b) is shown similarly, replacing $\frac{\mathbf{g}\mathbf{b}_k^T}{s-\lambda_k}$ in the argument above with $\frac{\mathbf{c}_k\mathbf{g}^T}{s-\lambda_k}$ for arbitrary $\mathbf{g} \in \mathbb{C}^m$.

To show (16c), suppose that $\left\langle \mathbf{G} - \mathbf{G}_r,\, \frac{\mathbf{c}_k\mathbf{b}_k^T}{(s-\lambda_k)^2} \right\rangle_{\mathcal{H}_2(W)} = \alpha_1 \neq 0$. and define $\theta_1 = \arg(\alpha_1)$. For $\varepsilon > 0$ sufficiently small, define

$$\widetilde{\mathbf{G}}_r^{(\varepsilon)}(s) = \frac{\mathbf{c}_k\mathbf{b}_k^T}{s - (\lambda_k + \varepsilon\, e^{-\imath\theta_1})} + \sum_{i \neq k} \frac{\mathbf{c}_i\mathbf{b}_i^T}{s - \hat{\lambda}_i}$$

As $\varepsilon \to 0$, we have

$$\begin{aligned}\|\mathbf{G}_r - \widetilde{\mathbf{G}}_r^{(\varepsilon)}\|_{\mathcal{H}_2(W)} &= \left\| \frac{-\varepsilon\, e^{-\imath\vartheta_1}\mathbf{c}_k\mathbf{b}_k^T}{(s-\lambda_k)(s-(\lambda_k + \varepsilon\, e^{-\imath\theta_1}))} \right\|_{\mathcal{H}_2(W)} \\ &= \mathcal{O}(\varepsilon)\end{aligned}$$

Following a similar argument as before, we find that $0 \leq -\varepsilon|\alpha_1| + \mathcal{O}(\varepsilon^2)$ as $\varepsilon \to 0$, which leads to a contradiction, forcing $\alpha_1 = 0$. This, in turn, implies

$$\begin{aligned}0 = \left\langle \mathbf{G} - \mathbf{G}_r,\, \frac{\mathbf{c}_k\mathbf{b}_k^T}{(s-\lambda_k)^2} \right\rangle_{\mathcal{H}_2(W)} &= \left\langle \mathfrak{F}[\mathbf{G} - \mathbf{G}_r],\, \frac{\mathbf{c}_k\mathbf{b}_k^T}{(s-\lambda_k)^2} \right\rangle_{\mathcal{H}_2} \\ &= \frac{d}{ds}\, \mathbf{c}_k^T \left(\mathfrak{F}[\mathbf{G} - \mathbf{G}_r](s)\right)\mathbf{b}_k \Big|_{s=-\lambda_k},\end{aligned}$$

which gives (16c). $\square$

We have one additional necessary condition for optimality that arises from the presence of the weighting filter. For $\mathbf{G}$, $\mathbf{G}_r \in \mathcal{H}_2(W)$, let $\mathbf{F}(t)$ and $\mathbf{F}_r(t)$ denote the impulse response functions associated respectively with $\mathfrak{F}[\mathbf{G}](s)$ and $\mathfrak{F}[\mathbf{G}_r](s)$. That is, $\mathfrak{F}[\mathbf{G}] = \mathcal{L}\{\mathbf{F}\}$ and $\mathfrak{F}[\mathbf{G}_r] = \mathcal{L}\{\mathbf{F}_r\}$, where $\mathcal{L}\{\cdot\}$ is the Laplace transform.

Theorem 7 *Assume the hypotheses and notation of Theorem 6. Then for all* $\mathbf{n} \in \mathsf{Ker}(\mathbf{D}_w^T)$,

$$\mathbf{F}(0)\mathbf{n} = \mathbf{F}_r(0)\mathbf{n}. \tag{16d}$$

Proof Pick $\mathbf{m} \in \mathbb{R}^p$ and $\mathbf{n} \in \mathsf{Ker}(\mathbf{D}_w^T)$, arbitrarily. From (8), $\mathbf{m}\,\mathbf{n}^T\,\mathbf{W}(s) = \displaystyle\sum_{k=1}^{n_w} (\mathbf{n}^T\mathbf{e}_k)\,\frac{\mathbf{m}\,\mathbf{f}_k^T}{s-\gamma_k}$ is evidently an $\mathcal{H}_2$ function. Hence, $\mathbf{m}\,\mathbf{n}^T \in \mathcal{H}_{2(W)}$. Suppose that

$$\left\langle \mathbf{G}-\mathbf{G}_r,\ \mathbf{m}\,\mathbf{n}^T \right\rangle_{\mathcal{H}_2(W)} = \alpha_0 \neq 0.$$

Define $\theta_0 = \arg(\alpha_0)$ and for arbitrary $\varepsilon > 0$, define a perturbation to $\mathbf{G}_r$ as

$$\widetilde{\mathbf{G}}_r^{(\varepsilon)}(s) = \varepsilon\, e^{-\imath\theta_0}\,\mathbf{m}\,\mathbf{n}^T + \mathbf{G}_r(s)$$

Arguments identical to those in the proof of Theorem 6 lead to

$$0 \leq -2\ \mathsf{Re}\left\langle \mathbf{G}-\mathbf{G}_r,\, \varepsilon\,\mathbf{m}\,\mathbf{n}^T \right\rangle_{\mathcal{H}_2(W)} + \|\varepsilon\,\mathbf{m}\,\mathbf{n}^T\|^2_{\mathcal{H}_2(W)},$$

implying that $0 \leq -\varepsilon|\alpha_0| + \mathcal{O}(\varepsilon^2)$, and leading to a contradiction as before; as a consequence, $\alpha_0 = 0$. But then

$$0 = \left\langle \mathbf{G}-\mathbf{G}_r,\ \mathbf{m}\,\mathbf{n}^T \right\rangle_{\mathcal{H}_2(W)} = \left\langle \mathfrak{F}[\mathbf{G}-\mathbf{G}_r],\ \mathbf{m}\,\mathbf{n}^T \right\rangle_{\mathcal{H}_2} = \mathbf{m}^T\left[\int_{-\infty}^{+\infty} \mathfrak{F}[\mathbf{G}-\mathbf{G}_r](\imath\omega)\,\mathrm{d}\omega\right]\mathbf{n}.$$

