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Model Order Reduction for Differential-Algebraic Equations: a Survey

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Abstract

In this paper, we discuss the model order reduction problem for descriptor systems, that is, systems with dynamics described by differential-algebraic equations. We focus on linear descriptor systems as a broad variety of methods for these exist, while model order reduction for nonlinear descriptor systems has not received sufficient attention up to now. Model order reduction for linear state-space systems has been a topic of research for about 50 years at this writing, and by now can be considered as a mature field. The extension to linear descriptor systems usually requires extra treatment of the constraints imposed by the algebraic part of the system. For almost all methods, this causes some technical difficulties, and these have only been thoroughly addressed in the last decade. We will focus on these developments in particular for the popular methods related to balanced truncation and rational interpolation. We will review efforts in extending these approaches to descriptor systems, and also add the extension of the so-called *stochastic balanced truncation* method to descriptor systems which so far cannot be found in the literature.

Keywords. Differential-algebraic equations, matrix pencils, model order reduction, balanced truncation, interpolation-based approximation, matrix equations.

Mathematics Subject Classification (2010) 15A22, 15A24, 34A09, 65D05, 65F30, 93C05

1 Introduction

Consider a linear time-invariant descriptor system

$$\begin{aligned} E\dot{x}(t) &= Ax(t)+Bu(t), && \text{(1a)}\\ y(t) &= Cx(t)+Du(t), && \text{(1b)} \end{aligned}$$

where E, $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{q\times n}$, $D \in \mathbb{R}^{q\times m}$, $x(t) \in \mathbb{R}^n$ is the generalized state space vector, $u(t) \in \mathbb{R}^m$ is the input, and $y(t) \in \mathbb{R}^q$ is the output. Here, (1a) represents a system of linear differential-algebraic equations (DAEs), while (1b) is an output equation, modeling observations or measurements of the system. Sometimes, the elements of $y(t)$ are also referred as the quantities-of-interest if the system is used in a (design) optimization context, and the output quantities in $y(t)$ are the subject of optimization.

Modeling by DAEs has become an ubiquitous tool in many engineering disciplines, in particular in structural dynamics and multi-body systems as well as in micro- and nanoelectronics, computational electromagnetics, and fluid mechanics, see, e.g., [22, 86, 93, 118], the DAE examples in [28, Part II], and the benchmarks provided at the Model Order Reduction Wiki [97]. In mechanics, algebraic constraints arise from holonomic or nonholonomic constraints, in circuit simulation and other network problems, among others, from Kirchhoff's laws, and in electromagnetics or fluid mechanics by the discretization of conservation laws like the preservation of mass in the incompressible Navier-Stokes equations. In these applications, the shear amount of equations like in the modeling of semiconductor devices or the fine-grain spatial discretization of partial differential equations like the already mentioned Navier-Stokes or Maxwell's equations in electromagnetics, leads to descriptor systems with n in the thousands to millions or even larger than this. A single forward simulation of such a system is certainly feasible on modern computer architectures, but simulating a couple of hundreds of times in the context of design optimization, varying input signals, and control design, is often out of scope. In these situations, replacing the descriptor system (1) by a system of the same structure, but of much smaller size $r \ll n$ by approximating the input-output relation to a desired accuracy, is beneficial.

A *model order reduction* problem consists in approximating (1) by a reduced-order model

$$\begin{aligned} \widetilde{E}\dot{\widetilde{x}}(t) &= \widetilde{A}\widetilde{x}(t)+\widetilde{B}u(t),\\ \widetilde{y}(t) &= \widetilde{C}\widetilde{x}(t)+\widetilde{D}u(t), \end{aligned} \qquad (2)$$

where $\widetilde{E}$, $\widetilde{A} \in \mathbb{R}^{r\times r}$, $\widetilde{B} \in \mathbb{R}^{r\times m}$, $\widetilde{C} \in \mathbb{R}^{q\times r}$, $\widetilde{D} \in \mathbb{R}^{q\times m}$ and $r \ll n$. Assume that the matrix pencil $\lambda E - A$ is *regular*, i.e., $\det(\lambda E - A) \neq 0$ for some $\lambda \in \mathbb{C}$. Applying the Laplace transform to system (1), it can be written in the frequency domain as

$$\hat{y}(s) = \boldsymbol{H}(s)\hat{u}(s) + C(sE-A)^{-1}x(0),$$

where $\hat{u}(s)$ and $\hat{y}(s)$ are the Laplace transforms of the input and output, respectively, and $\boldsymbol{H}(s) = C(sE-A)^{-1}B+D$ is a *transfer function* of (1). Then the model reduction problem can be formulated in the frequency domain as follows: given the transfer function $\boldsymbol{H}(s)$, find $\widetilde{\boldsymbol{H}}(s) = \widetilde{C}(s\widetilde{E}-\widetilde{A})^{-1}\widetilde{B}+\widetilde{D}$ of lower dimension that approximates $\boldsymbol{H}(s)$. The approximation quality can, for instance, be measured by the absolute error $\widetilde{\boldsymbol{H}}-\boldsymbol{H}$ or by the relative error $\boldsymbol{H}^{-1}(\widetilde{\boldsymbol{H}}-\boldsymbol{H})$ (provided $\boldsymbol{H}^{-1}$ exists).

The structure of this survey is as follows: in the following section, we provide the relevant systems and control theoretic basics for linear descriptor systems. In Section 3, we review the most common methods for model order reduction of linear descriptor systems: balanced

truncation and related methods in Subsection 3.1, and moment matching as well as other rational interpolation methods in Subsection 3.2. The computational bottleneck of many model reduction methods, in particular those related to balanced truncation, is the numerical solution of matrix equations (e.g., algebraic Lyapunov and Riccati equations). Therefore, we review the usually used methods and their adaptation to the DAE case in Section 4. Usually, descriptor systems have a certain block structure, often related to the differential and algebraic parts of the system. Exploiting these structures is mandatory for efficient methods for model order reduction and the associated matrix equations. This is discussed in Section 5, using some relevant example classes. In Section 6, we provide a brief outlook on topics not covered in depth in this survey and/or of current research interest.

Throughout the paper, $\mathbb{R}^{n\times m}$ and $\mathbb{C}^{n\times m}$ denote the spaces of $n\times m$ real and complex matrices, respectively. Furthermore, $\mathbb{C}_- = \{s\in\mathbb{C} : \mathrm{Re}(s) < 0\}$ and $\mathbb{C}_+ = \{s\in\mathbb{C} : \mathrm{Re}(s) > 0\}$ denote the open left and right half-planes, respectively, and $i = \sqrt{-1}$. The matrices A^T and A^* denote, respectively, the transpose and the conjugate transpose of $A\in\mathbb{C}^{n,m}$, and $A^{-T} = (A^{-1})^T$. We use $\mathrm{rank}(A)$, $\mathrm{im}(A)$ and $\ker(A)$ for the rank, the image and the kernel of A, respectively. A matrix $A\in\mathbb{C}^{n,n}$ is said to be positive semidefinite, if $v^*Av\geq 0$ for all $v\in\mathbb{C}^n$. Note that positive semidefiniteness of A does not require A to be Hermitian. For $A,B\in\mathbb{C}^{n,n}$, we write $A\geq B$ if $A-B$ is positive semidefinite.

2 DAE control systems

In this section, we provide necessary notation and fundamental matrix and control theoretic concepts for DAE systems.

Any regular matrix pencil $\lambda E - A$ can be transformed into the *Weierstrass canonical form*

$$E = T_l \begin{bmatrix} I_{n_f} & 0 \\ 0 & E_\infty \end{bmatrix} T_r, \qquad A = T_l \begin{bmatrix} A_f & 0 \\ 0 & I_{n_\infty} \end{bmatrix} T_r, \tag{3}$$

where T_l and T_r are the left and right nonsingular transformation matrices, E_∞ is nilpotent with index of nilpotency ν, and $n_f + n_\infty = n$, e.g., [58]. The number ν is called the *index* of $\lambda E - A$ and also of the DAE system (1). The eigenvalues of A_f are the finite eigenvalues of $\lambda E - A$, and $\lambda E_\infty - I$ has only eigenvalues at infinity. Thus, if E is singular, then $\lambda E - A$ has n_f finite and n_∞ infinite eigenvalues which together form a set of generalized eigenvalues.

The pencil $\lambda E - A$ is called *stable* if all its finite eigenvalues belong to the open left half-plane $\mathbb{C}_-$. In this case, the solution of system (1) with $u(t)\equiv 0$ tends to zero as $t\to\infty$, and, hence, the DAE system (1) is *asymptotically stable* .

We introduce now the *spectral projectors* onto the left and right deflating subspaces of the pencil $\lambda E - A$ corresponding to the finite eigenvalues along the left and right deflating subspaces corresponding to the eigenvalue at infinity as

$$P_l = T_l \begin{bmatrix} I_{n_f} & 0 \\ 0 & 0 \end{bmatrix} T_l^{-1}, \qquad P_r = T_r^{-1} \begin{bmatrix} I_{n_f} & 0 \\ 0 & 0 \end{bmatrix} T_r. \tag{4}$$

Furthermore, the matrices

$$Q_l = I - P_l = T_l \begin{bmatrix} 0 & 0 \\ 0 & I_{n_\infty} \end{bmatrix} T_l^{-1}, \qquad Q_r = I - P_r = T_r^{-1} \begin{bmatrix} 0 & 0 \\ 0 & I_{n_\infty} \end{bmatrix} T_r \tag{5}$$

define the complementary projectors. All these projectors play an important role in model reduction of DAE systems.

Using the Weierstrass canonical form (3) and introducing

$$T_r x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}, \qquad T_l^{-1} B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \qquad C T_r^{-1} = [C_1,\, C_2], \tag{6}$$

we can decouple the DAE system (1) into a *slow subsystem*

$$\begin{aligned} \dot{x}_1(t) &= A_f x_1(t) + B_1 u(t), \\ y_1(t) &= C_1 x_1(t), \end{aligned} \tag{7}$$

and a *fast subsystem*

$$\begin{aligned} E_\infty \dot{x}_2(t) &= x_2(t) + B_2 u(t), \\ y_2(t) &= C_2 x_2(t) + D u(t). \end{aligned} \tag{8}$$

The output of (1) is then determined as $y(t) = y_1(t) + y_2(t)$.

Next, we introduce some algebraic properties of matrix triplets related to the DAE system (1). The equivalent definitions in terms of controllability and observability concepts relating to the dynamic behavior of the DAE system can be found in [36, 40]. We restrict here to the definition of the algebraic properties as these are used in the rest of this paper.

Definition 1. *Let the matrices Z_l and Z_r be of full rank such that* $\mathrm{im}(Z_l) = \mathrm{im}(E^T)$ *and* $\mathrm{im}(Z_r) = \mathrm{im}(E)$. *Then the matrix triplet* $(E,A,B) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m}$ *is called*

1) controllable in the behavioral sense *(*R-controllable*), if* $\mathrm{rank}[\lambda E - A,\ B] = n$ *for all* $\lambda \in \mathbb{C}$*;*

2) stabilizable in the behavioral sense *(*R-stabilizable*), if* $\mathrm{rank}[\lambda E - A,\ B] = n$ *for all* $\lambda \in \mathbb{C} \setminus \mathbb{C}_-$*;*

3) impulse controllable *(*I-controllable*), if* $\mathrm{rank}[E,\ AZ_r,\ B] = n$*;*

4) controllable at infinity *(*Inf-controllable*), if* $\mathrm{rank}[E,\ B] = n$*;*

5) strongly controllable *(*S-controllable*), if it is R-controllable and I-controllable;*

6) strongly stabilizable *(*S-stabilizable*), if it is R-stabilizable and I-controllable;*

7) completely controllable *(*C-controllable*), if it is R-controllable and Inf-controllable.*

The matrix triplet $(E,A,C) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times n} \times \mathbb{R}^{q\times n}$ *is called*

8) observable in the behavioral sense *(*R-observable*), if* $\mathrm{rank}[\lambda E^T - A^T,\ C^T] = n$ *for all* $\lambda \in \mathbb{C}$*;*

9) detectable in the behavioral sense *(*R-detectable*), if* $\mathrm{rank}[\lambda E^T - A^T,\ C^T] = n$ *for all* $\lambda \in \mathbb{C} \setminus \mathbb{C}_-$*;*

10) impulse observable *(*I-observable*), if* $\mathrm{rank}[E^T,\ A^T Z_l,\ C^T] = n$*;*

11) observable at infinity *(*Inf-observable*), if* $\mathrm{rank}[E^T,\ C^T] = n$*;*

12) strongly observable *(*S-observable*), if it is R-observable and I-observable;*

13) strongly detectable *(*S-detectable*), if it is R-detectable and I-observable;*

14) completely observable *(*C-observable*), if it is R-observable and Inf-observable.*

In the following, we will not distinguish the algebraic and system-theoretic properties of the matrix triplets (E,A,B), (E,A,C) and the corresponding DAE system (1) and speak equivalently, e.g., of R-controllability of (E,A,B) and the DAE system (1).

In the frequency domain, the input-output behavior of the DAE system (1) is described by a transfer function $\boldsymbol{H}(s) = C(sE - A)^{-1}B + D$ which is a rational matrix-valued function. On the other side, for any rational matrix-valued function $\boldsymbol{H}(s)$, one can always find the

matrices E, A, B, C and D such that $\boldsymbol{H}(s) = C(sE - A)^{-1}B + D$, e.g., [40]. Such a quintuple $\boldsymbol{H} = (E, A, B, C, D)$ is called a *realization* of $\boldsymbol{H}(s)$. If (E, A, B, C, D) is a realization of $\boldsymbol{H}(s)$, then for any nonsingular matrices W and T, (WET, WAT, WB, CT, D) is also a realization of $\boldsymbol{H}(s)$. This implies that $\boldsymbol{H}(s)$ has many different realizations. Moreover, there exist realizations of arbitrarily high order which is defined by the dimension of the matrices E and A. A realization $\boldsymbol{H} = (E, A, B, C, D)$ is called *minimal* if E and A have the smallest possible dimension. One can show that $\boldsymbol{H} = (E, A, B, C, D)$ is minimal if and only if system (1) is C-controllable, C-observable and $A\ker(E) \subseteq \mathrm{im}(E)$, see [147]. The latter condition means that the nilpotent matrix E_∞ in the Weierstrass canonical form (3) does not have any 1×1 Jordan blocks.

The transfer function $\boldsymbol{H}(s)$ is called *proper* if $H_\infty = \lim_{s\to\infty} \boldsymbol{H}(s)$ exists, and *improper*, otherwise. If $H_\infty = 0$, then $\boldsymbol{H}(s)$ is called *strictly proper*. Using (3) and (6), the transfer function $\boldsymbol{H}(s)$ can additively be decomposed as $\boldsymbol{H}(s) = \boldsymbol{H}_{sp}(s) + \boldsymbol{P}(s)$, where

$$\boldsymbol{H}_{sp}(s) = C_1(sI - A_f)^{-1}B_1$$

is the *strictly proper part* of $\boldsymbol{H}(s)$, and

$$\boldsymbol{P}(s) = C_2(sE_\infty - I)^{-1}B_2 + D = \sum_{j=0}^{\nu-1} M_j s^j$$

with

$$M_j = -C_2 E_\infty^j B_2 + \delta_{0,j} D \tag{9}$$

is the *polynomial part* of $\boldsymbol{H}(s)$. Here, $\delta_{0,j}$ denotes the Kronecker delta. Note that $\boldsymbol{H}_{sp}(s)$ and $\boldsymbol{P}(s)$ are the transfer functions of the slow and fast subsystems (7) and (8), respectively. If the realization $\boldsymbol{H} = (E, A, B, C, D)$ is not minimal, then the degree of the polynomial $\boldsymbol{P}(s)$, denoted by $\deg(\boldsymbol{P})$, may be smaller than $\nu - 1$.

The transfer function $\boldsymbol{H}(s)$ can also be written as

$$\boldsymbol{H}(s) = \frac{\boldsymbol{N}(s)}{\boldsymbol{d}(s)},$$

where $\boldsymbol{N}(s)$ is a $q \times m$ matrix polynomial and $\boldsymbol{d}(s)$ is a scalar polynomial which is the least common denominator of the qm entries of $\boldsymbol{H}(s)$. The roots of the denominator $\boldsymbol{d}(s)$ are called the *finite poles* of $\boldsymbol{H}(s)$, and the roots of the numerator $\boldsymbol{N}(s)$ are called the *finite zeros* of $\boldsymbol{H}(s)$. The transfer function $\boldsymbol{H}(s)$ has a *pole* (*zero*) *at infinity* if $s = 0$ is a pole (zero) of $\boldsymbol{H}(1/s)$. If $\deg(\boldsymbol{N}) > \deg(\boldsymbol{d})$ or, equivalently, if $\boldsymbol{H}(s)$ is improper, then $\boldsymbol{H}(s)$ has a pole at infinity. If $\deg(\boldsymbol{N}) < \deg(\boldsymbol{d})$ or, equivalently, if $\boldsymbol{H}(s)$ is strictly proper, then $\boldsymbol{H}(s)$ has a zero at infinity. The poles of $\boldsymbol{H}(s)$ are generalized eigenvalues of the pencil $\lambda E - A$. The set of poles of $\boldsymbol{H}(s)$ coincides with the set of generalized eigenvalues of $\lambda E - A$ if and only if $\boldsymbol{H} = (E, A, B, C, D)$ is minimal. For the square transfer function $\boldsymbol{H}(s)$, the zeros of $\boldsymbol{H}(s)$ are generalized eigenvalues of the *system pencil*

$$\lambda \begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} A & B \\ C & D \end{bmatrix},$$

see [121]. If this pencil is regular, then $\boldsymbol{H}(s)$ is invertible and its inverse is given by

$$\boldsymbol{H}^{-1}(s) = [0,\ -I] \begin{bmatrix} sE - A & -B \\ -C & -D \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ I \end{bmatrix}.$$

This representation immediately follows from the relation

$$\begin{bmatrix} sE-A & 0 \\ 0 & \boldsymbol{H}(s) \end{bmatrix} = \begin{bmatrix} I & 0 \\ C(sE-A)^{-1} & I \end{bmatrix} \begin{bmatrix} sE-A & -B \\ -C & -D \end{bmatrix} \begin{bmatrix} I & -(sE-A)^{-1}B \\ 0 & -I \end{bmatrix}.$$

Note that if D is nonsingular, then $\boldsymbol{H}^{-1}(s)$ can also be realized as

$$\boldsymbol{H}^{-1}(s) = -D^{-1}C(sE-A+BD^{-1}C)BD^{-1}+D^{-1}.$$

An invertible transfer function $\boldsymbol{H}(s)$ is called (strictly) *minimum phase* if all its finite zeros have (negative) non-positive real part.

