MAX-PLANCK-GESELLSCHAFT

Martin Redmann Peter Benner

Approximation and Model Order Reduction for Second Order Systems with Levy-Noise

MAX–PLANCK–INSTITUT
FÜR DYNAMIK KOMPLEXER
TECHNISCHER SYSTEME
MAGDEBURG

Max Planck Institute Magdeburg
Preprints

MPIMD/14-16 September 15, 2014

Impressum:

Max Planck Institute for Dynamics of Complex Technical Systems, Magdeburg

Publisher:
Max Planck Institute for
Dynamics of Complex Technical Systems

Address:
Max Planck Institute for
Dynamics of Complex Technical Systems
Sandtorstr. 1
39106 Magdeburg

www.mpi-magdeburg.mpg.de/preprints

Abstract

We consider a controlled second order stochastic partial differential equation (SPDE) with Levy noise. To solve this system numerically, we apply a Galerkin scheme leading to a sequence of ordinary SDEs of large order. To reduce the high dimension we use balanced truncation.

1 Introduction

We introduce a controlled second order SPDE driven by Levy processes equipped with an output equation and transform it into a first order system. Regarding this transformation we follow Curtain [3] who deals with a similar SPDE but uncontrolled and with Wiener noise. For numerical purposes it is meaningful to do a finite dimensional approximation of the first order system we obtain at the end of Section 2. Therefore, we construct a sequence of finite dimensional processes using a Galerkin method in Section 3. The Galerkin scheme is for example also studied in Grecksch, Kloeden [5], Jentzen, Kloeden [7] and Hausenblas [6], where it is applied to other particular problems. Furthermore, we show the convergence of the finite dimensional sequence to the mild solution of the first order system. Finally, in Section 3 we provide an example which is covered by the second order system from the beginning and determine the coefficients of the corresponding Galerkin solution. Since the resulting Galerkin solution is of high dimension it is meaningful to treat it with model order reduction to save computational time. For that reason we introduce balanced truncation in Section 4 which Moore [8] considered first for linear deterministic systems. In addition, the book of Antoulas [1] summarizes all results concerning this method in the deterministic case. Benner, Damm [2] and Redmann, Benner [11] generalized this approach for linear systems with Wiener and Levy noise, respectively. We apply their results to the high order Galerkin equation of the example stated here. We compute an error bound and the exact error of balanced truncation to show the quality of this method and compare the output of the Galerkin system with the output of the reduced order model in a plot.

2 Second order systems

Let M_1 and M_2 be uncorrelated scalar square integrable Levy processes with zero mean being defined on a complete probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$.[1] In addition, we assume M_k $(k = 1, 2)$ to be $(\mathcal{F}_t)_{t\geq 0}$-adapted and the increments $M_k(t+h) - M_k(t)$ to be independent of $\mathcal{F}_t$ for $t, h \geq 0$. Suppose $\tilde{A} : D(\tilde{A}) \to H$ is a self adjoint and positive definite operator such that we can choose an orthonormal basis $\left(\tilde{h}_k\right)_{k\in\mathbb{N}}$ of the separable Hilbert space H consisting of eigenvectors of $\tilde{A}$:

$$\tilde{A}\tilde{h}_k = \lambda_k \tilde{h}_k,$$

where $0 < \lambda_1 \leq \lambda_2 \leq \dots$ are the corresponding eigenvalues. The well defined square root of $\tilde{A}$ we denote by $\tilde{A}^{\frac{1}{2}}$. $D(\tilde{A}^{\frac{1}{2}})$ equipped with the scalar product $\langle x, y\rangle_{D(\tilde{A}^{\frac{1}{2}})} = \left\langle \tilde{A}^{\frac{1}{2}}x, \tilde{A}^{\frac{1}{2}}y\right\rangle_H$ represents a Hilbert space. In this case, the norm $\|\cdot\|_{D(\tilde{A}^{\frac{1}{2}})}$ is equivalent to the graph norm of $\tilde{A}^{\frac{1}{2}}$.

The equation we consider next is also studied by Curtain in [3] for M_1, M_2 being Wiener processes and $u \equiv 0$. There the stability is analyzed for example. The system we focus on is the following (symbolic) second order stochastic differential equation:

$$\ddot{X}(t) + \alpha\dot{X}(t) + \tilde{A}X(t) + \tilde{B}u(t) + \tilde{D}_1X(t-)\dot{M}_1(t) + \tilde{D}_2\dot{X}(t-)\dot{M}_2(t) = 0 \qquad (1)$$

with initial conditions $X(0) = x_0$, $\dot{X}(0) = x_1$ and output equation

$$Y(t) = C\begin{pmatrix} X(t) \\ \dot{X}(t)\end{pmatrix}, \quad t \geq 0.$$

[1] We assume that $(\mathcal{F}_t)_{t\geq 0}$ is right continuous and that $\mathcal{F}_0$ contains all $\mathbb{P}$ null sets.