Since $\mathbf{m}$ was chosen arbitrarily, we must have

$$\mathbf{0} = \left[\int_{-\infty}^{+\infty} \mathfrak{F}[\mathbf{G}-\mathbf{G}_r](\imath\omega)\,\mathrm{d}\omega\right]\mathbf{n} = [\mathbf{F}(0) - \mathbf{F}_r(0)]\,\mathbf{n}.$$

which confirms (16d). $\square$

3 The Halevi optimality conditions

Following [13, Appendix A], the first-order necessary conditions for a locally optimal reduced model $\mathbf{G}_r$ can be stated in terms of solutions to linear matrix equations. Consider the set of matrix equations defined by $\mathbf{G}, \mathbf{G}_r \in \mathcal{H}_2(W)$ and $\mathbf{W} \in \mathcal{H}_\infty$ as follows:

$$\mathcal{A}_{\mathfrak{F}}\mathbf{X} + \mathbf{X}\mathbf{A}_r^T + \mathcal{B}_{\mathfrak{F}}\mathbf{B}_r^T = \mathbf{0}, \tag{17a}$$

$$\mathbf{A}_r\mathbf{P}_r + \mathbf{P}_r\mathbf{A}_r^T + \mathbf{B}_r\begin{bmatrix}\mathbf{0} & \mathbf{C}_w\end{bmatrix}\mathbf{X} + \left(\mathbf{X}^T\begin{bmatrix}\mathbf{0}\\ \mathbf{C}_w^T\end{bmatrix} + \mathbf{B}_r\mathbf{D}_w\mathbf{D}_w^T\right)\mathbf{B}_r^T = \mathbf{0}, \tag{17b}$$

$$\mathbf{A}_r^T\mathbf{Q}_r + \mathbf{Q}_r\mathbf{A}_r + \mathbf{C}_r^T\mathbf{C}_r = \mathbf{0}, \tag{17c}$$

$$\mathcal{A}_{\mathfrak{F}}^T\mathbf{Y} + \mathbf{Y}\mathbf{A}_r = \begin{bmatrix} \mathbf{C}^T \\ ((\mathbf{D}-\mathbf{D}_r)\mathbf{C}_w)^T \end{bmatrix}\mathbf{C}_r - \begin{bmatrix} \mathbf{0} \\ \mathbf{C}_w^T \end{bmatrix}\mathbf{B}_r^T\mathbf{Q}_r. \tag{17d}$$

If $\mathbf{G}_r$ is locally $\mathcal{H}_2(W)$-optimal, then:

$$\mathbf{Y}^T\mathbf{X} + \mathbf{Q}_r\mathbf{P}_r = \mathbf{0}, \tag{18a}$$

$$\mathcal{C}_{\mathfrak{F}}\mathbf{X} - \mathbf{C}_r\mathbf{P}_r - \mathbf{D}_r\begin{bmatrix}\mathbf{0} & \mathbf{C}_w\end{bmatrix}\mathbf{X} = \mathbf{0}, \tag{18b}$$

$$\mathbf{Y}^T\mathcal{B}_{\mathfrak{F}} + \mathbf{Q}_r\left(\mathbf{B}_r\mathbf{D}_w\mathbf{D}_w^T + \mathbf{X}^T\begin{bmatrix}\mathbf{0} \\ \mathbf{C}_w^T\end{bmatrix}\right) = \mathbf{0}, \tag{18c}$$

$$\mathbf{C}_r\mathbf{X}^T\begin{bmatrix}\mathbf{0} \\ \mathbf{C}_w^T\end{bmatrix}\mathbf{N} - \mathbf{C}\mathbf{Z}\mathbf{C}_w^T\mathbf{N} = (\mathbf{D}-\mathbf{D}_r)\mathbf{C}_w\mathbf{P}_w\mathbf{C}_w^T\mathbf{N}, \tag{18d}$$

where $\mathbf{N} = [\mathbf{n}_1, \ldots, \mathbf{n}_\ell]$ is a basis for $\mathsf{Ker}(\mathbf{D}_w^T)$.

Notice that for $\mathbf{W}(s) = \mathbf{I}$, conditions (18a)-(18c) coincide with the Wilson optimality conditions from [29], while the final condition (18d) is satisfied vacuously since in this case, $\mathsf{Ker}(\mathbf{D}_w^T) = \{0\}$.

3.1 Equivalence of the optimality conditions

The close connection between Sylvester equations and tangential interpolation in the unweighted case has been established in [8]. The model reduction bases that enforce tangential interpolation can be obtained as solutions to special Sylvester equations. Moreover, in [11], the necessary $\mathcal{H}_2$ optimality conditions in the form of Sylvester equations from [29] have been shown to be equivalent to the interpolatory ones from [16, 11]. For the weighted case, there are two frameworks as well: the interpolatory conditions (16a)-(16d) we developed here and the linear matrix equations based conditions (18a)-(18d) of Halevi [13]. Since these are only necessary conditions, their equivalence is not obvious. Next, we formally establish this equivalency.

Theorem 8 *Let $\mathbf{G}, \mathbf{G}_r \in \mathcal{H}_2(W)$ and $\mathbf{W} \in \mathcal{H}_\infty$. Assume that $\mathbf{G}_r$ has simple poles at $\{\lambda_1, \ldots, \lambda_{n_r}\}$. Then optimality conditions* (16a)-(16d) *and* (18a)-(18d) *are equivalent.*

Proof Assume $\mathbf{G}_r$ satisfies (18a)-(18d) and that $\mathbf{A}_r = \mathbf{R}\boldsymbol{\Lambda}\mathbf{R}^{-1}$ is an eigenvalue decomposition of $\mathbf{A}_r$. Multiplying (17a) with $\mathbf{R}^{-T}$ from right gives

$$\mathcal{A}_{\mathfrak{F}}\tilde{\mathbf{X}} + \tilde{\mathbf{X}}\boldsymbol{\Lambda} + \mathcal{B}_{\mathfrak{F}}\tilde{\mathbf{B}} = \mathbf{0},$$

where $\tilde{\mathbf{X}} = \mathbf{X}\mathbf{R}^{-T}$ and $\tilde{\mathbf{B}} = \mathbf{B}_r^T\mathbf{R}^{-T}$. This implies

$$\tilde{\mathbf{X}}\,\mathbf{s}_k = \tilde{\mathbf{X}}_k = (-\lambda_k\mathbf{I} - \mathcal{A}_{\mathfrak{F}})^{-1}\mathcal{B}_{\mathfrak{F}}\mathbf{b}_k, \tag{19}$$

where $\mathbf{s}_k$ is the k^{th} unit vector. Similarly, multiplying (17b) from right with $\mathbf{R}^{-T}$ yields

$$\mathbf{A}_r\tilde{\mathbf{P}} + \tilde{\mathbf{P}}\boldsymbol{\Lambda} + \mathbf{B}_r\begin{bmatrix}\mathbf{0} & \mathbf{C}_w\end{bmatrix}\tilde{\mathbf{X}} = -\left(\mathbf{X}^T\begin{bmatrix}\mathbf{0} \\ \mathbf{C}_w^T\end{bmatrix} + \mathbf{B}_r\mathbf{D}_w\mathbf{D}_w^T\right)\tilde{\mathbf{B}},$$

where $\tilde{\mathbf{P}} = \mathbf{P}_r\mathbf{R}^{-T}$. Since for $\mathbf{X} = \begin{bmatrix}\mathbf{X}_1\\ \mathbf{X}_2\end{bmatrix}$ we can conclude that $\mathbf{X}_2 = \mathbf{Z}_r^T$, where $\mathbf{Z}_r$ satisfies

$$\mathbf{A}_r\mathbf{Z}_r + \mathbf{Z}_r\mathbf{A}_w^T + \mathbf{B}_r(\mathbf{C}_w\mathbf{P}_w + \mathbf{D}_w\mathbf{B}_w^T) = \mathbf{0}. \tag{20}$$