Let $s_1,\ldots,s_k$ be the pairwise different finite poles of $\boldsymbol{H}(s)$ of order ℓ_j, $j=1,\ldots,k$, then $\boldsymbol{H}(s)$ can be represented using a partial fraction expansion as

$$\boldsymbol{H}(s) = \sum_{j=1}^{k}\sum_{i=1}^{\ell_j} \frac{R_j^{(i)}}{(s-s_j)^i} + \sum_{j=0}^{\nu-1} M_j s^j, \tag{10}$$

where $R_j \equiv R_j^{(1)}$ is the *residue* of $\boldsymbol{H}$ at s_j.

Another useful representation of the transfer function $\boldsymbol{H}(s)$ is given by its power series expansion at $s_0 \in \mathbb{C}$ being not a pole of $\boldsymbol{H}$:

$$\boldsymbol{H}(s) = \sum_{j=0}^{\infty} M_j(s_0)(s-s_0)^j, \tag{11}$$

where the coefficients $M_j(s_0)$, also called *(shifted) moments*[1], have the form

$$\begin{aligned} M_0(s_0) &= -C(A-s_0E)^{-1}B+D, \\ M_j(s_0) &= -C\big((A-s_0E)^{-1}E\big)^j(A-s_0E)^{-1}B, \quad j>0. \end{aligned}$$

For singular E, the Laurent expansion of $\boldsymbol{H}$ turns out to be beneficial as well:

$$\boldsymbol{H}(s) = \sum_{j=-\infty}^{\nu-1} M_j s^j, \tag{12}$$

where the coefficients M_j are the *Markov parameters* given by

$$\begin{aligned} M_j &= CT_r^{-1}\begin{bmatrix} A_f^{-j-1} & 0 \\ 0 & 0 \end{bmatrix} T_r^{-1}B = C_1A_f^{-j-1}B_1, && j<0, \\ M_j &= CT_r^{-1}\begin{bmatrix} 0 & 0 \\ 0 & -E_\infty^j \end{bmatrix} T_r^{-1}B+\delta_{0,j}D = -C_2E_\infty^jB_2+\delta_{0,j}D, && j\geq 0. \end{aligned}$$

Thus, the Markov parameters corresponding to the nonnegative powers are the same as the coefficients M_j in the partial fraction expansion (10) and in (9), and, therefore, they determine the polynomial part of $\boldsymbol{H}(s)$.

In order to measure the approximation error of reduced-order models, we will employ classical system norms. Let $\mathcal{H}_\infty$ denote the space of matrix-valued functions that are analytic and bounded in the open right half-plane. The $\mathcal{H}_\infty$-norm of $\boldsymbol{H}\in\mathcal{H}_\infty$ is defined as

$$\|\boldsymbol{H}\|_{\mathcal{H}_\infty} = \sup_{\omega\in\mathbb{R}} \|\boldsymbol{H}(i\omega)\|_2,$$

[1] Usually, the term *moments* is used to denote the coefficients of the Taylor series at $s_0=0$.

where $\|\cdot\|_2$ denotes the spectral matrix norm. Furthermore, we consider the space $\mathcal{H}_2$ of matrix-valued functions that are analytic in the open right half-plane. The $\mathcal{H}_2$-norm of $\boldsymbol{H} \in \mathcal{H}_2$ is defined as

$$\|\boldsymbol{H}\|_{\mathcal{H}_2} = \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} \|\boldsymbol{H}(i\omega)\|_F^2 d\omega \right)^{1/2},$$

where $\|\cdot\|_F$ denotes the Frobenius matrix norm. We note that the rational matrix-valued functions given by the transfer functions corresponding to (1) are in $\mathcal{H}_\infty$ if the system is stable and proper, and in $\mathcal{H}_2$ if, in addition, it is strictly proper.

3 Model order reduction techniques

Before describing different model reduction techniques, we would like to point out that most techniques are based on *(Petrov-)Galerkin projection*. The basic idea can simply be described as follows, where we use (1) as a model problem. Assuming the dynamics of the system evolves in a low dimensional subspace $\mathcal{T} \subset \mathbb{R}^n$ with basis matrix $T \in \mathbb{R}^{n \times r}$, we use the ansatz $x(t) \approx T\widetilde{x}(t)$. Hence, $\mathcal{T}$ is considered as *trial space*. Replacing $x(t)$ in the generalized state equation (the first equation in (1)), we obtain a residual

$$\widetilde{r}(t) := ET\dot{\widetilde{x}}(t) - AT\widetilde{x}(t) - Bu(t).$$

In general, the residual is not zero. Therefore, we demand it to at least vanish on an r-dimensional *test space* $\mathcal{W} \subset \mathbb{R}^n$ with basis matrix $W \in \mathbb{R}^{n \times r}$, so that T and W are bi-orthogonal, i.e., $W^T T = I_r$. The requirement $W^T \widetilde{r}(t) \equiv 0$ then leads to the reduced (generalized) state equation

$$W^T ET\dot{\widetilde{x}}(t) = W^T AT\widetilde{x}(t) + W^T Bu(t).$$

Applying the projection onto $\mathcal{T}$ also to the second equation in (1) leads to the reduced-order system

$$\left(\widetilde{E}, \widetilde{A}, \widetilde{B}, \widetilde{C}, \widetilde{D}\right) := \left(W^T ET, W^T AT, W^T B, CT, D\right).$$

This process is called *Petrov-Galerkin projection*, and TW^T defines an oblique projector onto $\mathcal{T}$. If one takes $\mathcal{W} \equiv \mathcal{T}$, necessitating to choose an orthogonal basis matrix T, i.e., $T^T T = I_r$, we speak of a *Galerkin projection* and TT^T defines an orthogonal projector onto $\mathcal{T}$.

Note that the method "balanced truncation" described in Subsection 3.1 is, in general, a Petrov-Galerkin projection (turning into a Galerkin projection for symmetric systems with $E = E^T$, $A = A^T$ and $C = B^T$), while the interpolatory approaches in Subsection 3.2 can either be Galerkin or Petrov-Galerkin projection methods.

3.1 Balanced truncation

Balanced truncation was initially introduced in the systems and control theory in the early eighties of the last century [47, 62, 96] and has been continuously developed ever since. Due to new developments in Numerical Linear Algebra, it becomes to be applicable to large-scale problems [12, 13, 14], and has already been used in many application areas including biochemical engineering [91], electrical circuit simulation [114, 116], mechanical systems [26, 113], computational fluid dynamics [21, 33, 71, 135] and power systems [52, 120].

A main idea of balanced truncation and its relatives is to transform a dynamical system to a balanced form defined in such a way that appropriately chosen controllability and observability Gramians are equal and diagonal. Then a reduced-order model is computed by truncating the states corresponding to the small diagonal elements of the Gramians. Depending on the choice of the Gramians, different balanced truncation techniques can be developed, see [14, 67] for surveys of balancing-related model reduction methods for standard state-space systems. In this section, we collect the extensions of these methods to DAE systems.

3.1.1 Lyapunov balanced truncation

The most commonly used balanced truncation method is based on balancing the *controllability* and *observability Gramians* G_c and G_o which are defined for system (1) with $E = I$ as unique symmetric, positive semidefinite solutions of the continuous-time Lyapunov equations

$$AG_c + G_cA^T = -BB^T, \qquad A^TG_o + G_oA = -C^TC,$$

provided all eigenvalues of the matrix A have negative real part. These Gramians characterize the controllability and observability properties of the control system and quantify the input and output energy [96]. The square roots of the eigenvalues of the product G_cG_o define the *Hankel singular values*, $\sigma_j = \sqrt{\lambda_j(G_cG_o)}$, which can be used to measure the importance of the state variables. We assume that σ_j are ordered decreasingly. Finding a balancing transformation T_b such that

$$T_bG_cT_b^T = T_b^{-T}G_oT_b^{-1} = \mathrm{diag}(\sigma_1, \ldots, \sigma_n)$$

and truncating $n-r$ components of the transformed state vector $T_bx(t)$, which correspond to small $\sigma_j < \sigma_r$, yields an asymptotically stable reduced-order model [104]. Another important property of this method is the presence of the computable error estimates

$$\begin{aligned} \|\widetilde{\boldsymbol{H}} - \boldsymbol{H}\|_{\mathcal{H}_\infty} &\leq 2(\sigma_{r+1} + \ldots + \sigma_n), \\ \|\,\widetilde{y} - y\,\|_{\mathcal{L}_2} &\leq \|\widetilde{\boldsymbol{H}} - \boldsymbol{H}\|_{\mathcal{H}_\infty}\|u\|_{\mathcal{L}_2} \leq 2(\sigma_{r+1} + \ldots + \sigma_n)\|u\|_{\mathcal{L}_2}, \end{aligned}$$

see [47, 62].

The Lyapunov-based balanced truncation approach was extended to DAEs in [15, 92, 103, 134]. A basic idea behind this extension is to decouple the DAE system (1) into the slow and fast subsystems (7) and (8), respectively, and reduce them separately. In the frequency domain, this corresponds to the separate approximation of the strictly proper part $\boldsymbol{H}_{sp}(s)$ and the polynomial part $\boldsymbol{P}(s)$ of the transfer function $\boldsymbol{H}(s) = \boldsymbol{H}_{sp}(s) + \boldsymbol{P}(s)$ resulting in an approximate system $\widetilde{\boldsymbol{H}}(s) = \widetilde{\boldsymbol{H}}_{sp}(s) + \widetilde{\boldsymbol{P}}(s)$. It should, however, be noticed that if $\widetilde{\boldsymbol{P}}(s) \neq \boldsymbol{P}(s)$ and $\deg(\boldsymbol{P}(s)) \geq 1$, then the error $\boldsymbol{H}(s) - \widetilde{\boldsymbol{H}}(s)$ is unbounded. Also in the time domain, a naive reduction of the order of the fast subsystem (8) which, actually, describes the constraints in the model, may lead to an inaccurate approximation, see [92, 138]. These difficulties have been resolved in [134] by determining a minimal realization of $\boldsymbol{P}(s)$. This guarantees that $\boldsymbol{P}(s) = \widetilde{\boldsymbol{P}}(s)$ and, hence, the error $\boldsymbol{H}(s) - \widetilde{\boldsymbol{H}}(s)$ will be small if the error in the slow subsystem $\boldsymbol{H}_{sp}(s) - \widetilde{\boldsymbol{H}}_{sp}(s)$ is small.

In practice, we do not need to compute the slow and fast subsystems explicitly. This is computationally expensive, especially for large-scale problems, and may be numerically ill-conditioned. Instead, we can define two pairs of controllability and observability Gramians in terms of the original data using the spectral projectors P_l, P_r and Q_l, Q_r given in (4) and

(5), respectively. Assume that the DAE system (1) is asymptotically stable. Then the *proper controllability* and *observability Gramians* G_{pc} and G_{po} of (1) are defined as unique symmetric, positive semidefinite solutions of the projected continuous-time Lyapunov equations

$$\begin{aligned} E\,G_{pc}\,A^T + A\,G_{pc}\,E^T &= -P_l B B^T P_l^T, \qquad G_{pc} = P_r G_{pc} P_r^T, \qquad (13)\\ E^T G_{po}\,A + A^T G_{po}\,E &= -P_r^T C^T C P_r, \qquad G_{po} = P_l^T G_{po} P_l, \qquad (14) \end{aligned}$$

respectively, whereas the *improper controllability* and *observability Gramians* G_{ic} and G_{io} of (1) are defined as unique symmetric, positive semidefinite solutions of the projected discrete-time Lyapunov equations

$$\begin{aligned} A\,G_{ic}A^T - E\,G_{ic}E^T &= Q_l B B^T Q_l^T, \qquad G_{ic} = Q_r G_{ic} Q_r^T, \qquad (15)\\ A^T G_{io}A - E^T G_{io}E &= Q_r^T C^T C Q_r, \qquad G_{io} = Q_l^T G_{io} Q_l, \qquad (16) \end{aligned}$$

respectively. The square roots of the largest n_f eigenvalues of $G_{pc}E^T G_{po}E$, denoted by σ_j, are called the *proper Hankel singular values* of (1), and the square roots of the largest n_∞ eigenvalues of $G_{ic}A^T G_{io}A$, denoted by θ_j, are called the *improper Hankel singular values*. System (1) is *balanced* if the Gramians satisfy

$$G_{pc} + G_{ic} = G_{po} + G_{io} = \operatorname{diag}(\sigma_1, \ldots \sigma_{n_f}, \theta_1, \ldots, \theta_{n_\infty}).$$

Thus, a reduced-order model (2) can be determined by truncating the states of the balanced system corresponding to the small proper Hankel singular values. In [134], it is shown that the states corresponding to the small eigenvalues of the proper controllability Gramian G_{pc} need the most energy to be reached. Also, the states corresponding to the small eigenvalues of the proper observability Gramian G_{po} contribute the least to the output energy

$$\mathbf{E}(y) = \int_0^\infty y(t)^T y(t)\,dt.$$

In balanced coordinates, the eigenvalues of G_{pc}, G_{po}, and the proper Hankel singular values coincide. Thus, the difficult-to-reach states coincide with those least involved in the output energy. Based on this energy interpretation of the proper Gramians, one can assert that these states are difficult to control and difficult to observe at the same time and can therefore be ignored in the system approximation. Furthermore, we can remove states which are not Inf-controllable and Inf-observable. Such states correspond to zero improper Hankel singular values.

Considering the Cholesky factorizations[2] of the Gramians

$$G_{pc} = Z_{pc}Z_{pc}^T, \qquad G_{po} = Z_{po}Z_{po}^T, \qquad G_{ic} = Z_{ic}Z_{ic}^T, \qquad G_{io} = Z_{io}Z_{io}^T,$$

and taking into account that the proper and improper Hankel singular values can be determined from the singular value decomposition of the matrices $Z_{po}^T E Z_{pc}$ and $Z_{io}^T A Z_{ic}$, respectively, we obtain the generalization of the square-root balanced truncation method [88, 143] for DAE systems shown in Algorithm 1. As in the standard state space case [47, 62], we have the error estimates

$$\begin{aligned} \|\widetilde{\boldsymbol{H}} - \boldsymbol{H}\|_{\mathcal{H}_\infty} &\le 2(\sigma_{r_f+1} + \ldots + \sigma_{n_f}),\\ \|\,\widetilde{y} - y\,\|_{\mathcal{L}_2} &\le \|\widetilde{\boldsymbol{H}} - \boldsymbol{H}\|_{\mathcal{H}_\infty}\|u\|_{\mathcal{L}_2} \le 2(\sigma_{r_f+1} + \ldots + \sigma_{n_f})\|u\|_{\mathcal{L}_2}. \end{aligned}$$

[2]It should be noted that by abuse of notation, these factors are neither necessarily upper triangular nor square, but we assume them to be of full rank. In particular, for non-minimal systems, these factors will in general be rectangular as then, the Gramians will be rank deficient.

Moreover, for $\boldsymbol{P}(s) \neq D$, one can show that the index of the reduced-order model is equal to $\deg(\boldsymbol{P})+1$ and does not exceed the index of the original system (1). If $\boldsymbol{P}(s) = D$, then the reduced-order model is an ODE system.

Algorithm 1 Lyapunov balanced truncation for DAE systems.

Input: an asymptotically stable system $\boldsymbol{H} = (E, A, B, C, D)$.

Output: a reduced-order asymptotically stable system $\widetilde{\boldsymbol{H}} = (\widetilde{E}, \widetilde{A}, \widetilde{B}, \widetilde{C}, \widetilde{D})$.

1: Compute the full rank Cholesky factors Z_{pc} and Z_{po} of the proper Gramians $G_{pc} = Z_{pc}Z_{pc}^T$ and $G_{po} = Z_{po}Z_{po}^T$ satisfying the projected Lyapunov equations (13) and (14), respectively.

2: Compute the full rank Cholesky factors Z_{ic} and Z_{io} of the improper Gramians $G_{ic} = Z_{ic}Z_{ic}^T$ and $G_{io} = Z_{io}Z_{io}^T$ satisfying the projected Lyapunov equations (15) and (16), respectively.

3: Compute the singular value decomposition $Z_{po}^T E Z_{pc} = [U_1, U_2]\mathrm{diag}(\Sigma_1, \Sigma_2)[V_1, V_2]^T$, where the matrices $[U_1, U_2]$ and $[V_1, V_2]$ have orthonormal columns, $\Sigma_1 = \mathrm{diag}(\sigma_1, \ldots, \sigma_{r_f})$ and $\Sigma_2 = \mathrm{diag}(\sigma_{r_f+1}, \ldots, \sigma_{n_f})$.

4: Compute the singular value decomposition $Z_{io}^T A Z_{ic} = U_3 \Theta V_3^T$, where U_3 and V_3 have orthonormal columns and Θ is nonsingular.

5: Compute the reduced-order system $(\widetilde{E}, \widetilde{A}, \widetilde{B}, \widetilde{C}, \widetilde{D}) = (W^T E T, W^T A T, W^T B, C T, D)$ with $W = [Z_{po} U_1 \Sigma_1^{-1/2}, Z_{io} U_3 \Theta^{-1/2}]$ and $T = [Z_{pc} V_1 \Sigma_1^{-1/2}, Z_{ic} V_3 \Theta^{-1/2}]$.

Using the Weierstrass canonical form (3), one notices that the improper Gramians G_{ic} and G_{io} have usually low rank which can be estimated as

$$r_c = \mathrm{rank}(G_{ic}) \leq \min(\nu m, n_\infty), \qquad r_o = \mathrm{rank}(G_{io}) \leq \min(\nu q, n_\infty),$$

where ν is the index of (1). Furthermore, if the eigenvalues of the proper Gramians G_{pc} and G_{po} decay fast, then G_{pc} and G_{po} have low numerical rank. In this case, they can be well approximated by low-rank matrices $G_{pc} \approx \widetilde{Z}_{pc}\widetilde{Z}_{pc}^T$ and $G_{po} \approx \widetilde{Z}_{po}\widetilde{Z}_{po}^T$, where $\widetilde{Z}_{pc} \in \mathbb{R}^{n \times n_c}$ and $\widetilde{Z}_{po} \in \mathbb{R}^{n \times n_o}$ with $n_c, n_o \ll n$. Replacing the full rank factors Z_{pc} and Z_{po} in Algorithm 1 by the low-rank matrices $\widetilde{Z}_{pc}$ and $\widetilde{Z}_{po}$, respectively, reduces significantly the computational complexity and storage requirements for the balanced truncation method, making it applicable to large-scale problems. In fact, apart from solving the projected Lyapunov equations, only the singular value decomposition of the small matrices $\widetilde{Z}_{po}^T E \widetilde{Z}_{pc} \in \mathbb{R}^{n_o \times n_c}$ and $Z_{io}^T A Z_{ic} \in \mathbb{R}^{r_o \times r_c}$ needs to be computed. The computation of the (low-rank) Cholesky factors of the Gramians will be discussed in Section 4.1.