Above, we make the following assumptions:

- The constant α is positive, $\tilde{D}_1 \in L(D(\tilde{A}^{\frac{1}{2}}), H)$ and $\tilde{D}_2 \in L(H)$.
- The process $u : \mathbb{R}_+ \times \Omega \to \mathbb{R}^m$ is $(\mathcal{F}_t)_{t\geq 0}$-adapted with

$$\|u\|_{L^2_T}^2 := \int_0^T \mathbb{E}\,\|u(s)\|_{\mathbb{R}^m}^2\,ds < \infty$$

  for every $T > 0$.
- $\tilde{B}$ is a linear and bounded operator on $\mathbb{R}^m$ with values in H and $C \in L(D(\tilde{A}^{\frac{1}{2}}) \times H, \mathbb{R}^p)$.

We introduce the Hilbert space $\mathcal{H} = D(\tilde{A}^{\frac{1}{2}}) \times H$ equipped with the scalar product

$$\left\langle \begin{pmatrix} \tilde{Z}_1 \\ \tilde{Z}_2 \end{pmatrix}, \begin{pmatrix} \bar{Z}_1 \\ \bar{Z}_2 \end{pmatrix} \right\rangle_{\mathcal{H}} = \left\langle \tilde{A}^{\frac{1}{2}}\tilde{Z}_1, \tilde{A}^{\frac{1}{2}}\bar{Z}_1 \right\rangle_H + \left\langle \tilde{Z}_2, \bar{Z}_2 \right\rangle_H.$$

An orthonormal basis of $\mathcal{H}$ is given by $(h_k)_{k\in\mathbb{N}}$ defined by

$$h_{2i-1} = \lambda_i^{-\frac{1}{2}} \begin{pmatrix} \tilde{h}_i \\ 0 \end{pmatrix} \text{ and } h_{2i} = \begin{pmatrix} 0 \\ \tilde{h}_i \end{pmatrix}$$

for $i \in \mathbb{N}$. The second order equation (1) can be expressed by the following first order system:

$$\begin{aligned} dZ(t) &= AZ(t) + Bu(t)dt + D_1 Z(t-)dM_1(t) + D_2 Z(t-)dM_2(t), \\ Y(t) &= CZ(t), \quad t \geq 0, \quad Z(0) = z_0 = \begin{pmatrix} x_0 \\ x_1 \end{pmatrix}, \end{aligned} \tag{2}$$

where

$$Z(t) = \begin{pmatrix} X(t) \\ \dot{X}(t) \end{pmatrix},\ A = \begin{bmatrix} 0 & I \\ -\tilde{A} & -\alpha I \end{bmatrix},\ B = \begin{bmatrix} 0 \\ \tilde{B} \end{bmatrix},\ D_1 = \begin{bmatrix} 0 & 0 \\ \tilde{D}_1 & 0 \end{bmatrix} \text{ and } D_2 = \begin{bmatrix} 0 & 0 \\ 0 & \tilde{D}_2 \end{bmatrix}.$$

Regarding this transformation we follow [3], where it is used as well. The next lemma from [10] provides a stability result and is also needed to define a cadlag mild solution of (2).

Lemma 2.1. *For every $\alpha > 0$ the linear operator A with domain $D(\tilde{A}) \times D(\tilde{A}^{\frac{1}{2}})$ generates an exponential stable contraction semigroup $(S(t))_{t\geq 0}$ with*

$$\|S(t)\|_{L(\mathcal{H})} \leq \mathrm{e}^{-ct},$$

where

$$c \geq \frac{2\alpha\lambda_1}{4\lambda_1 + \alpha(\alpha + \sqrt{\alpha^2 + 4\lambda_1})}.$$

The following definition is meaningful due to Theorem 9.29 in [9]. There it is stated that the mild solution of (2) has the cadlag property if $(S(t))_{t\geq 0}$ is a contraction semigroup.

Definition 2.2. An $(\mathcal{F}_t)_{t\geq 0}$-adapted cadlag process $(Z(t))_{t\geq 0}$ with values in $\mathcal{H}$ is called *mild solution* of (2) if $\mathbb{P}$-almost surely

$$Z(t) = S(t)z_0 + \int_0^t S(t-s)Bu(s)ds + \sum_{i=1}^{2} \int_0^t S(t-s)D_i Z(s-)dM_i(s)$$

holds for all $t \geq 0$.

3 Galerkin approximation

We approximate the mild solution of the infinite dimensional equation (2) by a sequence $(Z_n)_{n\in\mathbb{N}}$ of finite dimensional processes with values in $\mathcal{H}_n = \operatorname{span}\{h_1, \dots, h_n\}$ given by

$$\begin{aligned} dZ_n(t) &= [A_n Z_n(t) + B_n u(t)]\, dt + D_{1,n} Z_n(t-) dM_1(t) + D_{2,n} Z_n(t-) dM_2(t), \\ Z_n(0) &= z_{0,n}, \quad t \geq 0, \end{aligned} \tag{3}$$

where

- $A_n x = \sum_{k=1}^n \langle Ax, h_k\rangle_{\mathcal{H}} h_k \in \mathcal{H}_n$ holds for all $x \in D(A)$,
- $B_n x = \sum_{k=1}^n \langle Bx, h_k\rangle_{\mathcal{H}} h_k \in \mathcal{H}_n$ holds for all $x \in \mathbb{R}^m$,
- $D_{i,n} x = \sum_{k=1}^n \langle D_i x, h_k\rangle_{\mathcal{H}} h_k \in \mathcal{H}_n$ holds for all $x \in \mathcal{H}$ $(i = 1, 2)$,
- $z_{0,n} = \sum_{k=1}^n \langle z_0, h_k\rangle_{\mathcal{H}} h_k$ is an $\mathcal{H}_n$-valued and $\mathcal{F}_0$-measurable random variable.