It also follows

$$\begin{aligned}\tilde{\mathbf{P}}\,\mathbf{s}_k = \tilde{\mathbf{P}}_k &= (-\lambda_k\mathbf{I} - \mathbf{A}_r)^{-1}(\mathbf{Z}_r\mathbf{C}_w^T + \mathbf{B}_r\mathbf{D}_w\mathbf{D}_w^T)\mathbf{b}_k\\ &\quad + (-\lambda_k\mathbf{I} - \mathbf{A}_r)^{-1}\mathbf{B}_r\mathbf{C}_w(-\lambda_k\mathbf{I} - \mathbf{A}_w)^{-1}(\mathbf{P}_w\mathbf{C}_w^T + \mathbf{B}_w\mathbf{D}_w^T)\mathbf{b}_k. \end{aligned}\tag{21}$$

Right multiplication of (18b) with $\mathbf{R}^{-T}$, gives

$$\mathcal{C}_{\mathfrak{F}}\tilde{\mathbf{X}} - \mathbf{C}_r\tilde{\mathbf{P}} - \mathbf{D}_r\begin{bmatrix}\mathbf{0} & \mathbf{C}_w\end{bmatrix}\tilde{\mathbf{X}} = \mathbf{0}.$$

Hence, due to Lemma 3, each column is equivalent to (16a). Now postmultiply (17c) with $\mathbf{R}$ to obtain

$$\mathbf{A}_r^T\tilde{\mathbf{Q}} + \tilde{\mathbf{Q}}\boldsymbol{\Lambda} + \mathbf{C}_r^T\tilde{\mathbf{C}} = \mathbf{0},$$

where $\tilde{\mathbf{Q}} = \mathbf{Q}_r\mathbf{R}$ and $\tilde{\mathbf{C}} = \mathbf{C}_r\mathbf{R}$. Hence, it follows

$$\tilde{\mathbf{Q}}\,\mathbf{s}_k\tilde{\mathbf{Q}}_k = (-\lambda_k\mathbf{I} - \mathbf{A}_r^T)^{-1}\mathbf{C}_r^T\mathbf{c}_k. \tag{22}$$

Also, postmultiplication of (17d) with $\mathbf{R}$ leads to

$$\mathcal{A}_{\mathfrak{F}}^T\tilde{\mathbf{Y}} + \tilde{\mathbf{Y}}\boldsymbol{\Lambda} = \begin{bmatrix}\mathbf{C}^T\\ ((\mathbf{D} - \mathbf{D}_r)\mathbf{C}_w)^T\end{bmatrix}\tilde{\mathbf{C}} - \begin{bmatrix}\mathbf{0}\\ \mathbf{C}_w^T\end{bmatrix}\mathbf{B}_r^T\tilde{\mathbf{Q}}$$

where $\tilde{\mathbf{Y}} = \mathbf{Y}\mathbf{R}$. In particular, we get

$$\tilde{\mathbf{Y}}\,\mathbf{s}_k = \tilde{\mathbf{Y}}_k = (-\lambda_k\mathbf{I} - \mathcal{A}_{\mathfrak{F}})^{-T}\left(\begin{bmatrix}\mathbf{0}\\ \mathbf{C}_w^T\end{bmatrix}\mathbf{B}_r^T(-\lambda_k\mathbf{I} - \mathbf{A}_r^T)^{-1}\mathbf{C}_r^T + \mathbf{D}_r^T - \mathcal{C}_{\mathfrak{F}}^T\right)\mathbf{c}_k. \tag{23}$$

We further have $\tilde{\mathbf{Y}}^T\mathcal{B}_{\mathfrak{F}} + \tilde{\mathbf{Q}}\left(\mathbf{B}_r\mathbf{D}_w\mathbf{D}_w^T + \mathbf{Z}_r\mathbf{C}_w^T\right) = \mathbf{0}$ due to (18c). Together with (22) and (23), for each row it thus holds

$$\begin{aligned}\mathbf{0} &= -\mathbf{c}_k^T\mathcal{C}_{\mathfrak{F}}(-\lambda_k\mathbf{I} - \mathcal{A}_{\mathfrak{F}})^{-1}\mathcal{B}_{\mathfrak{F}}\\ &\quad + \mathbf{c}_k^T(\mathbf{C}_r(-\lambda_k\mathbf{I} - \mathbf{A}_r)^{-1}\mathbf{B}_r + \mathbf{D}_r)\mathbf{C}_w(-\lambda_k\mathbf{I} - \mathbf{A}_w)^{-1}(\mathbf{B}_w\mathbf{D}_w^T + \mathbf{P}_w\mathbf{C}_w^T)\\ &\quad + \mathbf{c}_k^T\mathbf{C}_r(-\lambda_k\mathbf{I} - \mathbf{A}_r)^{-1}\left(\mathbf{B}_r\mathbf{D}_w\mathbf{D}_w^T + \mathbf{Z}_r\mathbf{C}_w^T\right).\end{aligned}$$

Again, using Lemma 3, this leads to (16b). Finally, pre- and postmultiplication of (18a) with $\mathbf{R}^T$ and $\mathbf{R}^{-T}$ yields

$$\tilde{\mathbf{Y}}^T\tilde{\mathbf{X}} + \tilde{\mathbf{Q}}\tilde{\mathbf{P}} = \mathbf{0}. \tag{24}$$

Using (19) - (23) for the diagonal of (24), we find

$$\begin{aligned}\mathbf{0} &= -\mathbf{c}_k^T\mathcal{C}_{\mathfrak{F}}(-\lambda_k\mathbf{I} - \mathcal{A}_{\mathfrak{F}})^{-2}\mathcal{B}_{\mathfrak{F}}\mathbf{b}_k\\ &\quad + \mathbf{c}_k^T(\mathbf{C}_r(-\lambda_k\mathbf{I} - \mathbf{A}_r)^{-1}\mathbf{B}_r + \mathbf{D}_r)\mathbf{C}_w(-\lambda_k\mathbf{I} - \mathbf{A}_w)^{-2}(\mathbf{B}_w\mathbf{D}_w^T + \mathbf{P}_w\mathbf{C}_w^T)\mathbf{b}_k\\ &\quad + \mathbf{c}_k^T\mathbf{C}_r(-\lambda_k\mathbf{I} - \mathbf{A}_r)^{-2}\left(\mathbf{Z}_r\mathbf{C}_w^T + \mathbf{B}_r\mathbf{D}_w\mathbf{D}_w^T\right)\\ &\quad + \mathbf{c}_k^T\mathbf{C}_r(-\lambda_k\mathbf{I} - \mathbf{A}_r)^{-2}\mathbf{B}_r\mathbf{C}_w(-\lambda_k\mathbf{I} - \mathbf{A}_w)^{-1}(\mathbf{B}_w\mathbf{D}_w^T + \mathbf{P}_w\mathbf{C}_w^T)\mathbf{b}_k.\end{aligned}$$