Remark 1. *Unfortunately, in the literature [106, 142], one can often find the statement that the extension of balanced truncation from standard state-space systems to DAEs is as simple as to replace the identity matrix by E. In this case, the Lyapunov equations take the form*

$$A G_c E^T + E G_c A^T = -B B^T, \qquad A^T G_o E + E^T G_o A = -C^T C.$$

It should, however, be noted that for singular E, these equations may not be solvable even if the pencil $\lambda E - A$ is stable. Moreover, if the solutions exist, they are always non-unique. Hence, their use does not lead to a well-defined model reduction method.

In the Poor Man's truncated balanced reduction (PMTBR) method presented in [106], it

was proposed to define the Gramians of system (1) as

$$\begin{aligned} X &= \frac{1}{2\pi}\int_{-\infty}^{\infty}(i\omega E-A)^{-1}BB^T(-i\omega E-A)^{-T}d\omega, \\ Y &= \frac{1}{2\pi}\int_{-\infty}^{\infty}(-i\omega E-A)^{-T}C^TC(i\omega E-A)^{-1}d\omega. \end{aligned}$$

However, if E is singular, these integrals do not converge unless $B = P_lB$ and $C = CP_r$. Therefore, the correct definition should be

$$\begin{aligned} X &= \frac{1}{2\pi}\int_{-\infty}^{\infty}(i\omega E-A)^{-1}P_lBB^TP_l^T(-i\omega E-A)^{-T}d\omega, \\ Y &= \frac{1}{2\pi}\int_{-\infty}^{\infty}(-i\omega E-A)^{-T}P_r^TC^TCP_r(i\omega E-A)^{-1}d\omega. \end{aligned} \tag{17}$$

It is worth noting that these matrices solve the projected Lyapunov equations (13) and (14), respectively, that again justifies the above considerations.

3.1.2 Positive real balanced truncation

Positive real balanced truncation was first developed for standard state-space systems in [70, 98] as a model reduction method preserving passivity. It was then extended to DAEs in [115].

The DAE system (1) is called *passive* if $m = q$ and

$$\int_0^t u(\tau)^T y(\tau)\,d\tau \geq 0$$

for all $t > 0$ and all $u \in \mathcal{L}_2([0,t],\mathbb{R}^m)$ consistent with $x(0) = 0$. Physically, this property means that the system does not generate energy. It is of great importance especially for circuit equations. One can show that system (1) is passive if and only if its transfer function $\boldsymbol{H}(s)$ is *positive real*, i.e., $\boldsymbol{H}(s)$ is analytic in the open right half-plane $\mathbb{C}_+$ and $\boldsymbol{H}(s) + \boldsymbol{H}^*(s) \geq 0$ for all $s \in \mathbb{C}_+$, see [6]. Passivity of the DAE system (1) can also be characterized via the projected positive real Lur'e equations

$$\begin{aligned} AXE^T + EXA^T &= -K_cK_c^T, & X = P_rXP_r^T \geq 0, \\ EXC^T - P_lB &= -K_cJ_c^T, & M_0 + M_0^T = J_cJ_c^T, \end{aligned} \tag{18}$$

and

$$\begin{aligned} A^TYE + E^TYA &= -K_o^TK_o, & Y = P_l^TYP_l \geq 0, \\ E^TYB - P_r^TC^T &= -K_o^TJ_o, & M_0 + M_0^T = J_o^TJ_o, \end{aligned} \tag{19}$$

with M_0 as in (9) and unknowns $K_c, K_o^T \in \mathbb{R}^{n\times m}$, $J_c, J_o \in \mathbb{R}^{m\times m}$ and $X, Y \in \mathbb{R}^{n\times n}$. If system (1) is R-controllable, R-observable and passive, then the projected Lur'e equations (18) are solvable. Conversely, the solvability of (18) together with the conditions $M_1 = M_1^T \geq 0$ and $M_j = 0$ for $j > 1$ implies that (1) is passive. A similar result holds also for the dual Lur'e equations (19). Note that for some structured systems as they arise, for example, in modified nodal analysis (MNA) of electrical circuits, the existence of the solutions of the projected Lur'e equations can also be proved without R-controllability and R-observability conditions [114]. It should be emphasized that the solutions of (18) and (19) are not unique. There exist, however, unique extremal solutions satisfying

$$X_{\max} \geq X \geq X_{\min} \geq 0, \qquad Y_{\max} \geq Y \geq Y_{\min} \geq 0$$

for all symmetric solutions X and Y of (18) and (19), respectively. The minimal solutions $G_c^{PR} = X_{\min}$ and $G_o^{PR} = Y_{\min}$ are called, respectively, the *positive real controllability* and *observability Gramians* of system (1). Replacing the proper Gramians in the Lyapunov-based balanced truncation method by the positive real Gramians, we obtain the passivity-preserving model reduction method for DAE systems. In order to determine the positive real Gramians from the Lur'e equations (18) and (19), we need first to calculate M_0. This matrix can be obtained from the polynomial part $\boldsymbol{P}(s)$ whose realization is given by $\boldsymbol{P} = (W_\infty^T E T_\infty, W_\infty^T A T_\infty, W_\infty^T B, C T_\infty, D)$ with $W_\infty = Z_{io} U_3 \Theta^{-1/2}$ and $T_\infty = Z_{ic} V_3 \Theta^{-1/2}$. Since $W_\infty^T A T_\infty = I$, we have

$$M_0 = D - C T_\infty W_\infty^T B = D - C Z_{ic} V_3 \Theta^{-1} U_3^T Z_{io}^T B.$$

The resulting positive real balanced truncation method is presented in Algorithm 2.

Algorithm 2 Positive real balanced truncation for DAE systems.

Input: a passive system $\boldsymbol{H} = (E, A, B, C, D)$.

Output: a reduced-order passive system $\widetilde{\boldsymbol{H}} = (\widetilde{E}, \widetilde{A}, \widetilde{B}, \widetilde{C}, \widetilde{D})$.

1: Compute the full rank Cholesky factors Z_{ic} and Z_{io} of the improper Gramians $G_{ic} = Z_{ic} Z_{ic}^T$ and $G_{io} = Z_{io} Z_{io}^T$ satisfying the projected Lyapunov equations (15) and (16), respectively.

2: Compute the singular value decomposition $Z_{io}^T A Z_{ic} = U_3 \Theta V_3^T$ with nonsingular Θ.

3: Compute the matrix $M_0 = D - C Z_{ic} V_3 \Theta^{-1} U_3^T Z_{io}^T B$.

4: Compute the Cholesky factors Z_c^{PR} and Z_o^{PR} of the positive real Gramians $G_c^{PR} = Z_c^{PR} (Z_c^{PR})^T$ and $G_o^{PR} = Z_o^{PR} (Z_o^{PR})^T$ that are the minimal solutions of the positive real projected Lur'e equations (18) and (19), respectively.

5: Compute the singular value decomposition

$$(Z_o^{PR})^T E Z_c^{PR} = [U_1, U_2] \operatorname{diag}(\Sigma_1^{PR}, \Sigma_2^{PR}) [V_1, V_2]^T,$$

where $\Sigma_1^{PR} = \operatorname{diag}(\sigma_1^{PR}, \ldots, \sigma_{r_f}^{PR})$ and $\Sigma_2^{PR} = \operatorname{diag}(\sigma_{r_f+1}^{PR}, \ldots, \sigma_{n_f}^{PR})$.

6: Compute the reduced-order system $(\widetilde{E}, \widetilde{A}, \widetilde{B}, \widetilde{C}, \widetilde{D}) = (W^T E T, W^T A T, W^T B, C T, D)$ with $W = [Z_o^{PR} U_1 (\Sigma_1^{PR})^{-1/2}, Z_{io} U_3 \Theta^{-1/2}]$ and $T = [Z_c^{PR} V_1 (\Sigma_1^{PR})^{-1/2}, Z_{ic} V_3 \Theta^{-1/2}]$.

The values $\sigma_1^{PR} \geq \ldots \geq \sigma_{r_f}^{PR} > \sigma_{r_f+1}^{PR} \geq \ldots \geq \sigma_{n_f}^{PR}$ are called the *positive real characteristic values* of (1). Similar to the proper Hankel singular values, they can be used to estimate the approximation error. If $M_0 + M_0^T$ is nonsingular, we have the error bound

$$\|\widetilde{\boldsymbol{H}} - \boldsymbol{H}\|_{\mathcal{H}_\infty} \leq 2 \|(M_0 + M_0^T)^{-1}\|_2 \|\boldsymbol{H} + M_0^T\|_{\mathcal{H}_\infty} \|\widetilde{\boldsymbol{H}} + M_0^T\|_{\mathcal{H}_\infty} \sum_{j=r_f+1}^{n_f} \sigma_j^{PR}$$

that can be derived for DAE systems similarly to the standard state-space case [67].

The positive real balanced truncation method requires solving the projected Lur'e equations. The numerical solution of standard Lur'e equations based on deflating subspaces of a certain even pencil has been considered in [107, 108]. However, so far no numerical method has been developed for projected Lur'e equations. In the case where $R_0 = M_0 + M_0^T$ is nonsingular, the projected Lur'e equations (18) and (19) can be written as the projected positive real Riccati equations

$$A X E^T + E X A^T + (E X C^T - P_l B) R_0^{-1} (E X C^T - P_l B)^T = 0, \qquad X = P_r X P_r^T$$

and

$$A^TYE + E^TYA + (B^TYE - CP_r)^T R_0^{-1}(B^TYE - CP_r) = 0, \qquad Y = P_l^T Y P_l,$$

respectively. Such equations can be solved using Newton's method [32] briefly described in Section 4.2.

An alternative approach for passivity-preserving model reduction has been proposed in [145]. It relies on a combination of Lyapunov balancing and positive real balancing and involves solving only one Lyapunov equation and one Lur'e equation. However, there exists no error bound for this approach.

3.1.3 Bounded real balanced truncation

If, instead of passivity, we aim to preserve contractivity, an important property in $\mathcal{L}_2$-gain constraint controller design, then bounded real balanced truncation [98, 100, 115] has to be used. The DAE system (1) is called *contractive* if

$$\int_0^t \|u(\tau)\|^2 - \|y(\tau)\|^2 \, d\tau \geq 0$$

for all $t > 0$ and all $u \in \mathcal{L}_2([0,t],\mathbb{R}^m)$ consistent with $x(0) = 0$. This condition implies that the $\mathcal{L}_2$-norm of the output is bounded by the $\mathcal{L}_2$-norm of the input. In the frequency domain, contractivity is equivalent to *bounded realness* of the transfer function $\boldsymbol{H}(s)$ meaning that $\boldsymbol{H}(s)$ is analytic in $\mathbb{C}_+$ and $I - \boldsymbol{H}(s)^* \boldsymbol{H}(s) - I \geq 0$ for all $s \in \mathbb{C}_+$. The latter condition yields that the bounded real transfer function $\boldsymbol{H}(s)$ is necessarily proper.

To verify contractivity, we use the projected bounded real Lur'e equations

$$\begin{aligned} AXE^T + EXA^T + P_l BB^T P_l^T &= -K_c K_c^T, \quad & X = P_r X P_r^T \geq 0, \\ EXC^T + P_l BM_0^T &= -K_c J_c^T, \quad & I - M_0 M_0^T = J_c J_c^T, \end{aligned} \tag{20}$$

and

$$\begin{aligned} A^TYE + E^TYA + P_r^T C^T C P_r &= -K_o^T K_o, \quad & Y = P_l^T Y P_l \geq 0, \\ E^TYB + P_r^T C^T M_0 &= -K_o^T J_o, \quad & I - M_0^T M_0 = J_o^T J_o. \end{aligned} \tag{21}$$

Similarly to the positive real case, one can show that these equations have the minimal solutions $G_c^{BR} = X_{\min}$ and $G_o^{BR} = Y_{\min}$ that are called the *bounded real controllability* and *observability Gramians*, respectively. They can be used to characterize the required supply energy and the available storage energy for contractive systems [115]. This immediately leads to the bounded real balanced truncation method presented in Algorithm 3.

One can show that the reduced-order system computed by Algorithm 3 is contractive and has the error bound

$$\|\widetilde{\boldsymbol{H}} - \boldsymbol{H}\|_{\mathcal{H}_\infty} \leq 2 \sum_{j=r_f+1}^{n_f} \sigma_j^{BR}$$

with the *bounded real characteristic values* σ_j^{BR}.

If $R_c = I - M_0 M_0^T$ is nonsingular, then $R_o = I - M_0^T M_0$ is also nonsingular and the projected Lur'e equations (20) and (21) are equivalent to the projected bounded real Riccati equations

$$\begin{aligned} AXE^T + EXA^T + P_l BB^T P_l^T + (EXC^T + P_l BM_0^T) R_c^{-1} (EXC^T + P_l BM_0^T)^T &= 0, \\ X - P_r X P_r^T &= 0, \end{aligned}$$

Algorithm 3 Bounded real balanced truncation for DAE systems.

Input: a contractive system $\boldsymbol{H} = (E, A, B, C, D)$.

Output: a reduced-order contractive system $\widetilde{\boldsymbol{H}} = (\widetilde{E}, \widetilde{A}, \widetilde{B}, \widetilde{C}, \widetilde{D})$.

1: Compute the full rank Cholesky factors Z_{ic} and Z_{io} of the improper Gramians $G_{ic} = Z_{ic}Z_{ic}^T$ and $G_{io} = Z_{io}Z_{io}^T$ satisfying the projected Lyapunov equations (15) and (16), respectively.

2: Compute the singular value decomposition $Z_{io}^T A Z_{ic} = U_3 \Theta V_3^T$ with nonsingular Θ.

3: Compute the matrix $M_0 = D - CZ_{ic}V_3\Theta^{-1}U_3^T Z_{io}^T B$.

4: Compute the Cholesky factors Z_c^{BR} and Z_o^{BR} of the bounded real Gramians $G_c^{BR} = Z_c^{BR}(Z_c^{BR})^T$ and $G_o^{BR} = Z_o^{BR}(Z_o^{BR})^T$ that are the minimal solutions of the bounded real projected Lur'e equations (20) and (21), respectively.

5: Compute the singular value decomposition

$$(Z_o^{BR})^T E Z_c^{BR} = [U_1, U_2]\,\mathrm{diag}(\Sigma_1^{BR}, \Sigma_2^{BR})[V_1, V_2]^T,$$

where $\Sigma_1^{BR} = \mathrm{diag}(\sigma_1^{BR}, \ldots, \sigma_{r_f}^{BR})$ and $\Sigma_2^{BR} = \mathrm{diag}(\sigma_{r_f+1}^{BR}, \ldots, \sigma_{n_f}^{BR})$.

6: Compute the reduced-order system $(\widetilde{E}, \widetilde{A}, \widetilde{B}, \widetilde{C}, \widetilde{D}) = (W^T E T, W^T A T, W^T B, CT, M_0)$ with the projection matrices $W = Z_o^{BR} U_1 (\Sigma_1^{BR})^{-1/2}$ and $T = Z_c^{BR} V_1 (\Sigma_1^{BR})^{-1/2}$.

and

$$A^T Y E + E^T Y A + P_r^T C^T C P_r + (B^T Y E + M_0^T C P_r)^T R_o^{-1} (B^T Y E + M_0^T C P_r) = 0,$$
$$Y - P_l^T Y P_l = 0,$$

respectively. These equations can be solved using Newton's method described in Section 4.

Note that the bounded real systems are related to the positive real systems via a *Moebius transformation* defined as

$$\boldsymbol{H}_M(s) = \big(I - \boldsymbol{H}(s)\big)\big(I + \boldsymbol{H}(s)\big)^{-1}.$$

The transfer function $\boldsymbol{H}(s)$ is positive real if and only if the Moebius-transformed function $\boldsymbol{H}_M(s)$ is bounded real. For $\boldsymbol{H} = (E, A, B, C, D)$, a realization of $\boldsymbol{H}_M(s)$ is given by

$$\boldsymbol{H}_M = \big(E, A - B(I+D)^{-1}C, -\sqrt{2}B(I+D)^{-1}, \sqrt{2}(I+D)^{-1}C, (I-D)(I+D)^{-1}\big),$$

provided $I + D$ is invertible. This suggests another passivity-preserving balancing-related model reduction approach which consists of applying the bounded real balanced truncation method to $\boldsymbol{H}_M$ and computing the Moebius transformation $\widetilde{\boldsymbol{H}}(s) = \big(I - \widetilde{\boldsymbol{H}}_M(s)\big)\big(I + \widetilde{\boldsymbol{H}}_M(s)\big)^{-1}$ of the obtained reduced-order model $\widetilde{\boldsymbol{H}}_M$. This approach might be useful if the spectral projectors for the Moebius-transformed system are easier to compute than that for the original systems. Circuit equations belong, for example, to this class of problems [114].

3.1.4 Stochastic balanced truncation

Stochastic balanced truncation belongs to relative error model reduction methods attempting to minimize the relative error $\boldsymbol{H}^{-1}(\boldsymbol{H} - \widetilde{\boldsymbol{H}})$ in an appropriate norm. It was first introduced for discrete-time and continuous-time standard state space systems in [44, 70] and studied further in [30, 65, 66, 146]. The stochastic balanced truncation method relies on

an approximation of spectral factors of the *power spectrum* $\boldsymbol{\Phi}(s) = \boldsymbol{H}(s)\boldsymbol{H}^T(-s)$ and is known to preserve the right half-plane zeros of $\boldsymbol{H}(s)$. In this section, we present an extension of this method to DAEs.