We know that A_n generates a C_0 semigroup $(S_n(t))_{t\geq 0}$ on $\mathcal{H}_n$, which is defined by $S_n(t)x = \sum_{k=1}^n \langle S(t)x, h_k\rangle_{\mathcal{H}} h_k$ for all $x \in \mathcal{H}$, such that the mild solution of equation (3) is given by

$$X_n(t) = S_n(t) z_{0,n} + \int_0^t S_n(t-s) B_n u(s) ds + \sum_{i=1}^2 \int_0^t S_n(t-s) D_{i,n} Z_n(s-) dM_i(s)$$

for $t \geq 0$. Since every A_n also generates the C_0 semigroup $\mathrm{e}^{A_n t}$, $t \geq 0$, we know that $S_n(t) = \mathrm{e}^{A_n t}$ on $\mathcal{H}_n$. Furthermore, we consider the p-dimensional approximating output

$$Y_n(t) = C Z_n(t), \quad t \geq 0.$$

We now show that

$$\mathbb{E}\, \|Z_n(t) - Z(t)\|_{\mathcal{H}}^2 \to 0 \text{ and hence } \mathbb{E}\, \|Y_n(t) - Y(t)\|_{\mathbb{R}^p}^2 \to 0$$

for $n \to \infty$ and $t \geq 0$.

Theorem 3.1. *It holds*

$$\mathbb{E}\, \|Z_n(t) - Z(t)\|_{\mathcal{H}}^2 \to 0$$

for $n \to \infty$ and $t \geq 0$.

Proof.

$$\begin{aligned} \mathbb{E}\, \|Z(t) - Z_n(t)\|_{\mathcal{H}}^2 \leq & 4\mathbb{E}\, \|S(t) z_0 - S_n(t) z_{0,n}\|_{\mathcal{H}}^2 \\ & + 4\mathbb{E} \left\| \int_0^t (S(t-s)B - S_n(t-s)B_n) u(s) ds \right\|_{\mathcal{H}}^2 \\ & + 4\mathbb{E} \left\| \int_0^t S(t-s) D_1 Z(s-) - S_n(t-s) D_{1,n} Z_n(s-) dM_1(s) \right\|_{\mathcal{H}}^2 \\ & + 4\mathbb{E} \left\| \int_0^t S(t-s) D_2 Z(s-) - S_n(t-s) D_{2,n} Z_n(s-) dM_2(s) \right\|_{\mathcal{H}}^2 . \end{aligned}$$

Since $(S(t))_{t\geq 0}$ is a contraction semigroup, we obtain

$$\begin{aligned} 4\mathbb{E}\, \|S(t) z_0 - S_n(t) z_{0,n}\|_{\mathcal{H}}^2 &\leq 8\mathbb{E}\, \|S(t) z_0 - S(t) z_{0,n}\|_{\mathcal{H}}^2 + 8\mathbb{E}\, \|S(t) z_{0,n} - S_n(t) z_{0,n}\|_{\mathcal{H}}^2 \\ &\leq 8\mathbb{E}\, \|z_0 - z_{0,n}\|_{\mathcal{H}}^2 + 8\mathbb{E}\, \|S(t) z_{0,n} - S_n(t) z_{0,n}\|_{\mathcal{H}}^2 . \end{aligned} \tag{4}$$

By the representation $S_n(t)z_n = \sum_{i=1}^n \langle S(t)z_n, h_i\rangle_{\mathcal{H}} h_i$ ($z_n \in \mathcal{H}_n$) and Lebesgue's theorem, (4) tends to zero for $n \to \infty$. The Hölder inequality yields

$$\mathbb{E}\left\|\int_0^t (S(t-s)B - S_n(t-s)B_n)u(s)ds\right\|_{\mathcal{H}}^2 \leq t\mathbb{E}\int_0^t \|(S(t-s)B - S_n(t-s)B_n)u(s)\|_{\mathcal{H}}^2\, ds.$$

We obtain

$$\begin{aligned}
&\|S(t-s)Bu(s) - S_n(t-s)B_nu(s)\|_{\mathcal{H}}^2 \\
&\leq 2\|S(t-s)Bu(s) - S(t-s)B_nu(s)\|_{\mathcal{H}}^2 + 2\|S(t-s)B_nu(s) - S_n(t-s)B_nu(s)\|_{\mathcal{H}}^2 \\
&\leq 2\|Bu(s) - B_nu(s)\|_{\mathcal{H}}^2 + 2\|S(t-s)B_nu(s) - S_n(t-s)B_nu(s)\|_{\mathcal{H}}^2 \to 0
\end{aligned}$$