Then, due to Lemma 3, this implies (16c). Finally, due to (18d) we note that

$$\begin{bmatrix}\mathbf{C}_r & \mathbf{D}_r\mathbf{C}_w\end{bmatrix}\begin{bmatrix}\mathbf{Z}_r\mathbf{C}_w^T\\ \mathbf{P}_w\mathbf{C}_w^T\end{bmatrix}\mathbf{N} = \begin{bmatrix}\mathbf{C} & \mathbf{D}\mathbf{C}_w\end{bmatrix}\begin{bmatrix}\mathbf{Z}\mathbf{C}_w^T\\ \mathbf{P}_w\mathbf{C}_w^T\end{bmatrix}\mathbf{N}.$$

From [11], $\int_{-\infty}^{\infty}(i\omega\mathbf{I}-\mathbf{M})^{-1}\,\mathrm{d}\omega = \pi\mathbf{I}$, for any stable matrix $\mathbf{M}$, and we conclude that

$$\begin{aligned}&\frac{1}{\pi}\int_{-\infty}^{\infty}\begin{bmatrix}\mathbf{C}_r & \mathbf{D}_r\mathbf{C}_w\end{bmatrix}\begin{bmatrix}i\omega\mathbf{I}-\mathbf{A}_r & -\mathbf{B}_r\mathbf{C}_w\\ \mathbf{0} & i\omega\mathbf{I}-\mathbf{A}_w\end{bmatrix}^{-1}\begin{bmatrix}\mathbf{Z}_r\mathbf{C}_w^T\\ \mathbf{P}_w\mathbf{C}_w^T\end{bmatrix}\mathbf{N}\,\mathrm{d}\omega\\ &= \frac{1}{\pi}\int_{-\infty}^{\infty}\begin{bmatrix}\mathbf{C} & \mathbf{D}\mathbf{C}_w\end{bmatrix}\begin{bmatrix}i\omega\mathbf{I}-\mathbf{A} & -\mathbf{B}\mathbf{C}_w\\ \mathbf{0} & i\omega\mathbf{I}-\mathbf{A}_w\end{bmatrix}^{-1}\begin{bmatrix}\mathbf{Z}\mathbf{C}_w^T\\ \mathbf{P}_w\mathbf{C}_w^T\end{bmatrix}\mathbf{N}\,\mathrm{d}\omega\end{aligned}$$

Hence, for all $\mathbf{n}\in\mathsf{Ker}(\mathbf{D}_w^T)$,

$$\left[\int_{-\infty}^{\infty}\mathfrak{F}[\mathbf{G}_r](i\omega)\,\mathrm{d}\omega\right]\mathbf{n} = \left[\int_{-\infty}^{\infty}\mathfrak{F}[\mathbf{G}](i\omega)\,\mathrm{d}\omega\right]\mathbf{n},$$

which is equivalent to (16d). Reversing the arguments and using (14) for the offdiagonal entries of (18a) shows that (16a)-(16d) also imply (18a)-(18d). $\square$

4 Frequency-weighted rational interpolation

We henceforth assume that the feedthrough term of the original system, $\mathbf{G}$, is zero: $\mathbf{D}=\mathbf{0}$. This is without loss of generality since the general case may be recovered by subtracting the original $\mathbf{D}$ from the reduced system feedthrough term: $\mathbf{D}_r \leftarrow \mathbf{D}_r - \mathbf{D}$. From the previous discussion, we have seen that frequency-weighted $\mathcal{H}_2$ optimal approximants are mapped to Hermite interpolants via the mapping $\mathfrak{F}$ introduced in (9). This presents the practical problem of how to construct reduced order systems, $\mathbf{G}_r$, such that $\mathfrak{F}[\mathbf{G}_r](s)$ interpolates $\mathfrak{F}[\mathbf{G}](s)$ at selected points in $\mathbb{C}$, say at $\{\sigma_1, \sigma_2, \ldots, \sigma_{n_r}\}$, in selected tangent directions $\{\mathbf{b}_1,\ldots,\mathbf{b}_{n_r}\}$ and $\{\mathbf{c}_1,\ldots,\mathbf{c}_{n_r}\}$. Using the realization developed in Lemma 3 and standard interpolation results, we construct reduction subspaces that force interpolation:

$$\mathsf{Ran}\begin{bmatrix}\mathbb{V}^{(a)}\\ \mathbb{V}^{(b)}\end{bmatrix} = \underset{i=1,\ldots,n_r}{\mathrm{span}}\left\{(\sigma_i\mathbf{I}-\boldsymbol{\mathcal{A}}_{\mathfrak{F}})^{-1}\boldsymbol{\mathcal{B}}_{\mathfrak{F}}\mathbf{b}_i\right\}. \tag{25}$$

and

$$\mathsf{Ran}\begin{bmatrix}\mathbb{W}^{(a)}\\ \mathbb{W}^{(b)}\end{bmatrix} = \underset{i=1,\ldots,n_r}{\mathrm{span}}\left\{(\sigma_i\mathbf{I}-\boldsymbol{\mathcal{A}}_{\mathfrak{F}}^T)^{-1}\boldsymbol{\mathcal{C}}_{\mathfrak{F}}^T\mathbf{c}_i\right\}. \tag{26}$$

Define $\mathbf{V}_r,\,\mathbf{W}_r\in\mathbb{C}^{n\times n_r}$ so that $\mathbf{W}_r^T\mathbf{V}_r=\mathbf{I}$ and

$$\mathsf{Ran}(\mathbf{V}_r)\supset\mathsf{Ran}\left\{\mathbb{V}^{(a)}\right\},\quad \mathsf{Ran}(\mathbf{W}_r)\supset\mathsf{Ran}\left\{\mathbb{W}^{(a)}\right\}. \tag{27}$$

The reduced feedthrough term is computed from (18d):

$$\mathbf{D}_r = \mathbf{C}\,(\mathbf{Z}-\mathbf{V}_r\mathbf{Z}_r)\,\mathbf{C}_w^T\mathbf{N}(\mathbf{N}^T\mathbf{C}_w\mathbf{P}_w\mathbf{C}_w^T\mathbf{N})^{-1}\mathbf{N}^T, \tag{28}$$

where $\mathbf{N}$ is a basis for $\mathsf{Ker}(\mathbf{D}_w^T)$.