Assume that system (1) is asymptotically stable and has a square proper and invertible transfer function $\boldsymbol{H}(s)$. Using the spectral projectors P_l and P_r, $\boldsymbol{H}(s)$ can then be written as $\boldsymbol{H}(s) = CP_r(sE-A)^{-1}P_lB + M_0$. Then the power spectrum can be written as

$$\begin{aligned}\boldsymbol{\Phi}(s) &= \boldsymbol{H}(s)\boldsymbol{H}^T(-s)\\ &= \begin{bmatrix} CP_r, & M_0B^TP_l^T \end{bmatrix} \begin{bmatrix} sE-A & -P_lBB^TP_l^T \\ 0 & -sE^T-A^T \end{bmatrix}^{-1} \begin{bmatrix} P_lBM_0^T \\ P_r^TC^T \end{bmatrix} + M_0M_0^T.\end{aligned}$$

Taking into account that the proper controllability Gramian G_{pc} solves the Lyapunov equation (13), we obtain

$$\begin{bmatrix} sE-A & -P_lBB^TP_l^T \\ 0 & -sE^T-A^T \end{bmatrix} = \begin{bmatrix} I & -EG_{pc} \\ 0 & I \end{bmatrix} \begin{bmatrix} sE-A & 0 \\ 0 & -sE^T-A^T \end{bmatrix} \begin{bmatrix} I & -G_{pc}E^T \\ 0 & I \end{bmatrix}.$$

Therefore, introducing $B_0 = P_lBM_0^T + EG_{pc}C^T = P_lB_0$, we have

$$\begin{aligned}\boldsymbol{\Phi}(s) &= \begin{bmatrix} CP_r, & B_0^T \end{bmatrix} \begin{bmatrix} sE-A & 0 \\ 0 & -sE^T-A^T \end{bmatrix}^{-1} \begin{bmatrix} B_0 \\ P_r^TC^T \end{bmatrix} + M_0M_0^T\\ &= CP_r(sE-A)^{-1}B_0 + B_0^T(-sE-A)^{-T}P_r^TC^T + M_0M_0^T\\ &= \boldsymbol{Z}(s) + \boldsymbol{Z}^T(-s)\end{aligned}$$

with $\boldsymbol{Z}(s) = CP_r(sE-A)^{-1}B_0 + M_0M_0^T/2$. Since $\lambda E - A$ is stable and

$$\boldsymbol{Z}(i\omega) + \boldsymbol{Z}^*(i\omega) = \boldsymbol{H}(i\omega)\boldsymbol{H}^*(i\omega) \geq 0$$

for all $\omega \in \mathbb{R}$, it follows from [6, Theorem 2.7.2] that $\boldsymbol{Z}(s)$ is positive real. If $\boldsymbol{Z}$ is R-controllable and R-observable, then using the results from Section 3.1.2 we obtain that the corresponding positive real Lur'e equations

$$\begin{aligned} AXE^T + EXA^T &= -K_cK_c^T, & X &= P_rXP_r^T \geq 0,\\ EXC^T - B_0 &= -K_cJ_c^T, & M_0M_0^T &= J_cJ_c^T, \end{aligned} \tag{22}$$

and

$$\begin{aligned} A^TYE + E^TYA &= -K_o^TK_o, & Y &= P_l^TYP_l \geq 0,\\ E^TYB_0 - P_r^TC^T &= -K_o^TJ_o, & M_0M_0^T &= J_o^TJ_o \end{aligned} \tag{23}$$

are solvable. They have two extremal solutions satisfying

$$X_{\max} \geq X \geq X_{\min} \geq 0, \qquad Y_{\max} \geq Y \geq Y_{\min} \geq 0$$

for all symmetric solutions X and Y of (22) and (23), respectively. Moreover, one can also show that $X_{\max} = (E^TY_{\min}E)_r^-$, where $(M)_r^-$ denotes a *reflexive inverse* of M with respect to P_r^T and P_r which is defined as the unique solution of the matrix equations

$$(M)_r^-M(M)_r^- = (M)_r^-, \qquad M(M)_r^- = P_r^T, \qquad (M)_r^-M = P_r.$$

Consider now $\boldsymbol{W}(s)$ being a square right spectral factor of the power spectrum $\boldsymbol{\Phi}(s) = \boldsymbol{H}(s)\boldsymbol{H}^T(-s) = \boldsymbol{W}^T(-s)\boldsymbol{W}(s)$. Its realization can be determined using the matrix equations

(13) and (23). We have

$$\begin{aligned}\boldsymbol{\Phi}(s) &= \boldsymbol{Z}(s)+\boldsymbol{Z}^T(-s)\\ &= (CP_r-B_0^TYE)(sE-A)^{-1}B_0+B_0^T(-sE-A)^{-T}(CP_r-B_0^TYE)^T+M_0M_0^T\\ &\quad +B_0^TYE(sE-A)^{-1}B_0+B_0^T(-sE-A)^{-T}E^TYB_0\\ &= J_o^TK_o(sE-A)^{-1}B_0+B_0^T(-sE-A)^{-T}K_o^TJ_o+J_oJ_o^T\\ &\quad +B_0^T(-sE-A)^{-T}K_o^TK_o(sE-A)^{-1}B_0\\ &= \left(K_o(-sE-A)^{-1}B_0+J_o\right)^T\left(K_o(sE-A)^{-1}B_0+J_o\right),\end{aligned}$$

and, hence, $\boldsymbol{W}(s)=K_o(sE-A)^{-1}B_0+J_o$. Similarly to the standard state space case [110], we can show that for the minimal solution $Y_{\min}$ of (23), all finite eigenvalues of the pencil

$$\lambda\begin{bmatrix}E & 0\\ 0 & 0\end{bmatrix}-\begin{bmatrix}A & B_0\\ K_o & J_o\end{bmatrix}$$

have non-positive real part. Therefore, $\boldsymbol{W}(s)$ has no zeros in the open right half-plane meaning that $\boldsymbol{W}(s)$ is minimum phase. The matrices $G_c^S=G_{pc}$ and $G_o^S=Y_{\min}$ define the *stochastic controllability* and *observability Gramians* of system (1). A reduced-order model can then be computed by balancing these Gramians and truncating the states corresponding to small *stochastic characteristic values* σ_j^S defined as $\sigma_j^S=\sqrt{\lambda_j(G_c^SE^TG_o^SE)}$. The stochastic balanced truncation method is summarized in Algorithm 4.

Algorithm 4 Stochastic balanced truncation for DAE systems.

Input: an asymptotically stable system $\boldsymbol{H}=(E,A,B,C,D)$ with the proper and invertible transfer function.

Output: a reduced-order asymptotically stable system $\widetilde{\boldsymbol{H}}=(\widetilde{E},\widetilde{A},\widetilde{B},\widetilde{C},\widetilde{D})$.

1: Compute the full rank Cholesky factors Z_{ic} and Z_{io} of the improper Gramians $G_{ic}=Z_{ic}Z_{ic}^T$ and $G_{io}=Z_{io}Z_{io}^T$ satisfying the projected Lyapunov equations (15) and (16), respectively.

2: Compute the singular value decomposition $Z_{io}^TAZ_{ic}=U_3\Theta V_3^T$ with nonsingular Θ.

3: Compute $M_0=D-CZ_{ic}V_3\Theta^{-1}U_3^TZ_{io}^TB$.

4: Compute the Cholesky factors Z_c^S and Z_o^S of the stochastic controllability Gramian $G_c^S=Z_c^S(Z_c^S)^T=G_{pc}$ satisfying (13) and the stochastic observability Gramian $G_o^S=Z_o^S(Z_o^S)^T$ which is the minimal solution of the projected Lur'e equation (23).

5: Compute the singular value decomposition

$$(Z_o^S)^TEZ_c^S=[U_1,U_2]\operatorname{diag}(\Sigma_1^S,\Sigma_2^S)[V_1,V_2]^T,$$

where $\Sigma_1^S=\operatorname{diag}(\sigma_1^S,\ldots,\sigma_{r_f}^S)$ and $\Sigma_2^S=\operatorname{diag}(\sigma_{r_f+1}^S,\ldots,\sigma_{n_f}^S)$.

6: Compute the reduced-order system $(\widetilde{E},\widetilde{A},\widetilde{B},\widetilde{C},\widetilde{D})=(W^TET,W^TAT,W^TB,CT,M_0)$ with the projection matrices $W=Z_o^SU_1(\Sigma_1^S)^{-1/2}$ and $T=Z_c^SV_1(\Sigma_1^S)^{-1/2}$.

Since $X=G_{pc}$ solves (22), we have $(E^TY_{\min}E)_r^-=X_{\max}\geq G_{pc}$, and, hence, the eigenvalues of $G_{pc}E^TY_{\min}E=G_c^SE^TG_o^SE$ do not exceed one. This implies that the stochastic characteristic values of (1) satisfy $0\leq\sigma_j^S\leq 1$. Moreover, it follows from [65, Theorem 4.1] that $\boldsymbol{H}(s)$ has $k_z=\dim\left(\ker((E^TY_{\min}E)_r^--G_{pc})\right)-n_\infty$ infinite zeros and finite zeros in the closed right half-plane, and $\sigma_1^S=\ldots=\sigma_{k_z}^S=1$. Similarly to [65, 66], one can show that if

$r_f \geq k_z$ in Algorithm 4, then $\boldsymbol{H}(s)$ and $\widetilde{\boldsymbol{H}}(s)$ have the same zeros in the closed right half-plane, and the relative error bound

$$\|\boldsymbol{H}^{-1}(\boldsymbol{H}-\widetilde{\boldsymbol{H}})\|_{\mathcal{H}_\infty} \leq \prod_{j=r_f+1}^{n_f} \frac{1+\sigma_j^S}{1-\sigma_j^S} - 1$$

holds. Thus, if $\boldsymbol{H}(s)$ is minimum phase, then $\widetilde{\boldsymbol{H}}(s)$ is also minimum phase.

If M_0 is nonsingular, then the projected Lur'e equation (23) reduces to the projected Riccati equation

$$A^T Y E + E^T Y A + (B_0^T Y E - C P_r)^T (M_0 M_0^T)^{-1} (B_0^T Y E - C P_r) = 0, \quad Y = P_l^T Y P_l.$$

It has been shown in [159] that for standard state space systems with the invertible and strictly minimum phase transfer function $\boldsymbol{H}(s)$, the stochastic balanced truncation method is equivalent to a frequency-weighted balanced truncation approach with $\boldsymbol{H}^{-1}(s)$ as an output weight and I as an input weight. This approach is based on balancing the controllability Gramian of $\boldsymbol{H}$ against the observability Gramian of $\boldsymbol{H}^{-1}$. It can also be extended to the DAE system (1). If M_0 is nonsingular, then $\boldsymbol{H}^{-1}(s)$ can be realized as

$$\boldsymbol{H}^{-1} = (E,\, A - P_l B M_0^{-1} C P_r,\, P_l B M_0^{-1},\, -M_0^{-1} C P_r,\, M_0^{-1}).$$

The proper observability Gramian $\hat{G}_{po}$ of $\boldsymbol{H}^{-1}$ is defined as the solution of the projected Lyapunov equation

$$\begin{aligned}(A - P_l B M_0^{-1} C P_r)^T \hat{G}_{po} E + E^T \hat{G}_{po} (A - P_l B M_0^{-1} C P_r) &= -P_r^T C^T (M_0 M_0^T)^{-1} C P_r,\\ \hat{G}_{po} &= P_l^T \hat{G}_{po} P_l.\end{aligned}$$

The stochastic characteristic values σ_j^S are related to the new characteristic values $\hat{\sigma}_j = \sqrt{\lambda_j(G_{pc} E^T \hat{G}_{po} E)}$ via $\sigma_j^S = \hat{\sigma}_j / \sqrt{(1+\hat{\sigma}_j^2)}$, see [158]. Thus, if (1) is asymptotically stable, $\boldsymbol{H}(s)$ is strictly minimum phase and M_0 is nonsingular, then the stochastic balanced truncation method involves solving two projected Lyapunov equations, and, hence, it is as expensive as Lyapunov-based balanced truncation.

3.1.5 LQG balanced truncation

Another balancing-related model reduction approach is linear-quadratic Gaussian (LQG) balanced truncation developed first for unstable standard state-space systems in [78]. An extension of this method to DAEs was presented in [95] and further developed in [21] for flow control problems. The LQG balanced truncation method is based on the generalized Riccati equations

$$\begin{aligned}A X^T + X A^T + B B^T - (X C^T + B D^T)(I + D D^T)^{-1}(C X^T + D B^T) = 0,\\ E X^T - X E^T = 0,\end{aligned} \tag{24}$$

and

$$\begin{aligned}A^T Y + Y^T A + C^T C - (Y^T B + C^T D)(I + D^T D)^{-1}(B^T Y + D^T C) = 0,\\ E^T Y - Y^T E = 0,\end{aligned} \tag{25}$$

where the matrices $I + D D^T$ and $I + D^T D$ are assumed to be nonsingular. Note that these equations do not involve the spectral projectors. One can show that if the DAE system (1)

is S-stabilizable and S-detectable, then equations (24) and (25) have stabilizing solutions X and Y such that the pencils

$$\lambda E - \big(A - (XC^T + BD^T)(I + DD^T)^{-1}C\big),$$
$$\lambda E - \big(A - B(I + D^T D)^{-1}(B^T Y + D^T C)\big)$$

are both of index one and stable. The matrices $G_c^{LQG} = XE^T$ and $G_o^{LQG} = Y^T E$ are called the *LQG controllability* and *observability Gramians* of the DAE system (1). In contrast to X and Y, the Gramians G_c^{LQG} and G_o^{LQG} are symmetric, positive semidefinite and uniquely defined. The *LQG characteristic values* are defined as

$$\sigma_j^{LQG} = \sqrt{\lambda_j(G_c^{LQG}(E^+)^T G_o^{LQG} E^+)},$$

where E^+ denotes the Moore-Penrose pseudoinverse of E. Balancing the LQG Gramians and truncating the states corresponding to small LQG characteristic values provides the LQG balanced truncation model reduction method given in Algorithm 5.

Algorithm 5 LQG balanced truncation for DAE systems.

Input: $\boldsymbol{H} = (E, A, B, C, D)$

Output: a reduced-order model $\widetilde{\boldsymbol{H}} = (\widetilde{E}, \widetilde{A}, \widetilde{B}, \widetilde{C}, \widetilde{D})$.

1: Compute the full rank matrices Z_r and Z_l such that $\operatorname{im}(Z_r) = \ker(E)$ and $\operatorname{im}(Z_l) = \ker(E^T)$.
2: Compute the Cholesky factors Z_c^{LQG} and Z_o^{LQG} such that $EX^T = EZ_c^{LQG}(Z_c^{LQG})^T E^T$ and $E^T Y = E^T Z_o^{LQG}(Z_o^{LQG})^T E$, where X and Y are the stabilizing solutions of the generalized Riccati equations (24) and (25), respectively.
3: Compute a singular value decomposition $(Z_o^{LQG})^T E Z_c^{LQG} = [U_1, U_2]\operatorname{diag}(\Sigma_1^{LQG}, \Sigma_2^{LQG})[V_1, V_2]^T$ with $\Sigma_1^{LQG} = \operatorname{diag}(\sigma_1^{LQG}, \ldots, \sigma_r^{LQG})$ and $\Sigma_2^{LQG} = \operatorname{diag}(\sigma_{r+1}^{LQG}, \ldots, \sigma_k^{LQG})$.
4: Compute the reduced-order system $(\widetilde{E}, \widetilde{A}, \widetilde{B}, \widetilde{C}, \widetilde{D}) = (W^T E T, W^T A T, W^T B, CT, D)$ with the projection matrices $W = [Z_o^{LQG} U_1 (\Sigma_1^{LQG})^{-1/2}, Z_l]$ and $T = [Z_c^{LQG} V_1 (\Sigma_1^{LQG})^{-1/2}, Z_r]$.

For the LQG reduced-order system, there exists an error estimate in the gap metric [60] defined as follows. Let the DAE system (1) be S-stabilizable and S-detectable. Then its transfer function $\boldsymbol{H}(s)$ can be factored as $\boldsymbol{H}(s) = \boldsymbol{K}(s)\boldsymbol{M}^{-1}(s)$, where

$$\begin{aligned}\boldsymbol{K}(s) &= (C + DF)(sE - A - BF)^{-1}B(I + D^T D)^{-1/2} + D(I + D^T D)^{-1/2},\\ \boldsymbol{M}(s) &= F(sE - A - BF)^{-1}B(I + D^T D)^{-1/2} + (I + D^T D)^{-1/2}\end{aligned}$$

with $F = -(I + D^T D)^{-1}(B^T Y + D^T C)$ are stable proper rational functions called the *right coprime factors* of $\boldsymbol{H}(s)$. Obviously, $\begin{bmatrix} \boldsymbol{M} \\ \boldsymbol{K} \end{bmatrix} \in \mathcal{H}_\infty$, and we obtain the error estimate

$$\left\| \begin{bmatrix} \widetilde{\boldsymbol{M}} \\ \widetilde{\boldsymbol{K}} \end{bmatrix} - \begin{bmatrix} \boldsymbol{M} \\ \boldsymbol{K} \end{bmatrix} \right\|_{\mathcal{H}_\infty} \leq 2 \sum_{j=r+1}^{k} \frac{\sigma_j^{LQG}}{\sqrt{1 + \sigma_j^{LQG}}},$$

where $\widetilde{\boldsymbol{H}}(s) = \widetilde{\boldsymbol{K}}(s)\widetilde{\boldsymbol{M}}^{-1}(s)$ is the right coprime factorization of $\widetilde{\boldsymbol{H}}(s)$ and σ_j^{LQG} are the LQG characteristic values from Algorithm 5, see [95].

Projector-free generalized Riccati equations similar to (24) and (25) have also been studied in the context of linear-quadratic optimal control [79, 117, 154], spectral factorization

problems [80, 81], and extensions of the positive real and bounded real lemmas to DAE systems [57, 149, 150, 151, 156]. Stability and the index-1 property of (1) can also be characterized via the projector-free generalized Lyapunov equations

$$\begin{aligned} AX^T + XA^T + BB^T &= 0, & EX^T - XE^T &= 0, \\ A^TY + Y^TA + C^TC &= 0, & E^TY - Y^TE &= 0, \end{aligned}$$

see [74, 141]. All these matrix equations provide an alternative way to define different types of the Gramians for DAEs and also new balancing-related model reduction methods [112]. They might be advantageous if the spectral projectors are difficult to compute. It should, however, be noticed that currently existing numerical methods for such equations are restricted to small and medium-sized problems. Another disadvantage is that these new model reduction techniques would be limited, in most cases, to index one problems.

3.2 Interpolation-based approximation

Another family of methods for model reduction is based on (rational) interpolation. The unifying feature of the methods in this family is that the original transfer function $\mathbf{H}(s)$ is approximated by a rational matrix function $\widetilde{\mathbf{H}}(s)$ of lower degree satisfying some interpolation conditions (that is, the original and the reduced-order transfer function coincide, e.g. $\mathbf{H}(s_0) = \widetilde{\mathbf{H}}(s_0)$ at some predefined value s_0 such that $(A - s_0E)$ is nonsingular). Computationally, this is usually realized by certain Krylov subspace methods.

The classical approach is known under the name of *moment-matching* or *Padé(-type) approximation*. In these methods, the transfer functions of the original and the reduced-order systems are expanded into power series and the reduced-order system is then determined so that the first coefficients in the series expansions match. In this context, the coefficients of the power series expansion are called *moments*, which explains the term moment-matching. One speaks of Padé-approximation if the number of matching moments is maximized for a given degree of the approximating rational function.

Classically, the expansion of the transfer function in a power series about an expansion point s_0 as in (11) is used. Recall, that the moments $M_j(s_0), j = 0, 1, 2, \ldots$ are given by

$$M_j(s_0) = -C\left((A - s_0E)^{-1}E\right)^j(A - s_0E)^{-1}B + \delta_{0,j}D.$$

Note that s_0 is necessarily chosen such that $A - s_0E$ is nonsingular, and hence s_0 is neither an eigenvalue of the matrix pencil $\lambda E - A$ nor a pole of the transfer function $\boldsymbol{H}(s)$. Thus, the approach described in the following can be applied regardless whether E is singular or not, so that no special adaptation to DAE systems is necessary.