$\mathbb{P}$- almost surely for $n \to \infty$ and

$$8t\mathbb{E}\int_0^t \|Bu(s) - B_nu(s)\|_{\mathcal{H}}^2 + \|S(t-s)B_nu(s) - S_n(t-s)B_nu(s)\|_{\mathcal{H}}^2\, ds \to 0 \tag{5}$$

for $n \to \infty$ by Lebesgue's theorem. Applying the Ito isometry from Corollary 8.17 in [9], we have

$$\begin{aligned}
&4\mathbb{E}\left\|\int_0^t S(t-s)D_iZ(s-) - S_n(t-s)D_{i,n}Z_n(s-)dM_i(s)\right\|_{\mathcal{H}}^2 \\
&= 4\int_0^t \mathbb{E}\|S(t-s)D_iZ(s-) - S_n(t-s)D_{i,n}Z_n(s-)\|_{\mathcal{H}}^2\, ds\, \mathbb{E}[M_i^2(1)] \\
&\leq 8\mathbb{E}\left[\int_0^t \|D_iZ(s) - D_{i,n}Z(s)\|_{\mathcal{H}}^2\, ds\right] \mathbb{E}[M_i^2(1)] \qquad (6) \\
&\quad + 8\mathbb{E}\left[\int_0^t \|S(t-s)D_{i,n}Z(s) - S_n(t-s)D_{i,n}Z_n(s)\|_{\mathcal{H}}^2\, ds\right] \mathbb{E}[M_i^2(1)].
\end{aligned}$$

The term (6) converges to zero for $n \to \infty$ by Lebesgue's theorem. Moreover, it holds

$$\begin{aligned}
&8\mathbb{E}\left[\int_0^t \|S(t-s)D_{i,n}Z(s) - S_n(t-s)D_{i,n}Z_n(s)\|_{\mathcal{H}}^2\, ds\right] \mathbb{E}[M_i^2(1)] \\
&\leq 16\mathbb{E}\left[\int_0^t \|S(t-s)D_{i,n}Z(s) - S_n(t-s)D_{i,n}Z(s)\|_{\mathcal{H}}^2\, ds\right] \mathbb{E}[M_i^2(1)] \qquad (7) \\
&\quad + 16\mathbb{E}\left[\int_0^t \|S_n(t-s)D_{i,n}Z(s) - S_n(t-s)D_{i,n}Z_n(s)\|_{\mathcal{H}}^2\, ds\right] \mathbb{E}[M_i^2(1)].
\end{aligned}$$

Again, by Lebesgue's theorem the term (7) tends to zero for $n \to \infty$ and

$$\begin{aligned}
&16\mathbb{E}\left[\int_0^t \|S_n(t-s)D_{i,n}Z(s) - S_n(t-s)D_{i,n}Z_n(s)\|_{\mathcal{H}}^2\, ds\right] \mathbb{E}[M_i^2(1)] \\
&\leq 16\|D_i\|_{L(\mathcal{H})}^2\, \mathbb{E}\left[\int_0^t \|Z(s) - Z_n(s)\|_{\mathcal{H}}^2\, ds\right] \mathbb{E}[M_i^2(1)].
\end{aligned}$$

Summarizing everything we obtain

$$\mathbb{E}\|Z(t) - Z_n(t)\|_{\mathcal{H}}^2 \leq f_n(t) + k_1\int_0^t \mathbb{E}\|Z(s) - Z_n(s)\|_{\mathcal{H}}^2\, ds,$$

where $k_1 := 16\|D_i\|_{L(\mathcal{H})}^2\, \mathbb{E}[M_i^2(1)]$ and f_n is a sequence of functions consisting of the terms (4), (5), (6) and (7). Hence,

$$\mathbb{E}\,\|Z(t)-Z_n(t)\|_{\mathcal{H}}^2 \le f_n(t) + k_1 \int_0^t f_n(s)\,\mathrm{e}^{k_1(t-s)}\,ds \tag{8}$$

by Gronwall's inequality. The first term of the right hand side of inequality (8) converges to zero since $f_n(t)$ converges to zero for $n \to \infty$. In addition, f_n is bounded by the increasing function $\tilde{f}$ defined by

$$\tilde{f}(t) := k_2 \left(\mathbb{E}\,\|z_0\|_{\mathcal{H}}^2 + t \int_0^t \mathbb{E}\,\|u(s)\|_{\mathbb{R}^m}^2\,ds + \int_0^t \mathbb{E}\,\|Z(s)\|_{\mathcal{H}}^2\,ds \right)$$

with a suitable constant $k_2 > 0$. So, $f_n(s) \le \tilde{f}(t)$ for all $0 \le s \le t$ and every $n \in \mathbb{N}$. Hence, the second term of the right hand side of inequality (8) converges to zero by Lebesgue's theorem. □