Theorem 9 *Let* $\mathbf{A}_r = \mathbf{W}_r^T\mathbf{A}\mathbf{V}_r$*,* $\mathbf{B}_r = \mathbf{W}_r^T\mathbf{B}$*,* $\mathbf{C} = \mathbf{C}_r\mathbf{V}_r$*, with* $\mathbf{V}_r$ *and* $\mathbf{W}_r$ *constructed as in* (25)*,* (26)*, and* (27)*. Suppose* $\mathbf{D}_r$ *is determined by* (28)*. Then pick any interpolation point* $\sigma \in \{\sigma_1, \sigma_2, \ldots, \sigma_{n_r}\}$*, with associated tangent directions:* $\mathbf{b}$ *and* $\mathbf{c}$*. Provided* $\sigma \notin \{\Lambda(\mathbf{A}), \Lambda(\mathbf{A}_r)\}$*, we have*

$$\mathfrak{F}[\mathbf{G}](\sigma)\mathbf{b} - \mathfrak{F}[\mathbf{G}_r](\sigma)\mathbf{b} = \mathbf{H}_1(\sigma)\left(\mathbf{Z} - \mathbf{V}_r\mathbf{Z}_r\right)\mathbf{C}_w^T\mathbf{b} - \mathbf{C}\left(\mathbf{Z} - \mathbf{V}_r\mathbf{Z}_r\right)\mathbf{H}_2(\sigma)\mathbf{b}$$

$$\mathbf{c}^T\mathfrak{F}[\mathbf{G}](\sigma) - \mathbf{c}^T\mathfrak{F}[\mathbf{G}_r](\sigma) = \mathbf{c}^T\mathbf{H}_1(\sigma)\left(\mathbf{Z} - \mathbf{V}_r\mathbf{Z}_r\right)\mathbf{C}_w^T - \mathbf{c}^T\mathbf{C}\left(\mathbf{Z} - \mathbf{V}_r\mathbf{Z}_r\right)\mathbf{H}_2(\sigma),$$

$$\mathbf{c}^T\mathfrak{F}'[\mathbf{G}](\sigma)\mathbf{b} - \mathbf{c}^T\mathfrak{F}'[\mathbf{G}_r](\sigma)\mathbf{b} = \mathbf{c}^T\mathbf{H}_1'(\sigma)\left(\mathbf{Z} - \mathbf{V}_r\mathbf{Z}_r\right)\mathbf{C}_w^T\mathbf{b} - \mathbf{c}^T\mathbf{C}\left(\mathbf{Z} - \mathbf{V}_r\mathbf{Z}_r\right)\mathbf{H}_2'(\sigma)\mathbf{b},$$

and $$\mathbf{F}(0)\mathbf{n} = \mathbf{F}_r(0)\mathbf{n},$$

where $\mathbf{F}(t)$ *and* $\mathbf{F}_r(t)$ *are the impulse responses of* $\mathfrak{F}[\mathbf{G}]$ *and* $\mathfrak{F}[\mathbf{G}_r]$*, respectively,* $\mathbf{n} \in \mathsf{Ker}(\mathbf{D}_w^T)$ *is arbitrary, and*

$$\mathbf{H}_1(s) = \mathbf{C}_r(s\mathbf{I} - \mathbf{A}_r)^{-1}\mathbf{W}_r^T$$

$$\mathbf{H}_2(s) = \mathbf{C}_w^T\mathbf{N}(\mathbf{N}^T\mathbf{C}_w\mathbf{P}_w\mathbf{C}_w^T\mathbf{N})^{-1}\mathbf{N}^T\mathbf{C}_w(s\mathbf{I} - \mathbf{A}_w)^{-1}(\mathbf{P}_w\mathbf{C}_w^T + \mathbf{B}_w\mathbf{D}_w^T)$$

Proof We follow a pattern of proof given in [3]. Define $\mathbb{V} = \begin{bmatrix} \mathbf{V}_r & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix}$, $\mathbb{W} = \begin{bmatrix} \mathbf{W}_r & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix}$, and $\boldsymbol{\mathcal{A}}_{\mathfrak{Fr}} = \begin{bmatrix} \mathbf{A}_r & \mathbf{B}_r\mathbf{C}_w \\ \mathbf{0} & \mathbf{A}_w \end{bmatrix}$. Further define two (skew) projectors via

$$\mathcal{P}_r(s) = \mathbb{V}(s\mathbf{I} - \boldsymbol{\mathcal{A}}_{\mathfrak{Fr}})^{-1}\mathbb{W}^T(s\mathbf{I} - \boldsymbol{\mathcal{A}}_{\mathfrak{F}})$$

$$\mathcal{Q}_r(s) = (s\mathbf{I} - \boldsymbol{\mathcal{A}}_{\mathfrak{F}})\mathcal{P}_r(s)(s\mathbf{I} - \boldsymbol{\mathcal{A}}_{\mathfrak{F}})^{-1} = (s\mathbf{I} - \boldsymbol{\mathcal{A}}_{\mathfrak{F}})\mathbb{V}(s\mathbf{I} - \boldsymbol{\mathcal{A}}_{\mathfrak{Fr}})^{-1}\mathbb{W}^T.$$

For all s in a neighborhood of σ, we have $\mathcal{V} = \mathsf{Ran}(\mathcal{P}_r(s)) = \mathsf{Ker}(\mathbf{I} - \mathcal{P}_r(s))$ and $\mathcal{W}^\perp = \mathsf{Ker}(\mathcal{Q}_r(s)) = \mathsf{Ran}(\mathbf{I} - \mathcal{Q}_r(s))$. Now observe that

$$\begin{aligned}
\mathfrak{F}[\mathbf{G}_r](s) = {} & \begin{bmatrix} \mathbf{C}_r & \mathbf{0} \end{bmatrix} \begin{bmatrix} s\mathbf{I} - \mathbf{A}_r & -\mathbf{B}_r\mathbf{C}_w \\ \mathbf{0} & s\mathbf{I} - \mathbf{A}_w \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{W}_r^T\mathbf{Z}\mathbf{C}_w^T + \mathbf{B}_r\mathbf{D}_w\mathbf{D}_w^T \\ \mathbf{P}_w\mathbf{C}_w^T + \mathbf{B}_w\mathbf{D}_w^T \end{bmatrix} \\
& - \begin{bmatrix} \mathbf{C}_r & \mathbf{0} \end{bmatrix} \begin{bmatrix} s\mathbf{I} - \mathbf{A}_r & -\mathbf{B}_r\mathbf{C}_w \\ \mathbf{0} & s\mathbf{I} - \mathbf{A}_w \end{bmatrix}^{-1} \begin{bmatrix} (\mathbf{W}_r^T\mathbf{Z} - \mathbf{Z}_r)\mathbf{C}_w^T \\ \mathbf{0} \end{bmatrix} \\
& + \mathbf{D}_r\mathbf{C}_w(s\mathbf{I} - \mathbf{A}_w)^{-1}(\mathbf{P}_w\mathbf{C}_w^T + \mathbf{B}_w\mathbf{D}_w^T).
\end{aligned}$$