Now consider the *block Krylov subspace*

$$\mathcal{K}_k(F, G) = \text{blockspan}\{G, FG, F^2G, \ldots, F^{k-1}G\}$$

generated by $F = (A - s_0E)^{-1}E$ and $G = -(A - s_0E)^{-1}B$ with an appropriately chosen expansion point s_0 which may be real or complex. From the definitions of A, B and E, it follows that $F \in \mathbb{K}^{n\times n}$ and $G \in \mathbb{K}^{n\times m}$, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$ depending on whether s_0 is chosen in $\mathbb{R}$ or in $\mathbb{C}$. Considering $\mathcal{K}_k(F, G)$ columnwise, this leads to the observation that the number of column vectors in $[G, FG, F^2G, \ldots, F^{k-1}G]$ is given by $r = m \cdot k$, as there are k blocks $F^jG \in \mathbb{K}^{n\times m}$, $j = 0, \ldots, k-1$. In the case when all r column vectors are linearly independent, the dimension of the Krylov subspace $\mathcal{K}_k(F, G)$ is r. Assume that a unitary basis for this block Krylov subspace is generated such that the column-space of the resulting unitary matrix $T \in \mathbb{K}^{n\times r}$ spans $\mathcal{K}_k(F, G)$. Applying the Galerkin projection

$\Pi = TT^*$ to (1) yields a reduced system whose transfer function satisfies the following Hermite interpolation conditions

$$\widetilde{\boldsymbol{H}}^{(j)}(s_0) = \boldsymbol{H}^{(j)}(s_0), \qquad j = 0, 1, \ldots, k-1.$$

This means that the the transfer functions $\boldsymbol{H}$ and $\widetilde{\boldsymbol{H}}$ and their first k derivatives coincide at s_0. Considering the power series expansion (11) of the original and the reduced-order transfer function, this is equivalent to saying that at least the first k moments $\widetilde{M}_j(s_0)$ of the transfer function $\widetilde{\boldsymbol{H}}(s)$ of the reduced system (2) are equal to the first k moments $M_j(s_0)$ of the transfer function $\boldsymbol{H}(s)$ of the original system (1) at the expansion point s_0, i.e.,

$$M_j(s_0) = \widetilde{M}_j(s_0), \qquad j = 0, 1, \ldots, k-1.$$

If further the r columns of the unitary matrix W span the block Krylov subspace $\mathcal{K}_k(F,G)$ for $F = (A - s_0E)^{-T}E^T$ and $G = -(A - s_0E)^{-T}C^T$, applying the Petrov-Galerkin projection $\Pi = T(W^*T)^{-1}W^*$ to (1) yields a reduced system whose transfer function matches at least the first $2k$ moments of the transfer function $\boldsymbol{H}(s)$ of the original system.

Theoretically, the matrix T (and W) can be computed by explicitly forming the columns which span the corresponding Krylov subspace $\mathcal{K}_k(F,G)$ and using the Gram-Schmidt algorithm to generate unitary basis vectors for $\mathcal{K}_k(F,G)$. The forming of the moments (the Krylov subspace blocks F^jG) is numerically precarious and has to be avoided under all circumstances. Instead, it is recommended to use Krylov subspace methods to achieve an interpolation-based reduced-order model as described above. The unitary basis of a (block) Krylov subspace can be computed by employing a (block) Arnoldi or (block) Lanczos method, see e.g. [7, 63, 54].

In the case when an oblique projection is used, it is not necessary to compute two unitary bases as above. An alternative is then to use the nonsymmetric Lanczos process [63]. It computes bi-unitary bases for the above mentioned Krylov subspaces and the reduced-order model as a by product of the Lanczos process. An overview of the computational techniques for moment-matching and Padé approximation summarizing the work of a decade is given in [54] and the references therein.

The use of complex-valued expansion points will lead to a complex-valued reduced-order system (2). In some applications (in particular, if the original system is real-valued) this is undesired. In that case one can always use complex-conjugate pairs of expansion points as then the entire computations can be done in real arithmetic.

In general, the discussed model order reduction approaches are instances of rational interpolation. When the expansion point is chosen to be $s_0 = \infty$, the moments are called Markov parameters and the approximation problem is known as *partial realization*. Here, singularity of E obviously makes a difference as then the Laurent expansion (12) is used. For singular E, using the reflexive inverse of E, a partial realization method for descriptor systems was derived in [34].

As the use of one single expansion point s_0 leads to good approximation only close to s_0, it might be desirable to use more than one expansion point. This leads to *multi-point moment-matching* methods, which can also be interpreted as rational Krylov methods, see, e.g., [7, 54].

Assume that ℓ expansion points s_i, $i = 1, 2, \ldots, \ell$ are considered. The column vectors of the matrix T are determined from the ℓ block Krylov subspaces $\mathcal{K}_{k_i}(F_i, G_i)$ generated by $F_i = (A - s_iE)^{-1}E$ and $G_i = -(A - s_iE)^{-1}B$ for $i = 1, 2, \ldots, \ell$. From each of these subspaces,

the $m \cdot k_i$ column vectors are used to generate an $n \times r$ matrix

$$\hat{T} = \left[T_{[k_1]}, T_{[k_2]}, \ldots, T_{[k_\ell]} \right], \qquad r = m \sum_{i=1}^{\ell} k_i.$$

In order to obtain a unitary, full-rank matrix T, a rank-revealing QR decomposition can be used $\hat{T} = TR$, so that the numerical rank of $\hat{T}$ can be determined, $\hat{r} = \mathrm{rank}(\hat{T})$, and finally, $\hat{T}$ can be truncated to $T = [T(:, 1:\hat{r})]$ (employing MATLAB® notation). The columns of T span the same subspace as the span of the union of the Krylov subspaces $\mathcal{K}_{k_i}(F_i, G_i)$, that is, $\mathrm{span}(T) = \cup_{i=1}^{\ell} \mathcal{K}_{k_i}(F_i, G_i)$. Then at least k_i moments are matched per expansion point s_i:

$$M_j(s_i) = \widetilde{M}_j(s_i), \qquad j = 0, 1, \ldots, k_i - 1, \quad i = 1, 2, \ldots, \ell,$$

if the reduced system is generated by applying the Galerkin projection $\Pi = TT^*$. In this case, $\widetilde{\boldsymbol{H}}$ fulfils the Hermite interpolation conditions

$$\widetilde{\boldsymbol{H}}^{(j)}(s_i) = \boldsymbol{H}^{(j)}(s_i), \qquad j = 0, 1, \ldots, k_i - 1, \quad i = 1, 2, \ldots, \ell.$$

A Petrov-Galerkin projection can also be constructed following this idea. Then at least $2k_i$ moments are matched per expansion point s_i. It should be noted, that at each s_i a different number of moments k_i is matched.

In contrast to balanced truncation, these (rational) interpolation methods do not necessarily preserve stability. Remedies have been suggested, see, e.g. [54].

The methods just described provide good approximation quality around the expansion points. They do not aim at a global approximation as measured by the $\mathcal{H}_2$- or $\mathcal{H}_\infty$-norm. In [68], an iterative procedure is presented which determines, upon convergence[3], locally optimal expansion points with respect to the $\mathcal{H}_2$-norm approximation under the assumption that the order r of the reduced model is prescribed and such that only 0-th and 1-st order derivatives are matched. This is motivated by the necessary $\mathcal{H}_2$-norm optimality conditions for a stable, r-th order, rational interpolant $\widetilde{\boldsymbol{H}}$ of $\boldsymbol{H}$. In order for $\widetilde{\boldsymbol{H}}$ to be a local minimizer of the error measured in the $\mathcal{H}_2$-norm, it is necessarily an Hermite interpolant in the classical sense, i.e., interpolation of the function value and its first-order derivative at the mirror images (with respect to the imaginary axis) of the poles of $\widetilde{\boldsymbol{H}}$, see [94]. Also, for multi-input multi-output systems (that is, m and q in (1) are both larger than one), no full moment-matching is achieved, but only tangential interpolation

$$\boldsymbol{H}(s_j) b_j = \widetilde{\boldsymbol{H}}(s_j) b_j, \quad c_j^* \boldsymbol{H}(s_j) = c_j^* \widetilde{\boldsymbol{H}}(s_j), \quad c_j^* \boldsymbol{H}'(s_j) b_j = c_j^* \widetilde{\boldsymbol{H}}'(s_j) b_j$$

for certain vectors b_j and c_j determined together with the optimal s_j by the iterative procedure. The $\mathcal{H}_2$-optimal approximation procedure was extended to DAE systems in [69]. Though the interpolation properties of the reduced-order transfer function are the same for ODE and DAE systems, one needs to take special care of behavior at infinity for DAE systems. In order for the error function $\boldsymbol{H} - \widetilde{\boldsymbol{H}}$ to be an $\mathcal{H}_2$-function, it needs to be zero at infinity, which usually is not the case when only applying the necessary optimality conditions of the ODE case. In addition, it is necessary to "interpolate" at infinity. This requires some additional work and altering the realization of the reduced-order model without destroying the interpolation conditions in the mirror images of its poles. A procedure achieving this and requiring little extra effort is described in [69], but we refrain here from reproducing the technical details.

[3] For partial convergence results, see [51].

4 Solving large matrix equations

In this section, we discuss the numerical solution of projected Lyapunov and Riccati matrix equations arising in balancing-related model reduction of DAE systems. We assume that the spectral projectors in these equations are given, though their computation may be a challenging task, especially for large-scale problems. Fortunately, for some structured problems, the spectral projectors can either be constructed explicitly or the DAE system (and also the matrix equations) can be modified such that the projectors are not required any more. This issue will be addressed in Section 5.

4.1 Projected Lyapunov equations

We consider first the projected discrete-time Lyapunov equation

$$AXA^T - EXE^T = Q_l BB^T Q_l^T, \qquad X = Q_r X Q_r^T, \tag{26}$$

where $A, E \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$ with $m \ll n$. If the pencil $\lambda E - A$ is stable, i.e., all its finite eigenvalues have negative real part, and it has index ν, then A is nonsingular and the solution of (26) can be represented as

$$X = \sum_{j=0}^{\nu-1} (A^{-1}E)^j A^{-1} Q_l BB^T Q_l^T A^{-T} ((A^{-1}E)^T)^j = ZZ^T$$

with $Z = Q_r[A^{-1}B, (A^{-1}E)A^{-1}B, \ldots, (A^{-1}E)^{\nu-1}A^{-1}B]$. If the index of $\lambda E - A$ is unknown a priori, then this low-rank factor can be computed using the generalized Smith iteration [137] which converges in a finite number of steps, see Algorithm 6.

Algorithm 6 Smith method for projected discrete-time Lyapunov equations.

Input: $A, E \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, the spectral projector Q_r, and a convergence tolerance $tol > 0$.
Output: a low-rank factor Z_k such that $X = Z_k Z_k^T$ is an approximate solution of (26).

1: $V_0 = Q_r A^{-1} B$;
2: $Z_0 = [\ ]$;
3: $k = 0$;
4: **while** $\|V_k\|_F > tol$ **do**
5: $\quad Z_{k+1} = [Z_k, V_k]$;
6: $\quad V_{k+1} = A^{-1} E V_k$;
7: $\quad k \leftarrow k+1$;
8: **end while**

We consider now the projected continuous-time Lyapunov equation

$$EXA^T + AXE^T = -P_l BB^T P_l^T, \qquad X = P_r X P_r^T, \tag{27}$$

where $\lambda E - A$ is assumed to be stable. We aim to determine the solution of this equation in the factored form $X = ZZ^T$, avoiding the computation of the solution matrix X. For problems of small and moderate size (up to a few thousands), this can be achieved using the generalized Schur-Hammarling method [133] which relies on computing the generalized Schur form of the pencil $\lambda E - A$. One can also employ the matrix sign function method

which was initially developed for standard Lyapunov equations [29, 87, 119] and then extended to projected Lyapunov equations in [136]. This method is efficient, in particular, for large dense problems.

As mentioned in Section 3.1.1, to be able to apply the balanced truncation method to large-scale problems, we are rather interested in a low-rank approximation $X \approx \widetilde{Z}\widetilde{Z}^T$ with $\widetilde{Z} \in \mathbb{R}^{n\times k}$ and $k \ll n$. The most simple way to compute such an approximation is based on the integral representation (17) for the solution of (27). Computing this integral by a quadrature rule

$$\begin{aligned} X \quad \approx \quad & \sum_{j=1}^{p} f_j (i\omega_j E - A)^{-1} P_l B B^T P_l^T (-i\omega_j E - A)^{-T} \\ & + \sum_{j=1}^{p} f_j (-i\omega_j E - A)^{-1} P_l B B^T P_l^T (i\omega_j E - A)^{-T} \end{aligned}$$

with nonnegative nodes ω_j and positive weights f_j, we obtain the real low-rank factor

$$\widetilde{Z} = [\mathrm{Re}(B_1), \mathrm{Im}(B_1), \ldots, \mathrm{Re}(B_p), \mathrm{Im}(B_p)] \in \mathbb{R}^{n\times 2pm}$$

with $B_j = \sqrt{2f_j}(i\omega_j E - A)^{-1} P_l B$. For the dual projected Lyapunov equation, the low-rank factor can be calculated analogously. Using these factors in Algorithm 1 can be viewed as an extension of the frequency domain POD approach [153] and the PMTBR method [106] to DAE systems.

4.1.1 Alternating directions implicit method

A low-rank approximation to the solution of the projected Lyapunov equation (27) can also be computed iteratively using a low-rank version of the alternating directions implicit method known as the LR-ADI method [89, 102, 137]. In the last years, several modifications concerning the efficient computation of Lyapunov residuals, adaptive choice of ADI shift parameters and handling the complex shifts were proposed for Lyapunov equations with nonsingular E, which significantly improve the performance of the ADI iteration [25, 26, 27]. An extension of these results to the projected Lyapunov equation is straightforward [35, 137] and summarized in Algorithm 7.

One can see that this algorithm provides a real low-rank factor $Z_k \in \mathbb{R}^{n\times km}$ and the computational cost for the LR-ADI method is proportional to the cost of solving linear systems with the sparse matrix $E + \tau_k A$. The convergence rate of the ADI iteration is strongly influenced by the shift parameters $\tau_k \in \mathbb{C}_-$. Optimal parameters can be obtained by solving the minimax problem

$$\{\hat{\tau}_1, \ldots, \hat{\tau}_p\} = \operatorname*{arg\,min}_{\{\tau_1,\ldots,\tau_p\}\in\mathbb{C}_-} \max_{t\in \mathrm{Sp}(E,A)} \frac{|(1-\overline{\tau}_1 t)\cdots(1-\overline{\tau}_p t)|}{|(1+\tau_1 t)\cdots(1+\tau_p t)|},$$

where $\mathrm{Sp}(E,A)$ denotes the set of finite eigenvalues of the pencil $\lambda E - A$. Suboptimal ADI parameters can be determined from a set of largest and smallest in modulus approximate finite eigenvalues of $\lambda E - A$ computed by an Arnoldi or Lanczos procedure, or any other method to compute the extreme eigenvalues of a matrix pencil. Any other parameter selection technique developed for standard Lyapunov equations [27, 124, 148] can also be used for the projected Lyapunov equation.

Algorithm 7 LR-ADI method for projected continuous-time Lyapunov equations.

Input: $A, E \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, the spectral projector P_l, shifts $\tau_1, \ldots, \tau_p \in \mathbb{C}_-$, a tolerance tol, and $k_{\max} \in \mathbb{N}$.

Output: a low-rank factor Z_k such that $X \approx Z_k Z_k^T$ solves (27) approximately.

$W_0 = P_l B$;
$Z_0 = [\ ]$;
$k = 1$;
while $(\|W_{k-1}^T W_{k-1}\|_F / \|W_0^T W_0\|_F > tol$ **and** $k < k_{\max})$ **do**
  $V_k = (E + \tau_k A)^{-1} W_{k-1}$;
  if $\tau_k \in \mathbb{R}$ **then**
    $W_k = W_{k-1} - 2\tau_k A V_k$;
    $Z_k = [Z_{k-1},\ \sqrt{-2\tau_k}\, V_k]$;
  else
    $\alpha_k = \sqrt{-2\mathrm{Re}(\tau_k)}$, $\beta_k = \mathrm{Re}(\tau_k)/\mathrm{Im}(\tau_k)$;
    $W_{k+1} = W_{k-1} - 4\mathrm{Re}(\tau_k) A \big(\mathrm{Re}(V_k) + \beta_k \mathrm{Im}(V_k)\big)$;
    $Z_k = [Z_{k-1},\ \alpha_k \big(\mathrm{Re}(V_k) + \beta_k \mathrm{Im}(V_k)\big),\ \alpha_k \sqrt{\beta_k^2 + 1}\, \mathrm{Im}(V_k)]$;
    $k \leftarrow k + 1$;
  end if
  $k \leftarrow k + 1$;
end while

4.1.2 Krylov subspace methods

Alternative iterative methods for Lyapunov equations are Krylov subspace methods [37, 75, 77, 122] which become competitive with the ADI iteration due to recent developments on extended and rational Krylov subspaces [46, 83, 128], see also [45] for a comparative analysis of the Krylov subspace and ADI methods. Employing the ADI iteration as a preconditioner in Krylov subspace methods has been considered in [37, 76]. An extension of these methods to projected Lyapunov equations can be found in [37, 140]. The approaches differ in the way the linear matrix equation is solved by either interpreting them as classical linear systems using their Kronecker product representation in $\mathbb{R}^{n^2}$, as is the case, e.g., for [37, 76, 77], or by directly working on the matrix equation and building the Krylov subspaces in $\mathbb{R}^n$ as done in [46, 75, 83, 122, 128, 140]. The latter approach appears to be more efficient (though also the first approach uses Krylov subspaces in $\mathbb{R}^{n^2}$ only implicitly), and we will therefore concentrate on this concept here.

In the Krylov subspace methods, an approximate solution to the projected Lyapunov equation (27) is determined in the form $X \approx V\widetilde{Y}\widetilde{Y}^T V^T$, where columns of V span a certain Krylov subspace and $Y = \widetilde{Y}\widetilde{Y}^T$ solves the reduced Lyapunov equation

$$\widetilde{A}Y + Y\widetilde{A}^T = -\widetilde{B}\widetilde{B}^T,$$

where $\widetilde{A} = V^T A^{-1} E V$ and $\widetilde{B} = V^T A^{-1} P_l B$ or, alternatively, $\widetilde{A} = V^T E^- A V$ and $\widetilde{B} = V^T E^- B$. Here,

$$E^- = T_r^{-1} \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} T_l^{-1}$$

is a reflexive inverse of E with respect to the projectors P_l and P_r satisfying the matrix

equations

$$E^- E E^- = E^-, \qquad E E^- = P_l, \qquad E^- E = P_r.$$

The projection subspace $\mathrm{im}(V)$ can be chosen as an extended block Krylov subspace

$$\mathcal{K}_k(A^{-1}E, A^{-1}P_l B) \cup \mathcal{K}_k(E^- A, E^- B).$$

The resulting numerical procedure based on a block Arnoldi method for computing an orthogonal basis of this subspace and solving the projected Lyapunov equation (27) is given in Algorithm 8.