Moreover, notice that mild and strong solution of equation (3) are equivalent. This fact we use below but first we determine the components of Y_n. They are given by

$$Y_n^\ell(t) = \langle Y_n(t), e_\ell\rangle_{\mathbb{R}^p} = \langle CZ_n(t), e_\ell\rangle_{\mathbb{R}^p} = \sum_{k=1}^n \langle Ch_k, e_\ell\rangle_{\mathbb{R}^p} \langle Z_n(t), h_k\rangle_{\mathcal{H}}$$

for $\ell = 1, \dots, p$, where e_ℓ is the ℓth unit vector in $\mathbb{R}^p$. We set

$$\mathcal{Z}(t) = (\langle Z_n(t), h_1\rangle_{\mathcal{H}}, \dots, \langle Z_n(t), h_n\rangle_{\mathcal{H}})^T \text{ and } \mathcal{C} = (\langle Ch_k, e_\ell\rangle_{\mathbb{R}^p})_{\substack{\ell=1,\dots,p\\ k=1,\dots,n}}$$

and obtain

$$Y_n(t) = \mathcal{C}\mathcal{Z}(t), \quad t \ge 0. \tag{9}$$

The components $\mathcal{Z}_k(t) := \langle Z_n(t), h_k\rangle_{\mathcal{H}}$ of $\mathcal{Z}(t)$ fulfill the following:

$$d\mathcal{Z}_k(t) = [\langle A_n Z_n(t), h_k\rangle_{\mathcal{H}} + \langle B_n u(t), h_k\rangle_{\mathcal{H}}]\,dt + \sum_{i=1}^2 \langle D_{i,n} Z_n(t-), h_k\rangle_{\mathcal{H}}\,dM_i(t).$$

By using the Fourier series representation of Z_n, we obtain

$$\begin{aligned} d\mathcal{Z}_k(t) &= \left[\sum_{j=1}^n \langle A_n h_j, h_k\rangle_{\mathcal{H}} \mathcal{Z}_j + \sum_{j=1}^m \langle B_n e_j, h_k\rangle_{\mathcal{H}} \langle u(t), e_j\rangle_{\mathbb{R}^m} \right] dt \\ &\quad + \sum_{i=1}^2 \sum_{j=1}^n \langle D_{i,n} h_j, h_k\rangle_{\mathcal{H}} \mathcal{Z}_j(t-) dM_i(t) \\ &= \left[\sum_{j=1}^n \langle A h_j, h_k\rangle_{\mathcal{H}} \mathcal{Z}_j(t) + \sum_{j=1}^m \langle B e_j, h_k\rangle_{\mathcal{H}} \langle u(t), e_j\rangle_{\mathbb{R}^m} \right] dt \\ &\quad + \sum_{i=1}^2 \sum_{j=1}^n \langle D_i h_j, h_k\rangle_{\mathcal{H}} \mathcal{Z}_j(t-) dM_i(t), \end{aligned}$$

where e_j is the jth unit vector in $\mathbb{R}^m$. Hence, in compact form $\mathcal{Z}$ is given by

$$d\mathcal{Z}(t) = [\mathcal{A}\mathcal{Z}(t) + \mathcal{B}u(t)]\,dt + \sum_{i=1}^2 \mathcal{D}_i \mathcal{Z}(s-) dM_i(s), \tag{10}$$

where

- $\mathcal{A} = \big(\langle Ah_j, h_k\rangle_{\mathcal{H}}\big)_{k,j=1,\dots,n} = \operatorname{diag}(E_1,\dots,E_{\frac{n}{2}})$ with $E_\ell = \left(\begin{smallmatrix} 0 & \sqrt{\lambda_\ell} \\ -\sqrt{\lambda_\ell} & -\alpha \end{smallmatrix}\right)$ $(\ell = 1,\dots,\frac{n}{2})$,
- $\mathcal{B} = \big(\langle Be_j, h_k\rangle_{\mathcal{H}}\big)_{\substack{k=1,\dots,n \\ j=1,\dots,m}}$ and $\mathcal{D}_i = \big(\langle D_i h_j, h_k\rangle_{\mathcal{H}}\big)_{k,j=1,\dots,n}$.

Below, we present an example for system (1).

Example 1. Lateral displacement of an electricity cable impacted by wind can be modeled by

$$\frac{\partial^2}{\partial t^2}X(t,\zeta) + \alpha\frac{\partial}{\partial t}X(t,\zeta) + \mathrm{e}^{-(\zeta-\frac{\pi}{2})^2}\,u(t) + 2\,\mathrm{e}^{-(\zeta-\frac{\pi}{2})^2}\,X(t-,\zeta)\frac{\partial}{\partial t}M_1(t) = \frac{\partial^2}{\partial \zeta^2}X(t,\zeta)$$

for $t \in [0,T]$ and $\zeta \in [0,\pi]$. Here, we have

- $\tilde{A} = -\frac{\partial^2}{\partial\zeta^2}$, the operator $\tilde{B}$ is represented by the function $\mathrm{e}^{-(\cdot-\frac{\pi}{2})^2}$,
- $\tilde{D}_2 = 0$, $\tilde{D}_1$ is characterized by $2\,\mathrm{e}^{-(\cdot-\frac{\pi}{2})^2}$ and
- $H = L^2([0,\pi])$, $D(\tilde{A}^{\frac{1}{2}}) = H_0^1([0,\pi])$, $m = 1$.