Hence, we can write

$$\begin{aligned}
\mathfrak{F}[\mathbf{G}](s) - \mathfrak{F}[\mathbf{G}_r](s) = {} & \mathbf{H}_1(s)\left(\mathbf{Z} - \mathbf{V}_r\mathbf{Z}_r\right)\mathbf{C}_w^T - \mathbf{C}\left(\mathbf{Z} - \mathbf{V}_r\mathbf{Z}_r\right)\mathbf{H}_2(s) \\
& + \boldsymbol{\mathcal{C}}_{\mathfrak{F}}(s\mathbf{I} - \boldsymbol{\mathcal{A}}_{\mathfrak{F}})^{-1}(\mathbf{I} - \mathcal{Q}_r(s))(s\mathbf{I} - \boldsymbol{\mathcal{A}}_{\mathfrak{F}})(\mathbf{I} - \mathcal{P}_r(s))(s\mathbf{I} - \boldsymbol{\mathcal{A}}_{\mathfrak{F}})^{-1}\boldsymbol{\mathcal{B}}_{\mathfrak{F}}
\end{aligned} \tag{30}$$

Evaluating this expression at $s = \sigma$ and postmultiplying by $\mathbf{b}$ yields the first assertion; premultiplying by $\mathbf{c}^T$ yields the second. We find that

$$((\sigma + \varepsilon)\mathbf{I} - \boldsymbol{\mathcal{A}}_{\mathfrak{F}})^{-1} = (\sigma\mathbf{I} - \boldsymbol{\mathcal{A}}_{\mathfrak{F}})^{-1} - \varepsilon(\sigma\mathbf{I} - \boldsymbol{\mathcal{A}}_{\mathfrak{F}})^{-2} + \mathcal{O}(\varepsilon^2).$$

Evaluating (30) at $s = \sigma + \varepsilon$, premultiplying by $\mathbf{c}^T$, and postmultiplying by $\mathbf{b}$ together with $\varepsilon \to 0$ yields the third statement. The last statement results from the proof of Theorem 8 and the fact that $\mathbf{N}$ is a basis of $\mathsf{Ker}(\mathbf{D}_w^T)$. Note also that we have $\mathbf{D}_r\mathbf{D}_w = \mathbf{0}$. $\square$

Conditions for exact interpolation are now evident:

Corollary 10 *Let $\mathbf{G}_r$ denote the reduced order model of Theorem 9. If $\mathbf{G}_r$ is stable and $\mathsf{Ran}(\mathbf{Z}) \subset \mathsf{Ran}(\mathbf{V}_r)$ then $\mathfrak{F}[\mathbf{G}_r]$ is an exact bitangential Hermite interpolant to $\mathfrak{F}[\mathbf{G}]$ at each interpolation point, $\{\sigma_1, \sigma_2, \ldots, \sigma_{n_r}\}$ in corresponding tangent directions, $\{\mathbf{b}_1, \ldots, \mathbf{b}_{n_r}\}$ and $\{\mathbf{c}_1, \ldots, \mathbf{c}_{n_r}\}$.*

Proof Note first that under the hypotheses, $\mathbf{V}_r\mathbf{W}_r^T\mathbf{Z} = \mathbf{Z}$, Now, premultiply (12) by $\mathbf{W}_r^T$ and subtract (20) to obtain

$$\mathbf{A}_r\mathbf{W}_r^T\left(\mathbf{Z} - \mathbf{V}_r\mathbf{Z}_r\right) + \mathbf{W}_r^T\left(\mathbf{Z} - \mathbf{V}_r\mathbf{Z}_r\right)\mathbf{A}_w^T = \mathbf{0}.$$

Since $\mathbf{A}_r$ and $\mathbf{A}_w$ are both stable,

$$\mathbf{W}_r^T\left(\mathbf{Z} - \mathbf{V}_r\mathbf{Z}_r\right) = \mathbf{W}_r^T\mathbf{Z} - \mathbf{Z}_r = \mathbf{0}$$

and so, $\mathbf{Z} = \mathbf{V}_r\mathbf{Z}_r$.

The deviation from exact interpolation is quantified in Theorem 9 and depends on the deviation of $\mathbf{V}_r\mathbf{Z}_r$ from $\mathbf{Z}$. For shaping filters of modest order with $n_w \ll n$, exact interpolation can be induced since one may include $\mathsf{Ran}(\mathbf{Z})$ in the projection space, $\mathsf{Ran}(\mathbf{V}_r)$.

More generally, $\mathbf{V}_r\mathbf{Z}_r$ may be viewed as a Petrov-Galerkin approximation to the solution $\mathbf{Z}$ of the Sylvester equation (12) in the following sense: $\mathbf{Z}_r$ that solves (20) is a solution to the problem of finding $\mathfrak{Z} \in \mathbb{R}^{n_r \times n_w}$ such that with respect to the usual (Euclidean) inner product in $\mathbb{R}^n$,

$$\mathsf{Ran}\left(\mathbf{A}\left(\mathbf{V}_r\mathfrak{Z}\right) + \left(\mathbf{V}_r\mathfrak{Z}\right)\mathbf{A}_w^T + \mathbf{B}(\mathbf{C}_w\mathbf{P}_w + \mathbf{D}_w\mathbf{B}_w^T)\right) \perp \mathsf{Ran}\left(\mathbf{W}_r\right).$$

Since $m, m_w \ll n$, the singular values of the original solution, $\mathbf{Z}$, to (12) will typically decay rapidly [9, 18, 22, 23]; there will be good low rank approximations to $\mathbf{Z}$ and among them will be approximations of the form $\mathbf{V}_r\mathfrak{Z}$. Overall, this leads to the expectation that as n_r increases, $\mathbf{V}_r\mathbf{Z}_r \approx \mathbf{Z}$. If furthermore, the interpolation points that determine a reduced model coincide with the reflected poles of the model, then Theorem 9 asserts that the optimality conditions (16a)-(16d) will very nearly be satisfied; the reduced model draws closer to $\mathcal{H}_{2(W)}$-optimality as n_r increases.

The practical difficulty in constructing such *near optimal* reduced models is that one doesn't know *a priori* how to choose interpolation data determining a reduced model so as to coincide with the reflected poles of the model. The parallel circumstance for (unweighted) optimal $\mathcal{H}_2$ model reduction has been largely resolved with an iterative correction process [11]; we propose an analogous approach here: -.6cm

Algorithm NOWI:
Nearly Optimal Weighted Interpolation

Input: Interpolation points: $\{\sigma_1, \ldots, \sigma_{n_r}\}$;
Tangent directions: $\tilde{\mathbf{B}} = [\mathbf{b}_1, \ldots, \mathbf{b}_{n_r}]$ and $\tilde{\mathbf{C}} = [\mathbf{c}_1, \ldots, \mathbf{c}_{n_r}]$.
Output: $\mathbf{A}_r$, $\mathbf{B}_r$, $\mathbf{C}_r$, $\mathbf{D}_r$
1: **while** relative change in $\{\sigma_i\} >$ tol **do**
2: Compute $\mathbf{V}_r$ and $\mathbf{W}_r$ from (25), (26), and (27).
3: Update ROM: $\mathbf{A}_r = \mathbf{W}_r^T\mathbf{A}\mathbf{V}_r, \mathbf{B}_r = \mathbf{W}_r^T\mathbf{B}$, $\mathbf{C}_r = \mathbf{C}\mathbf{V}_r$, and $\mathbf{D}_r$ as in (28).
4: $\sigma_i = -\lambda_i(\mathbf{\Lambda}), \mathbf{A}_r = \mathbf{R}\mathbf{\Lambda}\mathbf{R}^{-1}, \tilde{\mathbf{B}} = \mathbf{B}_r^T\mathbf{R}^{-T}$, and $\tilde{\mathbf{C}} = \mathbf{C}_r\mathbf{R}$.
5: **end while**