Algorithm 8 Extended block Arnoldi method for projected Lyapunov equations.

Input: $A, E \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, the spectral projector P_r, and $k \in \mathbb{N}$.
Output: a low-rank factor Z_k such that $X \approx Z_k Z_k^T$ solves (27) approximately.

$\hat{V}_1 = \mathrm{orth}([E^- B, P_r A^{-1} B])$ {orthogonalization of the columns of $[E^- B, P_r A^{-1} B]$};
$V_1 = \hat{V}_1, \quad V_{1,1} = \hat{V}_1 [I_m, 0]^T, \quad V_{1,2} = \hat{V}_1 [0, I_m]^T$;
for $j = 1, 2, \ldots, k$ **do**
  $V^{(j)} = [E^- A V_{j,1},\ A^{-1} E V_{j,2}]$;
  for $i = 1, 2, \ldots, j$ **do**
    $H_{i,j} = \hat{V}_i^T V^{(j)}$;
    $V^{(j)} = V^{(j)} - \hat{V}_i H_{i,j}$;
  end for
  $\hat{V}_{j+1} = \mathrm{orth}(V^{(j)})$ {orthogonalization of the columns of $V^{(j)}$};
  $V_{j+1} = [V_j, \hat{V}_{j+1}], \quad V_{j+1,1} = \hat{V}_{j+1}[I_m, 0]^T, \quad V_{j+1,2} = \hat{V}_{j+1}[0, I_m]^T$;
  $\Phi_j = V_j^T E^- A V_j, \ B_j = V_j^T E^- B$;
  solve the Lyapunov equation $\Phi_j Y_j + Y_j \Phi_j^T = -B_j B_j^T$ for $Y_j = \widetilde{Y}_j \widetilde{Y}_j^T$;
end for
$Z_k = V_k \widetilde{Y}_k$.

The iteration in this algorithm can be terminated as soon as the normalized residual defined by

$$\eta(Z_j) = \frac{\|E Z_j Z_j^T A^T + A Z_j Z_j^T E^T\|_F}{\|P_l B B^T P_l^T\|_F}$$

satisfies the condition $\eta(Z_j) \le tol$ with a tolerance tol. Since the computation of the residual is expensive for large-scale problems, it has been proposed in [140] to use the following stopping criterion:

$$\frac{\|E^-(E Z_j Z_j^T A^T + A Z_j Z_j^T E^T)(E^-)^T\|_F}{\|E^- B B^T (E^-)^T\|_F} = \frac{\sqrt{2}\|V_{j+1,1}^T E^- A V_j Y_j\|_F}{\|(E^- B)^T (E^- B)\|_F} \le tol,$$

where the matrix $V_{j+1,1}^T E^- A V_j$ can be obtained as a by-product of the iteration with no additional matrix-vector products with E^- and A and inner products with long vectors.

In the rational Krylov subspace method, the projection subspace $\mathrm{im}(V)$ is taken as the rational block Krylov subspace defined as

$$\mathcal{K}_k(E, A, B; s_1, \ldots, s_k) = \mathrm{blockspan}\Big\{ (s_1 E - A)^{-1} P_l B,\ (s_2 E - A)^{-1} E (s_1 E - A)^{-1} P_l B,\ \ldots,\ (s_k E - A)^{-1} \prod_{j=1}^{k-1} E (s_j E - A)^{-1} P_l B \Big\}$$

for some shifts $s_1, \ldots, s_k$ which are not the eigenvalues of $\lambda E - A$. As in the LR-ADI method, these parameters should be chosen carefully to guarantee fast convergence [45, 46].

4.2 Projected Riccati equations

We consider now the projected Riccati equation in the general form

$$EXF^T + FXE^T + EXQ^TQXE^T + P_lRR^TP_l^T = 0, \qquad X = P_rXP_r^T, \tag{28}$$

where the matrices $F \in \mathbb{R}^{n\times n}$, $R \in \mathbb{R}^{n\times m}$ and $Q \in \mathbb{R}^{q\times n}$ vary depending on the balanced truncation method:

$$F = A - P_lBJ_c^{-T}J_c^{-1}CP_r, \quad Q = J_c^{-1}C, \quad R = BJ_c^{-T}, \quad M_0 + M_0^T = J_cJ_c^T$$

in the positive real case and

$$\begin{aligned} &F = A + P_lBM_0J_c^{-T}J_c^{-1}CP_r, \quad Q = J_c^{-1}C, \quad R = BJ_o^{-1}, \\ &I - M_0M_0^T = J_cJ_c^T, \quad I - M_0^TM_0 = J_o^TJ_o \end{aligned}$$

in the bounded real case. In the stochastic balanced truncation method, where a dual Riccati equation has to be solved, E, P_r and P_l should be replaced by E^T, P_l^T and P_r^T, respectively, and

$$\begin{aligned} &F = (A - B_0M_0^{-T}M_0^{-1}CP_r)^T, \quad Q = M_0^{-1}B_0^T, \quad R = C^TM_0^{-T}, \\ &B_0 = P_lBM_0^T + EG_{pc}C^T. \end{aligned}$$

We assume that (28) has a unique stabilizing solution X_* such that the matrix pencil $\lambda E - (F + EX_*Q^TQP_r)$ is stable. Since the first equation in (28) is nonlinear, we can solve it by Newton's method presented in [35]. For this purpose, we define a Riccati operator

$$\mathcal{R}(X) = EXF^T + FXE^T + EXQ^TQXE^T + P_lRR^TP_l^T$$

and compute its Frechét derivative

$$\mathcal{R}'_X(N) = EN(F + EXQ^TQP_r)^T + (F + EXQ^TQP_r)NE^T.$$

Then Newton's method for the projected Riccati equation (28) is given by

$$N_j = -(\mathcal{R}'_{X_j})^{-1}(\mathcal{R}(X_j)), \qquad X_{j+1} = X_j + N_j. \tag{29}$$

It has been shown in [35] that this iteration converges quadratically towards X_* for any stabilizing initial guess X_0. If $\lambda E - F$ is stable, then we can take $X_0 = 0$. However, for unstable problems, the computation of a stabilizing X_0 might be challenging. For some methods to find an initial stabilizing feedback for descriptor systems, see [16].

Note that the first equation in (29) is equivalent to the projected Lyapunov equation

$$EN_jF_j^T + F_jN_jE^T = -P_lK_jK_j^TP_l^T, \qquad N_j = P_rN_jP_r^T \tag{30}$$

with $F_j = F + EX_jQ^TQP_r$ and $K_j = EN_{j-1}Q^T$. This equation can now be solved for a low-rank factor using the LR-ADI method discussed above. The resulting low-rank Newton method is summarized in Algorithm 9. It should be mentioned that taking the advantage of the special structure of $F_j = F + (EX_jQ^T)(QP_r)$, the inverse of $E + \tau_kF_j$ required in the LR-ADI iteration can be written using the Sherman-Morrison-Woodburry formula [63, Section 2.1.3] as

$$(E + \tau_kF_j)^{-1} = F_{jk}^{-1} - F_{jk}^{-1}(EX_jQ^T)\big(I_q + QP_rF_{jk}^{-1}(EX_jQ^T)\big)^{-1}QP_rF_{jk}^{-1},$$

Algorithm 9 Low-rank Newton method for projected Riccati equations

Input: $E, F \in \mathbb{R}^{n\times n}$ such that $\lambda E - F$ is stable, $Q \in \mathbb{R}^{q\times n}$, $R \in \mathbb{R}^{n\times m}$, projectors P_r and P_l.

Output: an approximate low-rank factor of the stabilizing solution of (28).

1: Solve $EN_0F^T + FN_0E^T = -P_lRR^TP_l^T$, $N_0 = P_rN_0P_r^T$ for the low-rank factor $\widetilde{N}_0$ such that $N_0 \approx \widetilde{N}_0\widetilde{N}_0^T$;
2: $\widetilde{X}_1 = \widetilde{N}_0$;
3: $F_0 = F$;
4: **for** $j = 1, 2, \ldots$ **do**
5: $\quad K_j = E\widetilde{N}_{j-1}\widetilde{N}_{j-1}^TQ^T$;
6: $\quad F_j = F_{j-1} + K_jQP_r$;
7: $\quad$ solve (30) for the low-rank factor $\widetilde{N}_j$ such that $N_j \approx \widetilde{N}_j\widetilde{N}_j^T$;
8: $\quad \widetilde{X}_{j+1} = [\widetilde{X}_j,\ \widetilde{N}_j]$.
9: **end for**

with $F_{jk} = E + \tau_kF$. Thus, instead of solving the linear system with large and possibly dense $E + \tau_kF_j$, we can solve two large linear systems with sparse $E + \tau_kF$ and, additionally, one small system.

Substituting $N_j = X_{j+1} - X_j$ in (30), Newton's method can be reformulated as the Newton-Kleinman iteration, where the new approximation X_{j+1} to the solution of (28) is determined by solving the projected Lyapunov equation

$$EX_{j+1}F_j^T + F_jX_{j+1}E^T = -P_l(RR^T - EX_jQ^TQX_jE)P_l^T, \qquad X_{j+1} = P_rX_{j+1}P_r^T.$$

The low-rank version of the Newton-Kleinman iteration as well as a comparison of both the Newton-type techniques can be found in [35].

5 Structured DAE systems

The main difficulty in the model reduction methods for DAE systems involving the spectral projectors is the determination of these projectors themselves. This is often a numerically ill-conditioned problem since it requires the computation of the deflating subspaces corresponding to the finite eigenvalues of $\lambda E - A$. Fortunately, for some structured problems, the projectors P_l and P_r can be determined employing the block structures of E and A. Of course, we should avoid forming them explicitly as they are usually $n \times n$ dense matrices. Since the projectors often inherit the block structures of E and A, projector-vector products can be computed block-wise, where multiplication with sparse matrices and solving sparse linear systems is involved [137]. Furthermore, some structured DAE systems can be transformed into the ODE form such that the computation of the projectors can even be completely avoided.

5.1 Semi-explicit systems of index 1

First, we consider the semi-explicit DAE system

$$\begin{bmatrix} E_{11} & E_{12} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u(t), \tag{31}$$

$$y(t) = C_1x_1(t) + C_2x_2(t) + Du(t). \tag{32}$$

Such systems arise in computational fluid dynamics [152] and power systems modeling [52, 120]. In the latter case, we have additionally $E_{12} = 0$. If the matrices E_{11} and $A_{22} - A_{21}E_{11}^{-1}E_{12}$ are both nonsingular, then (31) is of index 1, and the spectral projectors are given by

$$P_l = \begin{bmatrix} I & -(A_{12} - A_{11}E_{11}^{-1}E_{12})(A_{22} - A_{21}E_{11}^{-1}E_{12})^{-1} \\ 0 & 0 \end{bmatrix},$$
$$P_r = \begin{bmatrix} I + E_{11}^{-1}E_{12}(A_{22} - A_{21}E_{11}^{-1}E_{12})^{-1}A_{21} & E_{11}^{-1}E_{12}(A_{22} - A_{21}E_{11}^{-1}E_{12})^{-1}A_{22} \\ -(A_{22} - A_{21}E_{11}^{-1}E_{12})^{-1}A_{21} & I - (A_{22} - A_{21}E_{11}^{-1}E_{12})^{-1}A_{22} \end{bmatrix},$$

see [137]. Furthermore, (31) can be rewritten as the ODE system

$$\begin{aligned} \hat{E}\dot{x}(t) &= \hat{A}x(t) + \hat{B}u(t), \\ y(t) &= \hat{C}x(t) + \hat{D}u(t), \end{aligned} \tag{33}$$

where $x(t) = x_1(t) + E_{11}^{-1}E_{12}x_2(t)$, $\hat{E} = E_{11}$, and

$$\begin{aligned} \hat{A} &= A_{11} - (A_{12} - A_{11}E_{11}^{-1}E_{12})(A_{22} - A_{21}E_{11}^{-1}E_{12})^{-1}A_{21}, \\ \hat{B} &= B_1 - (A_{12} - A_{11}E_{11}^{-1}E_{12})(A_{22} - A_{21}E_{11}^{-1}E_{12})^{-1}B_2, \\ \hat{C} &= C_1 - (C_2 - C_1E_{11}^{-1}E_{12})(A_{22} - A_{21}E_{11}^{-1}E_{12})^{-1}A_{21}, \\ \hat{D} &= D - (C_2 - C_1E_{11}^{-1}E_{12})(A_{22} - A_{21}E_{11}^{-1}E_{12})^{-1}B_2. \end{aligned}$$

We can now apply any model reduction method to system (33) with nonsingular $\hat{E}$, where the spectral projectors are not needed any more. In the LR-ADI method and the Krylov-based model reduction methods, one has to solve the shifted linear systems of the form $(\hat{E} + \tau\hat{A})z = f$. This can be done either by solving the linear systems

$$\begin{bmatrix} E_{11} + \tau A_{11} & A_{12} - A_{11}E_{11}^{-1}E_{12} \\ \tau A_{21} & A_{22} - A_{21}E_{11}^{-1}E_{12} \end{bmatrix} \begin{bmatrix} z \\ g \end{bmatrix} = \begin{bmatrix} f \\ 0 \end{bmatrix}$$

provided the coefficient matrix is sparse, or by using the Sherman-Morrison-Woodburry formula [63, Section 2.1.3]

$$\begin{aligned} z &= (\hat{E} + \tau\hat{A})^{-1}f \\ &= \left(I + \tau(E_{11} + \tau A_{11})^{-1}(A_{12} - A_{11}E_{11}^{-1}E_{12})\hat{A}_{22}^{-1}A_{21}\right)(E_{11} + \tau A_{11})^{-1}f \end{aligned}$$

with $\hat{A}_{22} = A_{22} - A_{21}(E_{11} + \tau A_{11})^{-1}(E_{12} + \tau A_{12})$ provided $E_{11} + \tau A_{11}$ is nonsingular. Here, the product $\hat{z} = \hat{A}_{22}^{-1}\hat{f}$ can be computed by solving the sparse linear system

$$\begin{bmatrix} E_{11} + \tau A_{11} & E_{12} + \tau A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} \hat{g} \\ \hat{z} \end{bmatrix} = \begin{bmatrix} 0 \\ \hat{f} \end{bmatrix}.$$

Another condition guaranteeing the index-1 property for (31) is nonsingularity of the matrices A_{22} and $E_{11} - E_{12}A_{22}^{-1}A_{21}$. In this case, the second equation in (31) gives

$$x_2(t) = -A_{22}^{-1}A_{21}x_1(t) - A_{22}^{-1}B_2u(t).$$

Substituting it in the first equation in (31) and in the output equation (32), we obtain the ODE system

$$\begin{aligned} \hat{E}_1\dot{x}_1(t) &= \hat{A}_1x_1(t) + \hat{B}_1u_1(t), \\ y(t) &= \hat{C}_1x_1(t) + \hat{D}_1u_1(t), \end{aligned} \tag{34}$$

where

$$\begin{array}{ll} \hat{E}_1 = E_{11} - E_{12}A_{22}^{-1}A_{21}, & \hat{A}_1 = A_{11} - A_{12}A_{22}^{-1}A_{21}, \\ \hat{B}_1 = [B_1 - A_{12}A_{22}^{-1}B_2,\ E_{12}A_{22}^{-1}B_2], & \hat{C}_1 = C_1 - C_2A_{22}^{-1}A_{21}, \\ \hat{D}_1 = [D - C_2A_{22}^{-1}B_2,\ 0], & u_1(t) = [u^T(t),\ \dot{u}^T(t)]^T \end{array}$$

provided u is continuously differentiable. It should be emphasized that the matrices $\hat{E}_1$ and $\hat{A}_1$ will never be computed explicitly since they may be dense even if all matrices E_{ij} and A_{ij} are sparse. The solution of $(\hat{E}_1 + \tau\hat{A}_1)z = f$ can be obtained by solving the sparse linear system

$$\begin{bmatrix} E_{11} + \tau A_{11} & E_{12} + \tau A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} z \\ g \end{bmatrix} = \begin{bmatrix} f \\ 0 \end{bmatrix}.$$

Note that if $E_{12} = 0$, then both systems (33) and (34) take the form

$$\begin{aligned} E_{11}\dot{x}_1(t) &= (A_{11} - A_{12}A_{22}^{-1}A_{21})x_1(t) + (B_1 - A_{12}A_{22}^{-1}B_2)u(t), \\ y(t) &= (C_1 - C_2A_{22}^{-1}A_{21})x_1(t) + (D - C_2A_{22}^{-1}B_2)u(t). \end{aligned}$$

Model reduction of such a system has been considered in [52, 120].