The boundary and initial conditions are

$$X(0,t) = 0 = X(\pi,t) \text{ and } X(0,\zeta),\ \frac{\partial}{\partial t}X(t,\zeta)\Big|_{t=0} \equiv 0.$$

The output is an approximation for the position of the middle of the string

$$Y(t) = \frac{1}{2\epsilon}\int_{\frac{\pi}{2}-\epsilon}^{\frac{\pi}{2}+\epsilon} X(t,\zeta)d\zeta,$$

where $\epsilon > 0$. Here, we set $C = \hat{C}\begin{bmatrix} I & 0\end{bmatrix}$ with $\hat{C}x = \frac{1}{2\epsilon}\int_{\frac{\pi}{2}-\epsilon}^{\frac{\pi}{2}+\epsilon} x(\zeta)d\zeta$ $(x \in D(\tilde{A}^{\frac{1}{2}}))$, such that $p = 1$.

Next, we study the Galerkin solution of Example 1. The orthonormal basis of eigenvectors of $-\frac{\partial^2}{\partial\zeta^2}$ for H is given by $\tilde{h}_k = \sqrt{\frac{2}{\pi}}\sin(k\cdot)$ and the corresponding eigenvalues are $\lambda_k = k^2$ for $k \in \mathbb{N}$. Hence, the matrices of the Galerkin system (10) are

- $\mathcal{A} = \operatorname{diag}\left(E_1,\dots,E_{\frac{n}{2}}\right)$ with $E_\ell = \left(\begin{smallmatrix} 0 & \ell \\ -\ell & -\alpha \end{smallmatrix}\right)$,
- $\mathcal{B} = (\langle B, h_k\rangle_{\mathcal{H}})_{k=1,\dots,n}$ with

$$\langle B, h_{2\ell-1}\rangle_{\mathcal{H}} = 0, \quad \langle B, h_{2\ell}\rangle_{\mathcal{H}} = \sqrt{\frac{2}{\pi}}\left\langle \mathrm{e}^{-(\cdot-\frac{\pi}{2})^2}, \sin(\ell\cdot)\right\rangle_H,$$

- $\mathcal{D}_2 = 0$ and $\mathcal{D}_1 = \big(\langle D_1 h_j, h_k\rangle_{\mathcal{H}}\big)_{k,j=1,\dots,n} = (d_{kj})_{k,j=1,\dots,n}$ with

$$d_{(2\ell-1)j} = 0, \quad d_{(2\ell)j} = \begin{cases} 0, & \text{if } j = 2v, \\ \frac{4}{\pi v}\left\langle \sin(\ell\cdot), \mathrm{e}^{-(\cdot-\frac{\pi}{2})^2}\sin(v\cdot)\right\rangle_H, & \text{if } j = 2v-1, \end{cases}$$

  for $j = 1,\dots,n$ and $v = 1,\dots,\frac{n}{2}$,
- the output matrix $\mathcal{C}$ in (9) is given by $\mathcal{C}^T = (Ch_k)_{k=1,\dots,n}$ with

$$Ch_{2\ell} = 0 \text{ and } Ch_{2\ell-1} = \frac{1}{\sqrt{2\pi}\ell^2}\left[\cos\left(\ell\left(\frac{\pi}{2}-\epsilon\right)\right) - \cos\left(\ell\left(\frac{\pi}{2}+\epsilon\right)\right)\right],$$

  where we assume n to be even and $\ell = 1,\dots,\frac{n}{2}$.

4 Balanced truncation and numerical results

Balanced truncation is a model order reduction technique which was first introduced for deterministic linear systems, see [1] and [8]. This scheme was generalized in [2] and [11], where linear systems with Wiener and Levy noise, respectively are considered. Below we use the results from [11] and apply balanced truncation to the Galerkin solution of Example 1. This model order reduction method only works for systems (10) being mean square asymptotically stable, which means that

$$\mathbb{E}\left\|\mathcal{Z}(t)\right\|^2_{\mathbb{R}^n} \to 0 \tag{11}$$

for $t \to \infty$ for every initial value if $u \equiv 0$. Below, we can ensure that property by setting $\alpha = 2$ and $M_1(t) = -(N(t) - t)$ with N being a Poisson process with parameter 1 such that we get $\mathbb{E}\left[M_1^2(1)\right] = 1$. Then the matrix equation

$$\mathcal{A}^T X + X\mathcal{A} + \mathcal{D}_1^T X \mathcal{D}_1\, \mathbb{E}\left[M_1^2(1)\right] = -I$$

has a positive definite solution $X > 0$, which we check numerically. Due to Theorem 3.6.1 in [4] this implies condition (11), such that the desired model order reduction technique can be used. The reduced order model by balanced truncation of state space dimension $r \ll n$ has the representation

$$\begin{aligned} d\tilde{\mathcal{Z}}(t) &= [\mathcal{A}_R\tilde{\mathcal{Z}}(t) + \mathcal{B}_R u(t)]dt + \mathcal{D}_R\tilde{\mathcal{Z}}(t-)dM_1(t), \\ \hat{Y}(t) &= \mathcal{C}_R\tilde{\mathcal{Z}}(t) \end{aligned} \tag{12}$$

with

$$\mathcal{A}_R = W^T\mathcal{A}V, \quad \mathcal{B}_R = W^T\mathcal{B}, \quad \mathcal{D}_R = W^T\mathcal{D}_1 V, \quad \mathcal{C}_R = \mathcal{C}V.$$