Computational complexity: Many issues enter in determining the computational resources necessary to produce an effective reduced order model. Estimates of computational complexity serve as a useful proxy for this expense, which may be then further refined according to problem-specific structure and implementation. Notice first that our NOWI Algorithm is an iterative process, requiring in each cycle the construction of left- and right- reduction subspaces. This requires first the solution of two linear matrix equations, (11) and (12) of orders $n_w \times n_w$ and $n \times n_w$, respectively. If $n_w \ll n$, this may be done directly with cost dominated by n_w linear solves of dimension n. For larger n_w, the numerical rank of $\mathbf{P}_w$ and $\mathbf{Z}$ is often relatively small allowing for very accurate approximations by low rank methods such as [17, 12, 19, 5, 14, 21]. Bases for the left- and right- reduction subspaces then may be computed exploiting the block triangular structure of the $\mathfrak{F}$-realization; this leads to $2n_r$ linear solves of dimension n and n_r linear solves of dimension n_w. Sparsity in $\mathbf{A}$ and $\mathbf{A}_w$ may be exploited with either direct or iterative linear solvers. Multiple right-hand sides and small changes among shifts offer further opportunities for efficiency from subspace and preconditioner recycling.

When compared to standard approaches for frequency-weighted balanced truncation (FWBT), we find that as long as the number of iterations of NOWI remains modest (which appears typical), the overhead associated with solving two large Lyapunov equations of dimension n, which is necessary for FWBT, has been eliminated. This creates a particularly dramatic advantage for NOWI in the case of a shaping filter where $n_w \ll n$. The computational advantages of NOWI are also significant when compared to Halevi's approach to weighted-$\mathcal{H}_2$ model reduction [13], which requires solving large-scale Riccati and Lyapunov equations of order $(n + n_w) \times (n + n_w)$ at every step of the iteration.

5 Numerical examples

We study the performance of our NOWI Algorithm for three different examples resulting from controller reduction. We compare the proposed method with frequency weighted balanced truncation (FWBT) of [7], and also with WIRKA of [2] for the SISO example.

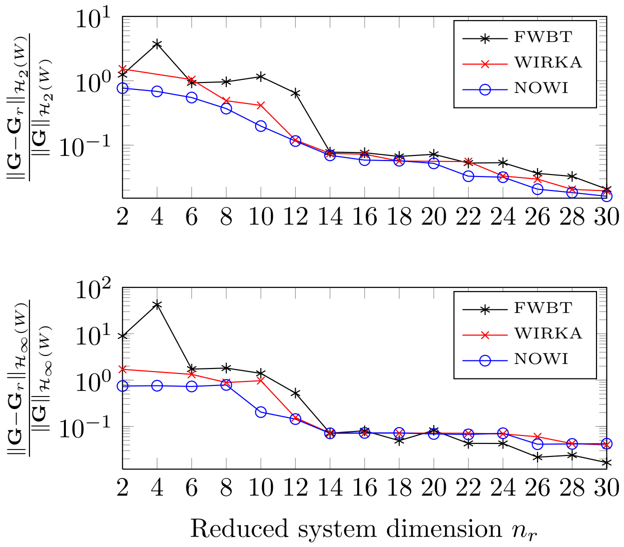

Figure 1: LA university hospital, $n = 48, n_w = 96$.

Los Angeles University Hospital: The plant is a linearized model for the Los Angeles University Hospital with order $n = 48$. An LQG-based controller of the same order as the original system is to be reduced, leading to a weighting $W(s)$ of order $n_w = 96$, see [2]. For a given n_r, we use the mirror images of the $\nu = 2$ most dominant poles of $W(s)$ and the mirror images $n_r - \nu$ most dominant poles of $G(s)$ as the initial interpolation points for WIRKA, as suggested in [2]. We use the same initialization for the NOWI Algorithm. Figure 1 shows the relative $\mathcal{H}_2(W)$- and $\mathcal{H}_\infty(W)$-errors obtained from NOWI, FWBT, and WIRKA for reduced system orders $n_r = 2, \ldots, 30$. For the $\mathcal{H}_2(W)$-case, NOWI outperforms FWBT and WIRKA for all n_r values except for $n_r = 18$, for which WIRKA is slightly better. The superiority of NOWI is especially evident for smaller n_r values. We find similar results for the $\mathcal{H}_\infty(W)$-error as well; FWBT yields the smallest $\mathcal{H}_\infty(W)$-errors for larger n_r, as expected. The fact that NOWI displays better $\mathcal{H}_\infty(W)$ performance than FWBT even for a subset of reduction orders suggests the effectiveness of the approach. NOWI produces reduced models that satisfy the $\mathcal{H}_2(W)$-optimality interpolation conditions (16) only approximately (see Theorem 9). Figure 2 shows how the relative interpolation error (deviation from (16)) in final reduced models produced by NOWI evolves with increasing n_r. As the figure shows, the relative error in the optimality conditions decreases as n_r increases. This confirms the expectations described in the discussion following Corollary 10. Figure 3 shows how the relative interpolation error in the the optimality conditions (16) evolve (for fixed reduction order, n_r) step to step in the NOWI Algorithm. Results for two cases are displayed: $n_r = 16$ and $n_r = 30$. In both cases, we observe that NOWI rapidly reduces interpolation error during the iteration. For example, for $n_r = 16$, relative

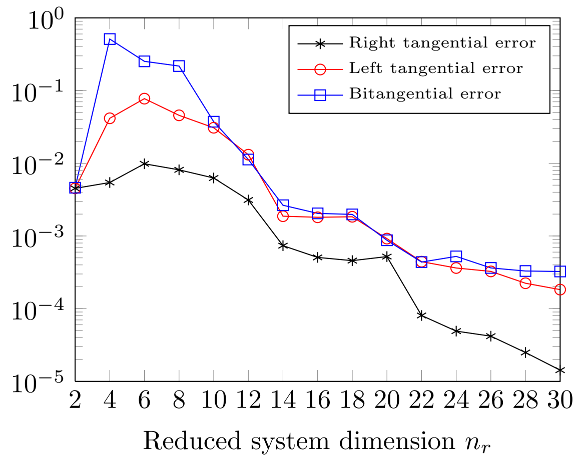

Figure 2: LA university hospital, $n = 48, n_w = 96$.

interpolation errors are in the order of 1 initially; however as the algorithm progresses, relative errors decline to levels of 10^{-3}, leading to near-optimal interpolation.