5.2 Magneto-quasistatic systems of index 1

Magneto-quasistatic field systems arise in modeling of electromagnetic devices such as induction machines and transformers by neglecting the displacement currents. A spatial discretization of Maxwell's equations in magnetic vector potential formulation together with the circuit coupling equations using the finite integration technique or the finite element method yields the DAE system

$$\begin{aligned} \begin{bmatrix} M_{11} & 0 & 0 \\ 0 & 0 & 0 \\ X_1^T & X_2^T & 0 \end{bmatrix} \begin{bmatrix} \dot{a}_1(t) \\ \dot{a}_2(t) \\ j(t) \end{bmatrix} &= \begin{bmatrix} -K_{11} & -K_{12} & X_1 \\ -K_{21} & -K_{22} & X_2 \\ 0 & 0 & -R \end{bmatrix} \begin{bmatrix} a_1(t) \\ a_2(t) \\ j(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ I \end{bmatrix} u(t), \\ y(t) &= j(t), \end{aligned} \tag{35}$$

where $[a_1^T, a_2^T]^T \in \mathbb{R}^{n_1+n_2}$ is a semidiscretized magnetic vector potential and $j(t) \in \mathbb{R}^m$ is a current vector, e.g., [126, 127]. The matrices M_{11}, K_{22} and R are symmetric, positive definite and X_2 is of full column rank. In this case, system (35) has index 1 [82]. Let the columns of Y form an orthonormal basis of the kernel of X_2^T and the columns of $Z = X_2(X_2^TX_2)^{-1/2}$ span the image of X_2. Multiplying the first equation in (35) with an orthogonal matrix

$$Q = \begin{bmatrix} I_{n_1} & 0 & 0 \\ 0 & Z^T & 0 \\ 0 & Y^T & 0 \\ 0 & 0 & I_m \end{bmatrix}$$

and introducing a vector $Qx(t) = \left[a_1^T(t),\ a_{21}^T(t),\ a_{22}^T(t),\ j^T(t)\right]^T$ partitioned according to Q, we obtain the ODE system

$$\begin{aligned} \hat{E}\dot{x}(t) &= \hat{A}x(t) + \hat{B}u(t), \\ y(t) &= \hat{C}x(t), \end{aligned} \tag{36}$$

where

$$\begin{aligned}
\hat{E} &= \begin{bmatrix} M_{11}+X_1R^{-1}X_1^T & X_1R^{-1}X_2^TZ \\ Z^TX_2R^{-1}X_1^T & Z^TX_2R^{-1}X_2^TZ \end{bmatrix}, \quad x(t) = \begin{bmatrix} a_1(t) \\ a_{21}(t) \end{bmatrix}, \\
\hat{A} &= -\begin{bmatrix} K_{11} & K_{12}Z \\ Z^TK_{21} & Z^TK_{22}Z \end{bmatrix} + \begin{bmatrix} K_{12} \\ Z^TK_{22} \end{bmatrix} Y\left(Y^TK_{22}Y\right)^{-1}Y^T[K_{21}, K_{22}Z], \\
\hat{B} &= \begin{bmatrix} X_1 \\ Z^TX_2 \end{bmatrix} R^{-1}, \\
\hat{C} &= -(X_2^TX_2)^{-1}X_2^T\left(I - K_{22}Y(Y^TK_{22}Y)^{-1}Y^T\right)[K_{21},\, K_{22}Z].
\end{aligned} \tag{37}$$

In order to be able to apply the balanced truncation model reduction method to system (36), we need to solve linear systems of the form

$$(\hat{E} + \tau\hat{A})z = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}.$$

Exploiting the block structure of the matrices $\hat{E}$ and $\hat{A}$ in (37), the solution of this system can be determined as $z = [z_1^T, (Z^Tz_2)^T]^T$, where z_1 and z_2 solve the sparse linear system

$$\begin{bmatrix} M_{11}-\tau K_{11} & -\tau K_{12} & \tau X_1 \\ -\tau K_{21} & -\tau K_{22} & \tau X_2 \\ X_1^T & X_2^T & -\tau R \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} f_1 \\ Zf_2 \\ 0 \end{bmatrix}.$$

Furthermore, the ADI shift parameters can be calculated by an Arnoldi procedure applied to the matrices $\hat{E}^{-1}\hat{A}$ and $\hat{A}^{-1}\hat{E}$. Again, the matrix-vector products $\hat{E}^{-1}\hat{A}v$ and $\hat{A}^{-1}\hat{E}v$ required in the Arnoldi procedure can be computed without the construction of the matrices $\hat{E}$, $\hat{A}$ and their inverses. A main difficulty here is the computation of the vector $z = Y(Y^TK_{22}Y)^{-1}Y^Tw$. Fortunately, this vector can be determined by solving the sparse linear system

$$\begin{bmatrix} K_{22} & X_2 \\ X_2^T & 0 \end{bmatrix} \begin{bmatrix} z \\ g \end{bmatrix} = \begin{bmatrix} w \\ 0 \end{bmatrix},$$

see [82] for details. This shows that the computation of the large dense matrix Y can completely be avoided that reduces the computational complexity significantly.

5.3 Circuit equations of index 1 and 2

Linear RLC circuits consisting of linear resistors, inductors, capacitors and independent current and voltage sources can be described using modified nodal analysis [72, 111]. Choosing currents through inductors and voltages of voltage sources as inputs, as well as voltages of current sources and currents through voltage sources as outputs, one obtains a DAE system of the form (1) described by

$$E = \begin{bmatrix} A_C\mathcal{C}A_C^T & 0 & 0 \\ 0 & \mathcal{L} & 0 \\ 0 & 0 & 0 \end{bmatrix}, \; A = \begin{bmatrix} -A_{\mathcal{R}}\mathcal{R}^{-1}A_{\mathcal{R}}^T & -A_{\mathcal{L}} & -A_{\mathcal{V}} \\ A_{\mathcal{L}}^T & 0 & 0 \\ A_{\mathcal{V}}^T & 0 & 0 \end{bmatrix}, \; B = \begin{bmatrix} -A_I & 0 \\ 0 & 0 \\ 0 & -I \end{bmatrix} = C^T,$$

$$D = 0, \quad x(t) = \begin{bmatrix} \eta(t) \\ j_{\mathcal{L}}(t) \\ j_{\mathcal{V}}(t) \end{bmatrix}, \quad u(t) = \begin{bmatrix} j_I(t) \\ v_{\mathcal{V}}(t) \end{bmatrix}, \quad y(t) = -\begin{bmatrix} v_I(t) \\ j_{\mathcal{V}}(t) \end{bmatrix}. \tag{38}$$

Here $\eta \in \mathbb{R}^{n_\eta}$ is a vector of node potentials, $j_\mathcal{L} \in \mathbb{R}^{n_\mathcal{L}}$, $j_I \in \mathbb{R}^{n_I}$ and $j_\mathcal{V} \in \mathbb{R}^{n_\mathcal{V}}$ are vectors of currents through inductors, current and voltage sources, respectively, and v_I and $v_\mathcal{V}$ are vectors of voltages of current and voltage sources, respectively. Furthermore, $A_\mathcal{C}$, $A_\mathcal{L}$, $A_\mathcal{R}$, $A_\mathcal{V}$ and A_I are the incidence matrices describing the topological structure of the circuit, and $\mathcal{C}$, $\mathcal{R}$ and $\mathcal{L}$ are the capacitance, resistance and inductance matrices. Under the assumptions that $A_\mathcal{V}$ has full column rank, $[A_\mathcal{C}, A_\mathcal{L}, A_\mathcal{R}, A_\mathcal{V}]$ has full row rank and $\mathcal{C}$, $\mathcal{R}$ and $\mathcal{L}$ are positive definite, system (1), (38) is of index at most 2 and passive [48, 109]. It has index 1 if, additionally, $[A_\mathcal{C}, A_\mathcal{L}, A_\mathcal{R}]$ has full row rank and $Z_\mathcal{C}^T A_\mathcal{V}$ has full column rank, where the columns of $Z_\mathcal{C}$ span $\ker(A_\mathcal{C}^T)$.

In model reduction of circuit equations, it is crucial to preserve passivity. This allows a back interpretation of the reduced-order model as an electrical circuit which has fewer electrical components than the original one [6, 109]. Passivity-preserving Krylov subspace based model reduction methods for structured circuit equations have been developed [53, 55, 56, 84, 99], whereas balancing-related methods have been considered in [15, 105, 114, 116, 155]. Unfortunately, the application of the positive real balanced truncation method is currently restricted to small and medium-sized problems, since there exists no explicit representation for the spectral projectors required in the positive real Lur'e equations (18) and (19). In contrast, for the Moebius-transformed system $\boldsymbol{H}_M = (E, A-BC, -\sqrt{2}B, \sqrt{2}C, I)$, the right and left spectral projectors are given by

$$P_r = \begin{bmatrix} H_5(H_4H_2 - I) & H_5H_4A_\mathcal{L}H_7 & 0 \\ 0 & H_7 & 0 \\ -A_\mathcal{V}^T(H_4H_2 - I) & -A_\mathcal{V}^T H_4 A_\mathcal{L} H_7 & 0 \end{bmatrix},$$

$$P_l = \begin{bmatrix} (H_2H_4 - I)H_6 & 0 & (H_2H_4 - I)A_\mathcal{V} \\ -H_8 A_\mathcal{L}^T H_4 H_6 & H_8 & -H_8 A_\mathcal{L}^T H_4 A_\mathcal{V} \\ 0 & 0 & 0 \end{bmatrix},$$

where

$$\begin{aligned}
&H_1 = Z_{\mathcal{CRIV}}^T A_\mathcal{L} \mathcal{L}^{-1} A_\mathcal{L}^T Z_{\mathcal{CRIV}}, \\
&H_2 = A_\mathcal{R} \mathcal{R}^{-1} A_\mathcal{R}^T + A_I A_I^T + A_\mathcal{V} A_\mathcal{V}^T + A_\mathcal{L} \mathcal{L}^{-1} A_\mathcal{L}^T Z_{\mathcal{CRIV}} H_1^{-1} Z_{\mathcal{CRIV}}^T A_\mathcal{L} \mathcal{L}^{-1} A_\mathcal{L}^T, \\
&H_3 = Z_\mathcal{C}^T H_2 Z_\mathcal{C}, \qquad H_4 = Z_\mathcal{C} H_3^{-1} Z_\mathcal{C}^T, \\
&H_5 = Z_{\mathcal{CRIV}} H_1^{-1} Z_{\mathcal{CRIV}}^T A_\mathcal{L} \mathcal{L}^{-1} A_\mathcal{L}^T - I, \quad H_6 = A_\mathcal{L} \mathcal{L}^{-1} A_\mathcal{L}^T Z_{\mathcal{CRIV}} H_1^{-1} Z_{\mathcal{CRIV}}^T - I, \\
&H_7 = I - \mathcal{L}^{-1} A_\mathcal{L}^T Z_{\mathcal{CRIV}} H_1^{-1} Z_{\mathcal{CRIV}}^T A_\mathcal{L}, \quad H_8 = I - A_\mathcal{L}^T Z_{\mathcal{CRIV}} H_1^{-1} Z_{\mathcal{CRIV}}^T A_\mathcal{L} \mathcal{L}^{-1}, \\
&Z_\mathcal{C} \text{ is a basis matrix for } \ker(A_\mathcal{C}^T), \\
&Z_{\mathcal{CRIV}} \text{ is a basis matrix for } \ker([A_\mathcal{C}, A_\mathcal{R}, A_I, A_\mathcal{V}]^T),
\end{aligned}$$

see [114, 139]. This allows us to compute the passive reduced-order model by applying the bounded real balanced truncation to $\boldsymbol{H}_M$ in the large-scale setting. Taking into account the block structure of the system matrices in (38), we can also determine the matrix $M_0 = \lim_{s\to\infty} \boldsymbol{H}_M(s)$ in the form

$$M_0 = \begin{bmatrix} I - 2A_I^T Z H_0^{-1} Z^T A_I & 2A_I^T Z H_0^{-1} Z^T A_\mathcal{V} \\ -2A_\mathcal{V}^T Z H_0^{-1} Z^T A_I & -I + 2A_\mathcal{V}^T Z H_0^{-1} Z^T A_\mathcal{V} \end{bmatrix},$$

where $H_0 = Z^T(A_\mathcal{R}\mathcal{R}^{-1}A_\mathcal{R}^T + A_I A_I^T + A_\mathcal{V} A_\mathcal{V}^T)Z$, $Z = Z_\mathcal{C} Z'_{\mathcal{RIV}-\mathcal{C}}$ and $Z'_{\mathcal{RIV}-\mathcal{C}}$ is a basis matrix for $\mathrm{im}([A_\mathcal{R}, A_I, A_\mathcal{V}]^T Z_\mathcal{C})$. Having this matrix, we do not need to compute the improper Gramians any more. Furthermore, if $\mathcal{C}$, $\mathcal{R}$ and $\mathcal{L}$ are symmetric, then $P_l = P_r^T$ and

the bounded real Gramians G_c^{BR} and G_o^{BR} are related by $G_c^{BR} = S_{\text{int}} G_o^{BR} S_{\text{int}}$ with a signature matrix $S_{\text{int}} = \text{diag}(I_{n_\eta}, -I_{n_\mathcal{L}}, -I_{n_\mathcal{V}})$. In this case, only one Lur'e equation has to be solved that reduces the computational cost.

A further cost reduction can be achieved for RC and RL circuits. The underlying equations for such circuits are either symmetric or they can be transformed to symmetric systems for which passivity-preserving model reduction can be performed employing the Lyapunov balancing [116].

5.4 Stokes-like systems of index 2

Another block structured DAE system arises in computational fluid dynamics, where the flow of an incompressible fluid is modeled by the Navier-Stokes equation. After a linearization along a stationary trajectory and discretization in space by the finite element method, one gets the Stokes-like system

$$\begin{aligned} \begin{bmatrix} E_{11} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{v}(t) \\ \dot{p}(t) \end{bmatrix} &= \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & 0 \end{bmatrix} \begin{bmatrix} v(t) \\ p(t) \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u(t), \\ y(t) &= C_1 v(t) + C_2 p(t) + Du(t), \end{aligned} \tag{39}$$

where $v(t)$ and $p(t)$ are the semidiscretized velocity and pressure vectors. Model reduction of such systems has been considered in [33, 69, 71, 135]. Note that unlike [71], we do not assume here that E_{11} is symmetric and $A_{21} = A_{12}^T$. If E_{11} and $A_{21}E_{11}^{-1}A_{12}$ are both nonsingular, then system (39) is of index 2, and the spectral projectors P_l and P_r have the form

$$P_l = \begin{bmatrix} \Pi_l & -\Pi_l A_{11} E_{11}^{-1} A_{12} (A_{21} E_{11}^{-1} A_{12})^{-1} \\ 0 & 0 \end{bmatrix},$$

$$P_r = \begin{bmatrix} \Pi_r & 0 \\ -(A_{21} E_{11}^{-1} A_{12})^{-1} A_{21} E_{11}^{-1} A_{11} \Pi_r & 0 \end{bmatrix},$$

where

$$\begin{aligned} \Pi_l &= I - A_{12}(A_{21}E_{11}^{-1}A_{12})^{-1} A_{21} E_{11}^{-1}, \\ \Pi_r &= I - E_{11}^{-1} A_{12} (A_{21} E_{11}^{-1} A_{12})^{-1} A_{21} = E_{11}^{-1} \Pi_l E_{11}. \end{aligned}$$

Note that the conditions for A_{12} and A_{21} to be of full rank are not enough for the index-2 property unless E_{11} is symmetric and positive definite. It has been shown in [71] that the velocity and pressure vectors can be determined as

$$\begin{aligned} v(t) &= v_0(t) - E_{11}^{-1} A_{12} (A_{21} E_{11}^{-1} A_{12})^{-1} B_2 u(t), \\ p(t) &= -(A_{21} E_{11}^{-1} A_{12})^{-1} \big(A_{21} E_{11}^{-1} A_{11} v_0(t) + A_{21} E_{11}^{-1} B_{12} u(t) + B_2 \dot{u}(t) \big), \end{aligned}$$

where $B_{12} = B_1 - A_{11} E_{11}^{-1} A_{12} (A_{21} E_{11}^{-1} A_{12})^{-1} B_2$ and $v_0(t) = \Pi_r v_0(t)$ solves the DAE system

$$\begin{aligned} \hat{E} \dot{v}_0(t) &= \hat{A} v_0(t) + \hat{B} u(t), \\ y(t) &= \hat{C} v_0(t) + \hat{D} u(t) + \hat{D}_1 \dot{u}(t), \end{aligned} \tag{40}$$

with

$$\begin{aligned} &\hat{E} = \Pi_l E_{11} \Pi_r, \qquad \hat{A} = \Pi_l A_{11} \Pi_r, \qquad \hat{B} = \Pi_l B_{12}, \\ &\hat{C} = C_1 - C_2 (A_{21} E_{11}^{-1} A_{12})^{-1} A_{21} E_{11}^{-1} A_{11}, \\ &\hat{D} = D - C_1 E_{11}^{-1} A_{12} (A_{21} E_{11}^{-1} A_{12})^{-1} B_2 - C_2 (A_{21} E_{11}^{-1} A_{12})^{-1} A_{21} E_{11}^{-1} B_{12}, \\ &\hat{D}_1 = -C_2 (A_{21} E_{11}^{-1} A_{12})^{-1} B_2. \end{aligned} \tag{41}$$

Note that the matrices $\hat{E}$ and $\hat{A}$ in (41) have a common nontrivial kernel, and, hence, $\lambda\hat{E}-\hat{A}$ is singular for all $\lambda\in\mathbb{C}$. At first glance, this renders the application of balanced truncation and interpolatory-based model reduction methods to (40) impossible since there the inversion of $\hat{E}+\tau_k\hat{A}$ (or $\hat{A}-s_k\hat{E}$) is required. Fortunately, these matrices can be inverted on a subspace. Then the LR-ADI iteration for the projected Lyapunov equation

$$\hat{A}X\hat{E}^T+\hat{E}X\hat{A}^T=-\hat{B}\hat{B}^T$$

associated with (40) can be reformulated as

$$\begin{aligned}
&\hat{W}_0=B_{12}, \qquad Z_0=[\ ],\\
&\hat{V}_k=(\hat{E}+\tau_k\hat{A})^-\hat{W}_{k-1},\\
&\hat{W}_k=\hat{W}_{k-1}-2\mathrm{Re}(\tau_k)A_{11}\hat{V}_k,\\
&\hat{Z}_k=[\hat{Z}_{k-1},\ \sqrt{-2\mathrm{Re}(\tau_k)}\hat{V}_k],
\end{aligned} \tag{42}$$

where $(\hat{E}+\tau_k\hat{A})^-$ is the reflexive inverse of $\hat{E}+\tau_k\hat{A}$ with respect to Π_l and Π_r. Taking into account the structure of $\hat{E}$ and $\hat{A}$, the matrices $\hat{V}_k=(\hat{E}+\tau_k\hat{A})^-\hat{W}_{k-1}$ can be computed by solving the linear matrix equation

$$\begin{bmatrix} E_{11}+\tau_kA_{11} & A_{12}\\ A_{21} & 0\end{bmatrix}\begin{bmatrix}\hat{V}_k\\ V\end{bmatrix}=\begin{bmatrix}\hat{W}_{k-1}\\ 0\end{bmatrix}$$

with sparse (if E_{11} and A_{ij} are sparse) coefficient matrix. The main advantage of the LR-ADI iteration (42) over those in Algorithm 7 is that the matrices $\hat{V}_k$, $\hat{W}_k$ and $\hat{Z}_k$ have smaller dimension than V_k, W_k and Z_k, respectively, and no multiplication with the projectors is required. For further details of this novel formulation of the ADI iteration and its specific implementation for Stokes-like equations, see [33], where also an extension of balanced truncation to unstable descriptor systems is considered. Further note that LQG balanced truncation for (Navier-)Stokes flow is discussed in [21].