Here, W^T are the first r rows of T and V are the first r columns of T^{-1}, where T is determined by

$$T = \Sigma^{\frac{1}{2}} K^T U^{-1} \tag{13}$$

with U coming from the Cholesky decomposition of $P = UU^T$ and K is an orthogonal matrix corresponding to the EVD (SVD respectively) of $U^T QU = K\Sigma^2 K^T$. The matrices P and Q are solutions of

$$\begin{aligned} \mathcal{A}^T Q + Q\mathcal{A} + \mathcal{D}_1^T Q\mathcal{D}_1\, \mathbb{E}\left[M_1^2(1)\right] &= -\mathcal{C}^T\mathcal{C}, \\ \mathcal{A}P + P\mathcal{A}^T + \mathcal{D}_1 P\mathcal{D}_1^T\, \mathbb{E}\left[M_1^2(1)\right] &= -\mathcal{B}\mathcal{B}^T. \end{aligned}$$

In the next theorem we state an error bound for the estimation (see [11]).

Theorem 4.1. *Let Y_n be the output of the Galerkin solution of Example 1 and $\hat{Y}$ be the output of the reduced order model (12), then*

$$\sup_{t\in[0,T]} \mathbb{E}\left\|\hat{Y}(t) - Y_n(t)\right\|_2 \leq \left(\operatorname{tr}\left(\mathcal{C}P\mathcal{C}^T\right) + \operatorname{tr}\left(\mathcal{C}_R P_R \mathcal{C}_R^T\right) - 2\ \operatorname{tr}\left(\mathcal{C}P_G\mathcal{C}_R^T\right)\right)^{\frac{1}{2}} \|u\|_{L^2_T}$$

for every $T > 0$. P_R and P_G satisfy the equations

$$\begin{aligned} \mathcal{A}_R P_R + P_R\mathcal{A}_R^T + \mathcal{D}_R P_R\mathcal{D}_R^T\, \mathbb{E}\left[M_1^2(1)\right] &= -\mathcal{B}_R\mathcal{B}_R^T, \\ \mathcal{A}P_G + P_G\mathcal{A}_R^T + \mathcal{D}_1 P_G\mathcal{D}_R^T\, \mathbb{E}\left[M_1^2(1)\right] &= -\mathcal{B}\mathcal{B}_R^T. \end{aligned}$$

We fix the dimension of the Galerkin solution to $n = 1000$ and compute exact errors and error bounds for different dimensions r of the reduced order model (ROM), where we choose particular normalized control functions $u_1(t) = \sqrt{\frac{2}{\pi}} 1_{[0,\frac{\pi}{2}]}(t)$ and $u_2(t) = \frac{\sqrt{8}}{\pi} 1_{[0,\frac{\pi}{2}]}(t) w(t)$ $(t \in [0, \pi])$. Moreover, w is a Wiener process and $\mathcal{E} := \left(\operatorname{tr}\left(\mathcal{C}P\mathcal{C}^T\right) + \operatorname{tr}\left(\mathcal{C}_R P_R \mathcal{C}_R^T\right) - 2\ \operatorname{tr}\left(\mathcal{C}P_G\mathcal{C}_R^T\right)\right)^{\frac{1}{2}}$.

Dim. ROM	Exact Error ($u = u_1$)	Exact Error ($u = u_2$)	Bound $\mathcal{E}$
40	$1.4484 \cdot 10^{-6}$	$1.1182 \cdot 10^{-6}$	$4.0103 \cdot 10^{-5}$
20	$7.2173 \cdot 10^{-6}$	$8.5996 \cdot 10^{-6}$	$1.2695 \cdot 10^{-4}$
10	$5.1396 \cdot 10^{-5}$	$3.8038 \cdot 10^{-5}$	$3.6395 \cdot 10^{-4}$
5	$5.2740 \cdot 10^{-4}$	$4.3632 \cdot 10^{-4}$	$2.3446 \cdot 10^{-3}$
3	0.0113	$8.6287 \cdot 10^{-3}$	0.0380

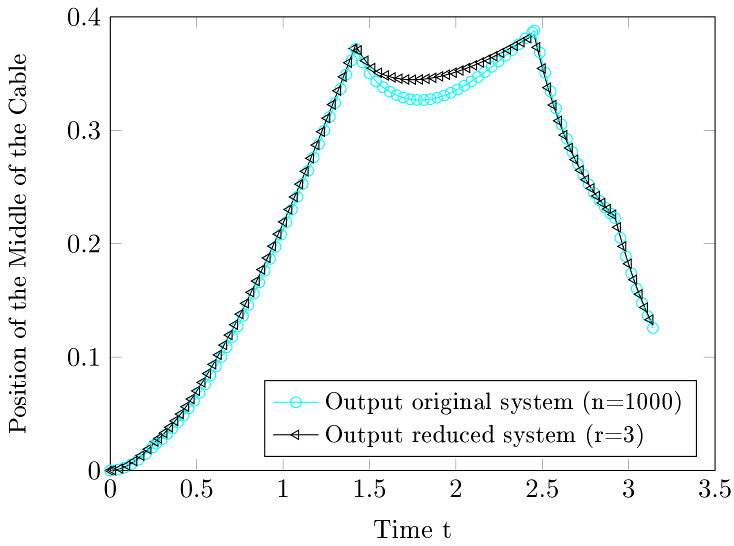

Figure 1:

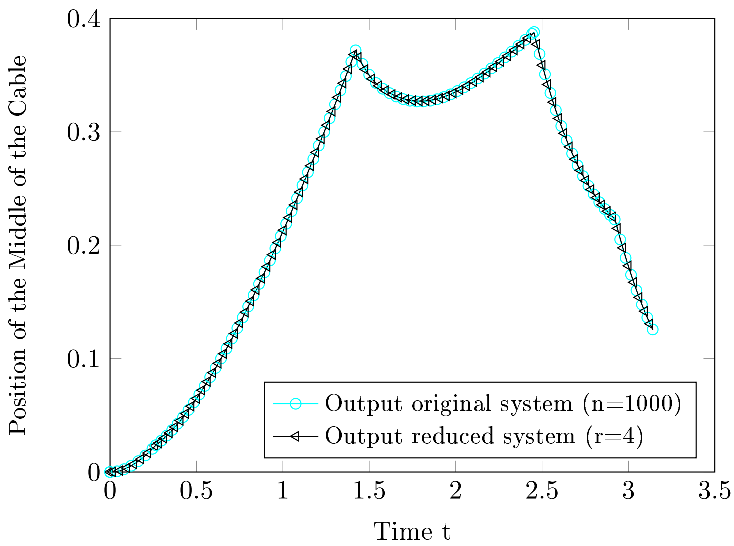

Figure 2:

In Figure 1 we plot the output Y_n of the Galerkin system with state space dimension $n = 1000$ and the output $\hat{Y}$ of the ROM with state space dimension $r = 3$, where we choose $u \equiv u_1$. Due to the input, which can be interpreted as electricity flowing through the cable, the curves are increasing first. Additionally, the cable is randomly hit by wind which is marked by the peaks in this picture. This effect pushes the cable in the opposite direction. After the electricity completely passed the cable, the graphs decrease to zero due to the stability of the system. It is also obvious that even after such a large reduction of the dimension the accuracy is quite good. In Figure 2 we increase the dimension of the reduced order model by one such that it is difficult to distinguish between the output of the ROM and the output of the Galerkin system. Hence, one can conclude that the output of the SPDE in Example 1 can be described by a system of ordinary SDEs of order four.

5 Conclusion

In this paper we dealt with a second order SPDE wit Levy noise and transformed it into a first order system. The corresponding mild solution we approximated with a Galerkin scheme. We obtained a large scale Galerkin system for a particular example and applied balanced truncation. Finally, we provided numerical results to show the performance of the model order technique used here.

References

[1] Athanasios C. Antoulas. *Approximation of large-scale dynamical systems.* Advances in Design and Control 6. Philadelphia, PA: Society for Industrial and Applied Mathematics (SIAM), 2005.

[2] Peter Benner and Tobias Damm. Lyapunov equations, energy functionals, and model order reduction of bilinear and stochastic systems. *SIAM J. Control Optim.*, 49(2):686–711, 2011.

[3] Ruth F. Curtain. Stability of Stochastic Partial Differential Equation. *J. Math. Anal. Appl.*, 79:352–369, 1981.

[4] Tobias Damm. *Rational Matrix Equations in Stochastic Control.* Lecture Notes in Control and Information Sciences 297. Berlin: Springer, 2004.

[5] Wilfried Grecksch and Peter E. Kloeden. Time-discretised Galerkin approximations of parabolic stochastic PDEs. *Bull. Aust. Math. Soc.*, 54(1):79–85, 1996.

[6] Erika Hausenblas. Approximation for Semilinear Stochastic Evolution Equations. *Potential Anal.*, 18(2):141–186, 2003.

[7] Arnulf Jentzen and Peter E. Kloeden. Overcoming the order barrier in the numerical approximation of stochastic partial differential equations with additive space-time noise. *Proc. R. Soc. A 2009*, 465:649–667, 2009.

[8] Bruce C. Moore. Principal component analysis in linear systems: Controllability, observability, and model reduction. *IEEE Trans. Autom. Control*, 26:17–32, 1981.

[9] Szymon Peszat and Jerzy Zabczyk. *Stochastic Partial Differential Equations with Lévy Noise. An evolution equation approach.* Encyclopedia of Mathematics and Its Applications 113. Cambridge: Cambridge University Press, 2007.

[10] Anthony J. Pritchard and Jerzy Zabczyk. Stability and Stabilizability of Infinite-Dimensional Systems. *SIAM Rev.*, 23:25–52, 1981.

[11] Martin Redmann and Peter Benner. Model Reduction for Stochastic Systems. Preprint MPIMD/14-03, Max Planck Institute Magdeburg, 2014.