CD player: The plant is a model for a CD player and belongs to the SLICOT benchmark collection. We consider the original MIMO version with $n = 120$ and $m = p = 2$. As in the previous example, we design an LQG-based controller having the same order as the plant, leading to a weight $\mathbf{W}(s)$ with $n_w = 240$. Since WIRKA has been proposed only for SISO systems and a MIMO extension is not immediate, we show comparisons only between FWBT and NOWI, using a random initialization. Figure 4 again compares the quality of reduction in terms of the $\mathcal{H}_2(W)$-error and $\mathcal{H}_\infty(W)$-error. Both methods perform equally well with slight advantages for NOWI in the case of the $\mathcal{H}_2(W)$-error and for FWBT in the case of the $\mathcal{H}_\infty$-error. Similar to the previous example, Figure 5 shows how the relative error in the optimal interpolation conditions (16) vary as n_r varies. Once again, the relative residual of the optimality conditions decreases as n_r increases, yielding near-optimal interpolation.

ISS: The final example is the component 1r of the International Space Station from the SLICOT benchmark collection. The plant is a MIMO system with $n = 270$, and $m = p = 3$. The controller to be reduced is an LQG-based controller as before. We compare NOWI and FWBT for $n_r = 2, 4, \ldots, 40$. For $n_r \leq 30$, we use logarithmically spaced interpolation points for initializing NOWI. For larger values of n_r, we aggregate the optimal points from smaller reduced models. The relative $\mathcal{H}_2(W)$ errors are shown in Figure 6. The full model is hard to reduce with slowly decaying Hankel singular values. This is apparent from Figure 6 where FWBT hardly reduces the error for smaller n_r values. The proposed method clearly outperforms FWBT for every reduction order.

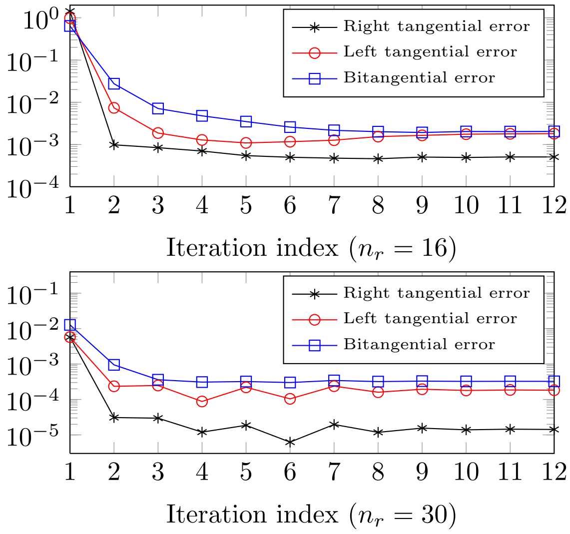

Figure 3: LA university hospital, $n = 48, n_w = 96$.

6 Conclusions

We have extended an interpolatory framework for weighted-$\mathcal{H}_2$ model reduction to include MIMO dynamical systems with feed-forward terms. The main tool was a new representation of the weighted-$\mathcal{H}_2$ inner product in MIMO settings (using $\mathfrak{F}[\cdot]$) which led to associated first-order necessary conditions for an optimal weighted-$\mathcal{H}_2$ reduced-order model. These conditions were found to be equivalent with necessary conditions established earlier by Halevi. An examination of realizations for systems defined by $\mathfrak{F}[\cdot]$ then led to an algorithm that remains tractable for large state-space dimension. There are a variety of refinements of the ideas presented here that can exploit the flexibility afforded by the interpolatory model reduction framework. One direction that has been fruitful in the unweighted case is trust-region based descent approaches, as described in [4] and extended to frequency-weighted settings in [6]. We have presented here several numerical examples that illustrate the effectiveness of our basic approach and its competitiveness with weighted balanced truncation.

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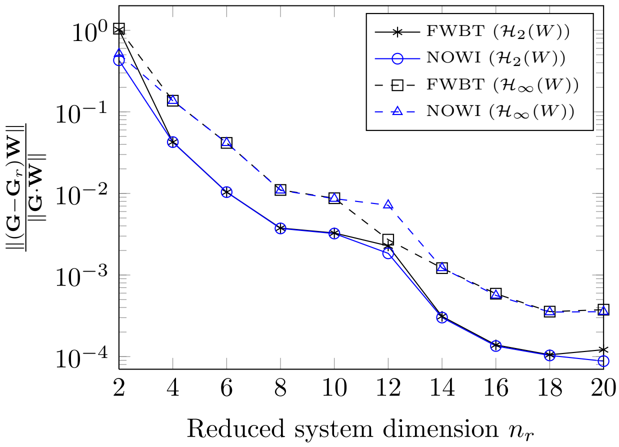

Figure 4: CD player, $n = 120, n_w = 240$.

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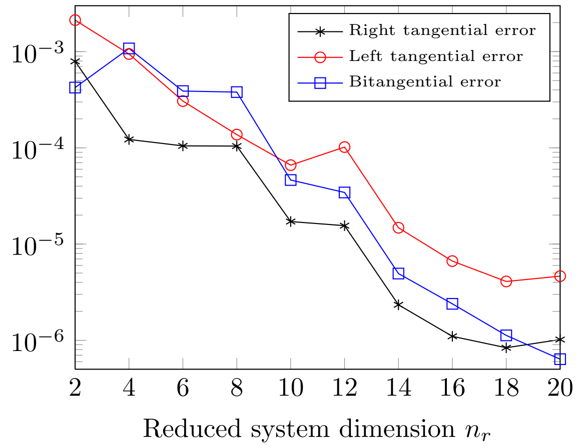

Figure 5: CD player, $n = 120, n_w = 240$.

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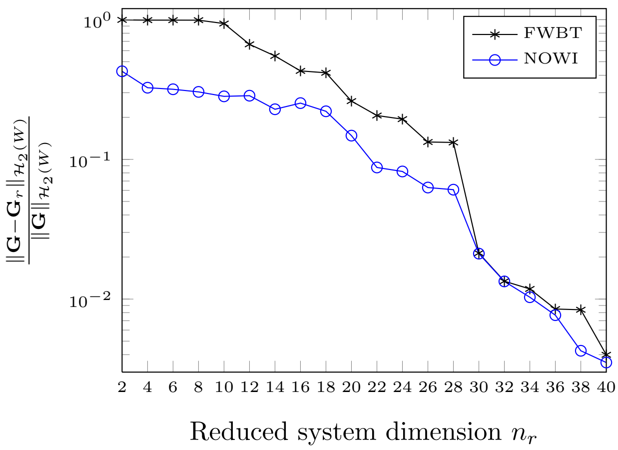

Figure 6: ISS, $n = 270, n_w = 540$.

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