5.5 Mechanical systems of index 1 and 3

Consider a second-order DAE system

$$\begin{aligned}
\begin{bmatrix}M_{11} & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}\ddot{p}(t)\\ \ddot{\eta}(t)\end{bmatrix}+\begin{bmatrix}\mathcal{D}_{11} & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}\dot{p}(t)\\ \dot{\eta}(t)\end{bmatrix}+\begin{bmatrix}K_{11} & K_{12}\\ K_{21} & K_{22}\end{bmatrix}\begin{bmatrix}p(t)\\ \eta(t)\end{bmatrix} &= \begin{bmatrix}B_1\\ B_2\end{bmatrix}u(t),\\
C_1p(t)+C_2\eta(t) &= y(t),
\end{aligned} \tag{43}$$

where $p(t)$ is a displacement vector and $\eta(t)$ is a vector of electrical potentials. Such systems frequently arise in mechatronics, where micro-electromechanical devices are of great interest, e.g., [144]. Introducing $x(t)=[p^T(t),\dot{p}^T(t),\eta^T(t)]^T$, system (43) can be written as the first-order DAE system (1) with

$$E=\begin{bmatrix}I & 0 & 0\\ 0 & M_{11} & 0\\ 0 & 0 & 0\end{bmatrix},\ A=\begin{bmatrix}0 & I & 0\\ -K_{11} & -\mathcal{D}_{11} & -K_{12}\\ -K_{21} & 0 & -K_{22}\end{bmatrix},\ B=\begin{bmatrix}0\\ B_1\\ B_2\end{bmatrix},\ C=[C_1,0,C_2],\ D=0.$$

If M_{11} and K_{22} are both nonsingular, then this system (and also (43)) is of index 1. Similarly to the semi-explicit DAE system (31), (32), system (43) can be rewritten in the compact form

$$\begin{aligned}
M_{11}\ddot{p}(t)+\mathcal{D}_{11}\dot{p}(t)+\hat{K}_{11}p(t) &= \hat{B}u(t),\\
y(t) &= \hat{C}p(t)+\hat{D}u(t),
\end{aligned}$$

where $\hat{K}_{11} = K_{11} - K_{12}K_{22}^{-1}K_{21}$, $\hat{B} = B_1 - K_{12}K_{22}^{-1}B_2$, $\hat{C} = C_1 - C_2K_{22}^{-1}K_{21}$ and $\hat{D} = C_2K_{22}^{-1}B_2$. Applying the second-order balanced truncation method as proposed in [26, 31] or the second-order Krylov subspace methods [8, 125] requires the solution of the linear systems $(\tau^2 M_{11} \pm \tau\mathcal{D}_{11} + \hat{K}_{11})z = f$. Employing the structure of the involved matrices, the vector z can be determined by solving the sparse system

$$\begin{bmatrix} \tau^2 M_{11} \pm \tau\mathcal{D}_{11} + K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} \begin{bmatrix} z \\ g \end{bmatrix} = \begin{bmatrix} f \\ 0 \end{bmatrix}$$

using a sparse LU factorization or Krylov subspace methods [123].

The dynamical behaviour of linear multibody systems with holonomic constraints is described by the Euler-Lagrange equations

$$\begin{aligned} \begin{bmatrix} I & 0 & 0 \\ 0 & M & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{p}(t) \\ \dot{v}(t) \\ \dot{\lambda}_p(t) \end{bmatrix} &= \begin{bmatrix} 0 & I & 0 \\ -K & -\mathcal{D} & -G^T \\ G & 0 & 0 \end{bmatrix} \begin{bmatrix} p(t) \\ v(t) \\ \lambda_p(t) \end{bmatrix} + \begin{bmatrix} 0 \\ B \\ 0 \end{bmatrix} u(t), \\ y(t) &= C_p p(t) + C_v v(t), \end{aligned} \tag{44}$$

where $p(t)$ and $v(t)$ are the position and velocity vectors, $\lambda_p(t)$ is the Lagrange multiplier, M, $\mathcal{D}$ and K are the mass, stiffness and damping matrices, respectively, and G is a matrix of constraints. If M and $GM^{-1}G^T$ are both nonsingular, then system (44) is of index 3. Exploiting the block structure of the system matrices, the spectral projectors P_l and P_r can be computed as

$$\begin{aligned} P_l &= \begin{bmatrix} \Pi_r & 0 & \Pi_r M^{-1}\mathcal{D}G_1 \\ \Pi_r^T\mathcal{D}(I-\Pi_r) & \Pi_r^T & \Pi_r^T(K - \mathcal{D}\Pi_r M^{-1}\mathcal{D})G_1 \\ 0 & 0 & 0 \end{bmatrix}, \\ P_r &= \begin{bmatrix} \Pi_r & 0 & 0 \\ \Pi_r M^{-1}\mathcal{D}(I-\Pi_r) & \Pi_r & 0 \\ -G_1^T(K\Pi_r + \mathcal{D}\Pi_r M^{-1}\mathcal{D}(I-\Pi_r)) & -G_1^T\mathcal{D}\Pi_r & 0 \end{bmatrix}, \end{aligned}$$

where $G_1 = M^{-1}G^T(GM^{-1}G^T)^{-1}$ and

$$\Pi_r = I - M^{-1}G^T(GM^{-1}G^T)^{-1}G = I - G_1G$$

is a projector onto the constraint manifold $\ker(G)$. Instead of using the spectral projectors P_l and P_r explicitly, one can reformulate the DAE system (44) in such a way that only the implicit projection is needed. This can be achieved by the Gear-Gupta-Leimkuhler formulation [59] given by

$$\begin{aligned} \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & M & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{p}(t) \\ \dot{v}(t) \\ \dot{\lambda}_p(t) \\ \dot{\lambda}_v(t) \end{bmatrix} &= \begin{bmatrix} 0 & I & 0 & -G^T \\ -K & -\mathcal{D} & -G^T & 0 \\ G & 0 & 0 & 0 \\ 0 & G & 0 & 0 \end{bmatrix} \begin{bmatrix} p(t) \\ v(t) \\ \lambda_p(t) \\ \lambda_v(t) \end{bmatrix} + \begin{bmatrix} 0 \\ B \\ 0 \\ 0 \end{bmatrix} u(t), \\ y(t) &= C_p p(t) + C_v v(t) \end{aligned} \tag{45}$$

which has index 2. In computational multibody dynamics, (45) is also known as stabilized index-2 formulation of the equations of motion, e.g., [38]. It can be obtained by differentiating the position-level constraint $Gp(t) = 0$ and adding the resulting velocity-level constraint equation $Gv(t) = 0$ to (44) by introducing an additional Lagrange multiplier λ_v. It

was shown in [59] that if (p,v,λ_p) is a solution of (44), then $(p,v,\lambda_p,\lambda_v)$ with $\lambda_v = 0$ is a solution of (45). Conversely, if $(p,v,\lambda_p,\lambda_v)$ solves (45), then $\lambda_v = 0$ and (p,v,λ_p) satisfies (44). Observe that system (45) has the Stokes-like form (39) with

$$E_{11} = \begin{bmatrix} I & 0 \\ 0 & M \end{bmatrix}, A_{11} = \begin{bmatrix} 0 & I \\ -K & -\mathcal{D} \end{bmatrix}, A_{12} = \begin{bmatrix} 0 & -G^T \\ -G^T & 0 \end{bmatrix}, A_{21} = \begin{bmatrix} G & 0 \\ 0 & G \end{bmatrix},$$
$$B_1 = \begin{bmatrix} 0 \\ B \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad C_1 = [C_p,\ C_v], \quad C_2 = 0, \quad D = 0. \tag{46}$$

Therefore, all results of Section 5.4 can be applied to the constrained mechanical system (45). Exploiting the block structure of the matrices in (46), we obtain the second-order system

$$\begin{aligned} \hat{M}\ddot{p}(t) + \hat{\mathcal{D}}\dot{p}(t) + \hat{K}p(t) &= \hat{B}u(t), \\ y(t) &= \hat{C}_p p(t) + \hat{C}_v \dot{p}(t) \end{aligned} \tag{47}$$

for the position vector $p(t) = \Pi p(t)$, where

$$\begin{aligned} \hat{M} &= \Pi_l M \Pi, & \hat{\mathcal{D}} &= \Pi_l \mathcal{D} \Pi, & \hat{K} &= \Pi_l K \Pi, \\ \hat{B} &= \Pi_l B, & \hat{C}_p &= C_p \Pi, & \hat{C}_v &= C_v \Pi, \\ \Pi_l &= M \Pi_r M^{-1}, & \Pi &= I - G^T (G G^T)^{-1} G. \end{aligned}$$

Combining the balanced truncation technique from [71] with the second-order LR-ADI method presented in [26, 31], we can derive an efficient computational procedure for model reduction of system (47) which does not require forming the first-order system. This procedure involves solving projected linear systems

$$\Pi_l (\tau^2 M - \tau \mathcal{D} + K) \Pi z = \Pi_l f$$

whose solution $z = \Pi z$ can be determined from the saddle point linear system

$$\begin{bmatrix} \tau^2 M - \tau \mathcal{D} + K & G^T \\ G & 0 \end{bmatrix} \begin{bmatrix} z \\ g \end{bmatrix} = \begin{bmatrix} f \\ 0 \end{bmatrix}$$

without computing the projectors Π_l and Π.

6 Other model reduction topics

In this section, we briefly discuss other works related to model reduction of DAE systems. This list is far from complete and rather provides a very short overview of recent developments in this active research area.

Model reduction of periodic discrete-time descriptor systems

The balanced truncation model reduction method can also be formulated for discrete-time DAEs. In this case, instead of projected continuous-time Lyapunov equations (13) and (14), one has to solve the projected discrete-time Lyapunov equations

$$A X A^T - E X E^T = -P_l B B^T P_l^T, \qquad X = P_r X P_r^T, \tag{48}$$
$$A^T Y A - E^T Y E = -P_r^T C^T C P_r, \qquad Y = P_l^T Y P_l, \tag{49}$$

introduced in [133].

Model reduction of periodic discrete-time descriptor systems

$$\begin{aligned} E_k x_{k+1} &= A_k x_k + B_k u_k, \\ y_k &= C_k x_k, \end{aligned} \tag{50}$$

where $E_k \in \mathbb{R}^{\mu_{k+1}\times n_{k+1}}$, $A_k \in \mathbb{R}^{\mu_{k+1}\times n_k}$, $B_k \in \mathbb{R}^{\mu_{k+1}\times m_k}$, $C_k \in \mathbb{R}^{q_k\times n_k}$ are periodic with a period $K \geq 1$, $\sum_{k=0}^{K-1} \mu_k = \sum_{k=0}^{K-1} n_k = n$, $\sum_{k=0}^{K-1} m_k = m$ and $\sum_{k=0}^{K-1} q_k = q$, has been considered in [24, 39]. The Gramians for such systems can be determined as solutions of periodic projected Lyapunov equations. Using a lifted representation [132] for the periodic descriptor system (50), these equations can be written in the form (15), (16) and (48), (49) with block structured matrices $E, A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$ and $C \in \mathbb{R}^{q\times n}$. The efficient solution of these lifted systems using methods from Section 4.1 adapted to exploit the block sparsity in the lifted system matrices is considered in [23, 24, 73].

Index-aware model reduction for DAEs

In [1, 2], an index-aware model reduction approach was proposed for DAE systems which is based on splitting the DAE into an ODE system and a system of algebraic equations. It was shown in [1] that the index-1 DAE system (1) can be written in the form

$$\begin{aligned} \dot{x}_1(t) &= A_{11}x_1(t) + B_1 u(t), & y_1(t) &= C_1 x_1(t), \\ x_2(t) &= A_{21}x_1(t) + B_2 u(t), & y(t) &= y_1(t) + C_2 x_2(t) + Du(t), \end{aligned} \tag{51}$$

where

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} W_1^T \\ W_2^T \end{bmatrix} x(t), \quad \begin{bmatrix} A_{11} \\ A_{21} \end{bmatrix} = \begin{bmatrix} W_1^T \\ W_2^T \end{bmatrix} E_1^{-1} A T_1, \quad \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} = \begin{bmatrix} W_1^T \\ W_2^T \end{bmatrix} E_1^{-1} B,$$

$$E_1 = E - A T_2 W_2^T, \qquad [C_1,\, C_2] = C[T_1,\, T_2], \qquad [W_1,\, W_2]^T = [T_1,\, T_2]^{-1},$$

and the columns of the matrices T_1 and T_2 form the basis of $\mathrm{im}(E^T)$ and $\ker(E)$, respectively. Then the ODE system (51) is approximated by a reduced-order model

$$\dot{\widetilde{x}}_1(t) = \widetilde{A}_{11}\widetilde{x}_1(t) + \widetilde{B}_1 u(t), \qquad \widetilde{y}_1(t) = \widetilde{C}_1 \widetilde{x}_1(t)$$

with $\widetilde{A}_{11} = V^T A_{11} V$, $\widetilde{B}_1 = V^T B_1$ and $\widetilde{C}_1 = C_1 V$ using any projection-based model reduction method, and $\widetilde{y}(t) = \widetilde{y}_1(t) + C_2 A_{21} V \widetilde{x}_1(t) + (C_2 B_2 + D)u(t)$ approximates the output $y(t)$. The transformation matrix $[T_1,\, T_2]$ can be determined from the sparse LUQ factorization [85] of E^T as a product of a permutation matrix and a sparse lower triangular matrix, and its inverse is computed by forward substitution, see [157] for a detailed discussion. The index-aware model reduction approach was also extended in [2] to DAEs of index 2. It should be noted that this approach does not require any special structure of the matrices E and A, but its efficiency strongly relies on sparsity of the matrix $A_{11} = W_1^T E_1^{-1} A T_1$. Even if $[T_1,\, T_2]$ is sparse, the multiplication with E_1^{-1} may result in a full matrix that makes this approach unfeasible for large-scale problems.

Parametric model reduction

In the last years, model reduction of parameterized systems has received a lot of attention, see [20] for an overview and numerous references. Here, we are only going to provide a brief sketch on some approaches, noting that a lot remains to be done to adapt some of them to descriptor systems, and to exploit special structures as in Section 5.

Consider a linear parametric DAE system

$$\begin{aligned} E(p)\dot{x}(t,p) &= A(p)x(t,p)+B(p)u(t), \\ y(t,p) &= C(p)x(t,p), \end{aligned} \tag{52}$$

where the system matrices and, hence, the state and the output depend on a parameter $p \in \mathbb{P} \subset \mathbb{R}^d$. Such systems appear frequently in control design and optimization problems, where parameters describe varying geometric configurations and material characteristics. When approximating the parametric system, it is important to preserve the parameter dependence in the reduced-order model. For parametric model reduction, different techniques have been developed over the years, that are, in some sense, extensions of traditional non-parametric model reduction approaches, see [20] for a survey of state-of-the-art parametric model reduction methods. In Krylov subspace based methods [9, 17, 41, 49, 90], the transfer function

$$\mathbf{H}(s,p) = C(p)(sE(p)-A(p))^{-1}B(p)$$

of (52) is approximated by

$$\widetilde{\mathbf{H}}(s,p) = \widetilde{C}(p)(s\widetilde{E}(p)-\widetilde{A}(p))^{-1}\widetilde{B}(p)$$

of lower dimension that satisfies (tangential) interpolation conditions with respect to s and p. Another class of the parametric model reduction methods is based on interpolation. For selected parameters $p_1,\ldots,p_k \in \mathbb{P}$, one computes first the reduced-order local models

$$\begin{aligned} \widetilde{E}_j\dot{\widetilde{x}}_j(t) &= \widetilde{A}_j\widetilde{x}_j(t)+\widetilde{B}_j u(t), \\ \widetilde{y}_j(t) &= \widetilde{C}_j\widetilde{x}_j(t), \end{aligned}$$

where $\widetilde{E}_j = W_j^T E(p_j)T_j$, $\widetilde{A}_j = W_j^T A(p_j)T_j$, $\widetilde{B}_j = W_j^T B(p_j)$ and $\widetilde{C}_j = C(p_j)T_j$. Then a parameter-dependent reduced-order model is constructed by using one of the following interpolation approaches:

1. *interpolation in the frequency domain* [10, 50, 129], where the reduced transfer function is obtained by interpolation of the reduced local transfer functions

$$\widetilde{\boldsymbol{H}}(s,p) = \sum_{j=1}^{k} f_j(p)\widetilde{C}_j(s\widetilde{E}_j-\widetilde{A}_j)^{-1}\widetilde{B}_j;$$

2. *interpolation in the time domain* [3, 5, 42, 61, 101], where the reduced-order model is derived by interpolation of the reduced system matrices

$$\begin{aligned} \widetilde{E}(p) &= \sum_{j=1}^{k} f_j(p)\widetilde{E}_j, & \widetilde{A}(p) &= \sum_{j=1}^{k} f_j(p)\widetilde{A}_j, \\ \widetilde{B}(p) &= \sum_{j=1}^{k} f_j(p)\widetilde{B}_j, & \widetilde{C}(p) &= \sum_{j=1}^{k} f_j(p)\widetilde{C}_j; \end{aligned}$$

3. *interpolation of the projection subspaces* [4, 130], where the reduced-order model is determined by projection

$$\begin{aligned} \widetilde{E}(p) &= W^T(p)E(p)T(p), & \widetilde{A}(p) &= W^T(p)A(p)T(p), \\ \widetilde{B}(p) &= W^T(p)B(p), & \widetilde{C}(p) &= C(p)T(p), \end{aligned}$$

and the projection matrices $W(p)$ and $T(p)$ are obtained by interpolation of $W_1,\ldots,W_k$ and $T_1,\ldots,T_k$, respectively, on the Grassmann manifolds.

For an extension of these methods to descriptor systems and a comparative analysis of them, with particular focus on their application to circuit equations, we refer to [131].

Model reduction of nonlinear parametric DAEs arising in circuit simulation using a reduced bases method was considered in [43].

7 Conclusions

We have surveyed model order reduction methods for linear descriptor systems, i.e., systems with input-output structure and dynamics described by systems of differential-algebraic equations. We have seen that most methods based on system-theoretic approaches such as balanced truncation and the related family of balancing-based methods as well as methods based on rational interpolation of the associated transfer function can be adapted to descriptor systems by using appropriate spectral projectors. As an extension of the available literature, we have extended the method of balanced stochastic truncation to descriptor systems. The presented approaches rely on the availability of the spectral projectors. Often, in applications, these can be formed explicitly without additional computation by a smart usage of the structure arising from the different applications. Moreover, the explicit formation of the spectral projectors can usually be avoided using clever implementations of the algorithms needed, e.g., to compute the factors of the system Gramians used in balancing-based methods. These Gramians are solution of projected algebraic Lyapunov or Riccati equations. We have shown recent advances in the numerical methods to solve these projected matrix equations. Details of the projector-avoiding strategies have been discussed for various engineering problems leading to descriptor systems of index 1, 2, or 3, resulting in specialized implementations of the model order reduction methods.

Future work in this area will address extensions of the methods discussed to nonlinear systems. Such extensions of the system-theoretic methods for nonlinear systems described by ordinary differential equations have been surveyed recently in [11]. First attempts focusing on bilinear descriptor systems as discussed in [19, 64] show that in particular the interpolatory approaches carry over directly when the underlying structure is carefully exploited. The extension of these results to more general classes of nonlinear descriptor systems will require further research efforts in the future.

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